Intuitionistic fuzzy generalized normed spaces

1S. G. Dapke, 2C. T. Aage and 3J. N. Salunke

Received 5 May 2013; Revised 27 June 2013; Accepted 17 July 2013

Abstract. The aim of this paper is to study the generalization of the intuitionistic fuzzy normed spaces such as intuitionistic fuzzy 2-normed space. In this structure, we have discussed the intuitionistic fuzzy 2-continuity and intuitionistic fuzzy 2-boundedness. Also, we have introduced the intuitionistic fuzzy ψ-2-normed space which is a generalization of intuitionistic fuzzy 2-normed space. We have discussed some results in this new set up.


Keywords: t-norm, t-conorm, intuitionistic fuzzy 2-normed space, intuitionistic fuzzy 2-continuity, intuitionistic fuzzy 2-boundedness, intuitionistic fuzzy ψ-2-normed space.

1. INTRODUCTION

The theory of fuzzy sets was introduced by Zadeh[19] in 1965. After the pioneer work of Zadeh, many researchers have extended this concept in various branches of mathematics and introduced new theories like fuzzy group theory, fuzzy differential equation, fuzzy topology, fuzzy normed spaces [14] etc. We are especially interested in theory of fuzzy normed spaces and their generalizations. Atanassov[3] introduced the concept of intuitionistic fuzzy sets which is further studied by Coker[4]. Park[13] has introduced the concept of intuitionistic fuzzy metric space. Saadati and Park[14] coined the notion of intuitionistic fuzzy normed space. Hee Won Kang, Jeong-Gon Lee, Kul Hur[8] studied some fundamental properties of intuitionistic fuzzy mapping. Certainly, there are some situations where the ordinary norm does not work and the concept of intuitionistic fuzzy norm seems to be more suitable. A lot of works have been done in intuitionistic fuzzy normed spaces see in [17],[11],[12],[14], [6], [18].

Recently, M. Mursaleen[10] defined the new structure intuitionistic fuzzy 2-normed space and studied some basic results of normed linear spaces. In this paper, we have studied the continuity and boundedness in intuitionistic fuzzy 2-normed spaces. T.K. Samanta and Sumit Mohinta[15] have introduced the concept of intuitionistic fuzzy ψ-normed space and discussed continuity and boundedness in this structure. We
have coined the concept of intuitionistic fuzzy $\psi$-2-normed space which is generalization of intuitionistic fuzzy 2-normed space. It shall provide more suitable framework to deal with the inexactness of the norm or 2-norm in some situations.

2. Preliminaries

We recall some notations and basic definitions used in this paper.

**Definition 2.1.** [16] A binary operation $\ast : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(a) $\ast$ is associative and commutative;
(b) $\ast$ is continuous;
(c) $a \ast 1 = a$ for all $a \in [0, 1]$;
(d) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

**Example 2.2.** Two typical examples of continuous t-norms are $a \ast b = ab$ and $a \ast b = \min\{a, b\}$.

**Definition 2.3.** [16] A binary operation $\cdot : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(a) $\cdot$ is associative and commutative;
(b) $\cdot$ is continuous;
(c) $a \cdot 0 = a$ for all $a \in [0, 1]$;
(d) $a \cdot b \leq c \cdot d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

**Example 2.4.** Two typical examples of continuous t-conorms are $a \cdot b = \min\{a + b, 1\}$ and $a \cdot b = \max\{a, b\}$.

**Definition 2.5.** [14] The five-tuple $(V, \mu, \nu, \ast, \cdot)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if $V$ is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $\ast$ is a continuous t-norm, $\cdot$ is a continuous t-conorm, and $\mu, \nu$ are fuzzy sets on $V \times (0, \infty)$ satisfying the following conditions. For every $x, y \in V$ and $s, t > 0$,

(a) $\mu(x, t) + \nu(x, t) \leq 1$;
(b) $\mu(x, t) > 0$;
(c) $\mu(x, t) = 1$ if and only if $x = 0$;
(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(e) $\mu(x, t) \ast \mu(y, s) \leq \mu(x + y, t + s)$;
(f) $\mu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
(g) $\lim_{t \rightarrow \infty} \mu(x, t) = 1$ and $\lim_{t \rightarrow 0} \mu(x, t) = 0$;
(h) $\nu(x, t) < 1$;
(i) $\nu(x, t) = 0$ if and only if $x = 0$;
(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
(k) $\nu(x, t) \ast \nu(y, s) \geq \nu(x + t, y + s)$;
(l) $\nu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
(m) $\lim_{t \rightarrow \infty} \nu(x, t) = 0$ and $\lim_{t \rightarrow 0} \nu(x, t) = 1$.

In this case $(\mu, \nu)$ is called an intuitionistic fuzzy norm.
Example 2.6. Let \((V, \| \cdot \|)\) be normed space over \(F\). Denote \(a \ast b = ab\) and \(a \odot b = \min\{a + b, 1\}\), \(\forall a, b \in [0, 1]\) and let \(\mu_0\) and \(\nu_0\) be fuzzy sets on \(V \times (0, \infty)\) defined as follows \(\mu_0(x, t) = \frac{t}{t + \|x\|}, \nu_0(x, t) = \frac{\|x\|}{t + \|x\|}\), for all \(t \in \mathbb{R}^+\). Then \((V, \mu_0, \nu_0, *, \odot)\) is an intuitionistic fuzzy normed space over \(F \in \{\mathbb{R}, \mathbb{C}\}\).

Definition 2.7. \([10]\) Let \(V\) be a real vector space of dimension \(d\), where \(2 \leq d < \infty\). A 2-norm on \(V\) is a function \(\| \cdot , \cdot \| : V \times V \rightarrow \mathbb{R}\) which satisfies, for every \(x, y, z \in V\)

\[
\begin{align*}
(a) & \quad \|x, y\| = 0 \text{ if and only if } x \text{ and } y \text{ are linearly dependent}; \\
(b) & \quad \|x, y\| = \|y, x\|; \\
(c) & \quad \|\alpha x, y\| = |\alpha| \|x, y\|; \\
(d) & \quad \|x, y \pm z\| \leq \|x, y\| + \|y, z\|.
\end{align*}
\]

The pair \((V, \| \cdot , \cdot \|)\) is then called a 2-normed space.

As an example of a 2-normed space take \(V = \mathbb{R}^2\) being equipped with the 2-norm \(\|x, y\| := \text{the area of the parallelogram spanned by the vectors } x \text{ and } y\), which may be given explicitly by the formula \(\|x, y\| = |x_1y_2 - x_2y_1|, x = (x_1, x_2), y = (y_1, y_2)\).

Definition 2.8. \([10]\) The five-tuple \((V, \mu, \nu, *, \odot)\) is said to be an intuitionistic fuzzy 2-normed space (for short, IF 2-NS) if \(V\) is a vector space over \(F \in \{\mathbb{R}, \mathbb{C}\}\), * is a continuous t-norm, \(\odot\) is a continuous t-conorm, and \(\mu, \nu\) are fuzzy sets on \(V \times V \times (0, \infty)\) satisfying the following conditions. For every \(x, y, z \in V\) and \(s, t > 0\),

\[
\begin{align*}
(a) & \mu(x, y, t) + \nu(x, y, t) \leq 1; \\
(b) & \mu(x, y, t) > 0; \\
(c) & \mu(x, y, t) = 1 \text{ if and only if } x \text{ and } y \text{ are linearly dependent}; \\
(d) & \mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|}) \text{ for each } \alpha \neq 0; \\
(e) & \mu(x, y, t) * \mu(x, z, s) \leq \mu(x, y + z, t + s); \\
(f) & \mu(x, y, \cdot) : (0, \infty) \rightarrow [0, 1] \text{ is continuous}; \\
(g) & \lim_{t \rightarrow \infty} \mu(x, y, t) = 1 \text{ and } \lim_{t \rightarrow 0} \mu(x, y, t) = 0; \\
(h) & \nu(x, y, t) = \mu(y, x, t); \\
(i) & \nu(x, y, t) < 1; \\
(j) & \nu(x, y, t) = 0 \text{ if and only if } x \text{ and } y \text{ are linearly dependent}; \\
(k) & \nu(\alpha x, y, t) = \nu(x, y, \frac{t}{|\alpha|}) \text{ for each } \alpha \neq 0; \\
(l) & \nu(x, y, t) \odot \nu(x, z, s) \geq \nu(x, y + z, t + s); \\
(m) & \nu(x, y, \cdot) : (0, \infty) \rightarrow [0, 1] \text{ is continuous}; \\
(n) & \lim_{t \rightarrow \infty} \nu(x, y, t) = 0 \text{ and } \lim_{t \rightarrow 0} \nu(x, y, t) = 1, \\
(o) & \nu(x, y, t) = \nu(y, x, t).
\end{align*}
\]

In this case \((\mu, \nu)_2\) is called an intuitionistic fuzzy 2-norm on \(V\). We denote it by \((\mu, \nu)_2\).

Example 2.9. \([10]\) Let \((V, \| \cdot , \cdot \|)\) be 2-normed space over \(F\) and let \(a \ast b = ab\) and \(a \odot b = \min\{a + b, 1\}\), for all \(a, b \in [0, 1]\) and every \(t > 0\), consider \(\mu(x, y, t) = \frac{t}{t + \|x, y\|}, \nu(x, y, t) = \frac{\|x, y\|}{t + \|x, y\|}\). Then \((V, \mu, \nu, *, \odot)\) is an intuitionistic fuzzy 2-normed space.

Definition 2.10. \([10]\) Let \((V, \mu, \nu, *, \odot)\) be an intuitionistic fuzzy 2-normed space and let \(r \in (0, 1), t > 0\) and \(x \in X\). The set \(B(x, r, t) = \{y \in V : \mu(y - x, z, t) > 0\}\) is an intuitionistic fuzzy 2-ball centered at \(x\) with radius \(r\) and thickness \(t\).
1 - r, \( \nu(y - x, z, t) < r, \forall z \in V \) is called the open ball with center \( x \) and radius \( r \) with respect to \( t \).

**Definition 2.11.** [10] Let \( (V, \mu, \nu, *, \diamond) \) be an intuitionistic fuzzy 2-normed space. A set \( U \subseteq V \) is said to be an open set if each of its points is the centre of some open ball contained in \( U \). The open set in an intuitionistic fuzzy 2-normed space \( (V, \mu, \nu, *, \diamond) \) is denoted by \( U \).

**Definition 2.12.** [10] Let \( (V, \mu, \nu, *, \diamond) \) be an intuitionistic fuzzy 2-normed space. A sequence \( \{x_n\} \) in \( V \) is said to be Cauchy if for each \( r > 0 \) and each \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( \mu(x_n - x_m, z, t) > 1 - r \) and \( \nu(x_n - x_m, z, t) < r \) for all \( n, m \geq n_0 \) and for all \( z \in V \).

**Definition 2.13.** [10] Let \( (V, \mu, \nu, *, \diamond) \) be an intuitionistic fuzzy 2-normed space. A sequence \( \{x_k\} \) is said to be convergent to \( L \in V \) with respect to the intuitionistic fuzzy 2-norm \( (\mu, \nu)_2 \), if for every \( \epsilon > 0 \) and \( t > 0 \), there exists \( k_0 \in \mathbb{N} \) such that \( \mu(x_k - L, z, t) > 1 - \epsilon \) and \( \nu(x_k - L, z, t) < \epsilon \) for all \( k \geq k_0 \) and for all \( z \in V \).

**Definition 2.14.** [15] Let \( \psi \) be a function defined on the real field \( \mathbb{R} \) into itself satisfying the following properties:

- (a) \( \psi(-t) = \psi(t) \) for all \( t \in \mathbb{R} \)
- (b) \( \psi(1) = 1 \)
- (c) \( \psi \) is strictly increasing and continuous on \( (0, \infty) \)
- (d) \( \lim_{\alpha \to 0} \psi(\alpha) = 0 \) and \( \lim_{\alpha \to \infty} \psi(\alpha) = \infty \).

**Example 2.15.** [15] Consider \( \psi(\alpha) = |\alpha|; \psi(\alpha) = |\alpha|^p, p \in \mathbb{R}^+; \psi(\alpha) = \frac{2^n}{|\alpha|+1}, n \in \mathbb{N}^+ \). The function \( \psi \) allows us to generalize fuzzy metric and normed space.

**Definition 2.16.** [15] The five-tuple \( (V, \mu, \nu, *, \diamond) \) is said to be an intuitionistic fuzzy \( \psi \)-normed space if \( V \) is a vector space over \( \mathbb{F} \in \{ \mathbb{R}, \mathbb{C} \} \), * is a continuous t-norm, \( \diamond \) is a continuous t-conorm and \( \mu, \nu \) are fuzzy sets on \( V \times (0, \infty) \) satisfying the following conditions. For every \( x, y \in V \) and \( s, t > 0 \),

- (a) \( \mu(x, t) + \nu(x, t) \leq 1 \);
- (b) \( \mu(x, t) > 0 \);
- (c) \( \mu(x, t) = 1 \) if and only if \( x = 0 \);
- (d) \( \mu(ax, t) = \mu(x, \frac{t}{\psi(\alpha)}) \) for each \( \alpha \neq 0 \);
- (e) \( \mu(x, t) * \mu(y, s) \leq \mu(x+y, t+s) \);
- (f) \( \mu(x, \cdot) : (0, \infty) \to [0, 1] \) is continuous;
- (g) \( \lim_{t \to \infty} \mu(x, t) = 1 \) and \( \lim_{t \to 0} \mu(x, t) = 0 \);
- (h) \( \nu(x, t) < 1 \);
- (i) \( \nu(x, t) = 0 \) if and only if \( x = 0 \);
- (j) \( \nu(ax, t) = \nu(x, \frac{t}{\psi(\alpha)}) \) for each \( \alpha \neq 0 \);
- (k) \( \nu(x, t) * \nu(y, s) \geq \nu(x + t, y + s) \);
- (l) \( \nu(x, \cdot) : (0, \infty) \to [0, 1] \) is continuous;
- (m) \( \lim_{t \to \infty} \nu(x, t) = 0 \) and \( \lim_{t \to 0} \nu(x, t) = 1 \).

In this case \( (\mu, \nu) \) is called an intuitionistic fuzzy \( \psi \)-norm.
3. INTUITIONISTIC FUZZY 2-NORMED SPACE

Theorem 3.1. In an intuitionistic fuzzy 2-normed space \((V, \mu, \nu, *, \circ)\), if \(\{x_n\}_{n=1}^{\infty} \xrightarrow{\mu, \nu, \circ} x\) and \(\{y_n\}_{n=1}^{\infty} \xrightarrow{\mu, \nu, \circ} y\) then \(\{x_n + y_n\}_{n=1}^{\infty}\) is convergent to \(x + y\). In other word, if \((V, \mu, \nu, *, \circ)\) be an intuitionistic fuzzy 2-normed space then the addition is continuous in \((V, \mu, \nu, *, \circ)\).

Theorem 3.2. In an intuitionistic fuzzy 2-normed space \((V, \mu, \nu, *, \circ)\), if \(\lambda_n, \lambda \in \mathbb{R}^+, \lambda_n \to \lambda\) as \(n \to \infty\) and \(\{x_n\}_{n=1}^{\infty} \xrightarrow{\mu, \nu, \circ} x\) as \(n \to \infty\) then \(\{\lambda x_n\}_{n=1}^{\infty} \xrightarrow{\mu, \nu, \circ} \lambda x\). In other word, if \((V, \mu, \nu, *, \circ)\) be an IF-2-NS then the scalar multiplication is continuous in \((V, \mu, \nu, *, \circ)\).

Proof. The proof of theorems (3.1) and (3.2) directly follows from definitions. \(\square\)

Lemma 3.3. Let \(\{x_n\}_{n=1}^{\infty} \xrightarrow{\mu, \nu, \circ} x\) as \(n \to \infty\) in intuitionistic fuzzy 2-normed space \((V, \mu, \nu, *, \circ)\). Then for every \(t > 0\) as \(n \to \infty\),

\[
(3.1) \quad \mu(x_n, z, t) \to \mu(x, z, t), \quad \nu(x_n, z, t) \to \nu(x, z, t).
\]

Proof. Let \(\{x_n\}_{n=1}^{\infty} \xrightarrow{\mu, \nu, \circ} x\) as \(n \to \infty\) in \((V, \mu, \nu, *, \circ)\). Then \(t > 0\), \(\forall k \in \mathbb{N}^+\),

\[
\mu(x_n, z, t) = \mu(x_n - x + x, z, \frac{t}{k+1} + \frac{kt}{k+1}) \\
\geq \mu(x_n - x, z, \frac{t}{k+1}) * \mu(x, z, \frac{kt}{k+1}) \\
\to 1 * \mu(x, z, \frac{kt}{k+1})(n \to \infty) \\
= \mu(x, z, \frac{kt}{k+1}),
\]

so \(\lim_{n \to \infty} \mu(x_n, z, t) \geq \mu(x, z, \frac{kt}{k+1}), (k = 1, 2, \cdots)\).

Letting \(k \to +\infty\) yields that,

\[
(3.2) \quad \lim_{n \to \infty} \mu(x_n, z, t) \geq \mu(x, z, t).
\]

On the other hand, for all \(k \in \mathbb{N}^+\), \(\mu(x - x_n, z, \frac{1}{k+1}) \to 1 > \frac{k}{k+1} > 0\), as \(n \to \infty\). So there exists an \(N\) such that, \(\mu(x - x_n, z, \frac{1}{k+1}) > \frac{k}{k+1}\), \(\forall n > N\). Thus, \(\forall n > N\) and \(\forall t > 0\), we have

\[
\mu(x_n, z, t) * \frac{k}{k+1} \leq \mu(x_n, z, t) * \mu(x - x_n, z, \frac{1}{k+1}) \leq \mu(x, z, t + \frac{1}{k+1}).
\]

Thus,

\[
\mu(x_n, z, t) * \frac{k}{k+1} \leq \mu(x, z, t + \frac{1}{k+1}), (\forall n > N).
\]

Hence,

\[
\lim_{n \to \infty} \mu(x_n, z, t) * \frac{k}{k+1} \leq \mu(x, z, t + \frac{1}{k+1}).
\]
for all \( k = 1, 2, 3, \cdots \). Letting \( k \to +\infty \) yields that
\[
\lim_{n \to \infty} \mu(x_n, z, t) \leq \mu(x, z, t).
\]

Now (3.2), (3.3) implies that \( \lim_{n \to \infty} \mu(x_n, z, t) = \mu(x, z, t) \).

Similarly, we get, \( \lim_{n \to \infty} \nu(x_n, z, t) = \nu(x, z, t) \). The proof is completed. \( \square \)

**Theorem 3.4.** In an intuitionistic fuzzy 2-normed space \((V, \mu, \nu, *, \odot)\), the mappings \( \mu, \nu : V \times V \times (0, \infty) \to [0, 1] \) are continuous.

**Proof.** Let \( x \in V \) and \( t > 0 \) with \((x_n, z, t_n) \to (x, z, t)\) as \( n \to \infty \) in \( V \times V \times (0, \infty) \). Then \( x_n \to_\mu x \) as \( n \to \infty \) in \( V \) and \( t_n \to t \) as \( n \to \infty \) in \((0, \infty)\). Thus, for every \( \delta > 0 \) such that \( x_n(z, t_n, \delta) \to \infty \) in \( V \times \mu, \nu \): \( t \), there is \( n_0 \in N \) such that for all \( n \geq n_0 \),
\[
(3.4) \quad t - \delta < t_n < t + \delta \quad \text{and} \quad \mu(x - x_n, z, \delta) > 1 - \delta, \quad \nu(x - x_n, z, \delta) < \delta.
\]

Hence, for all \( n \geq n_0 \), we see from (3.4)
\[
\mu(x_n, z, t_n) \geq \mu(x_n, z, t - \delta)
\]
\[
= \mu(x_n - x + x, z, \delta + t - 2\delta)
\]
\[
\geq \mu(x_n - x, z, \delta) * \mu(x, z, t - 2\delta)
\]
\[
\geq (1 - \delta) * \mu(x, z, t - 2\delta)
\]
and
\[
\nu(x_n, z, t_n) \leq \nu(x_n, z, t - \delta)
\]
\[
= \nu(x_n - x + x, z, \delta + t - 2\delta)
\]
\[
\leq \nu(x - x_n, z, \delta) \circ \nu(x, z, t - 2\delta)
\]
\[
\leq \delta \circ \nu(x, z, t - 2\delta).
\]

Thus, for all \( n \geq n_0 \), \( \mu(x_n, z, t_n) \geq (1 - \delta) * \mu(x, z, t - 2\delta) \) and \( \nu(x_n, z, t_n) \leq \delta \circ \nu(x, z, t - 2\delta) \). This shows that
\[
(3.5) \quad \lim_{n \to \infty} \mu(x_n, z, t_n) \geq (1 - \delta) * \mu(x, z, t - 2\delta)
\]
and
\[
(3.6) \quad \lim_{n \to \infty} \nu(x_n, z, t_n) \leq \delta \circ \nu(x, z, t - 2\delta).
\]
Letting \( \delta \to 0^+ \), in (3.5), (3.6) yields that
\[
(3.7) \quad \lim_{n \to \infty} \mu(x_n, z, t_n) \geq 1 * \mu(x, z, t) = \mu(x, z, t)
\]
and
\[
(3.8) \quad \lim_{n \to \infty} \nu(x_n, z, t_n) \leq 0 \circ \nu(x, z, t) = \nu(x, z, t).
\]

On the other hand, when \( n \geq n_0 \). It follows from Lemma (3.3) that
\[
\mu(x_n, z, t_n) \leq \mu(x_n, z, t + \delta) \to \mu(x, z, t + \delta) \quad \text{as} \quad n \to \infty,
\]
and
\[
\nu(x_n, z, t_n) \geq \nu(x_n, z, t - \delta) \to \nu(x, z, t - \delta) \quad \text{as} \quad n \to \infty.
\]

Hence,
\[
(3.9) \quad \lim_{n \to \infty} \mu(x_n, z, t_n) \leq \mu(x, z, t + \delta)
\]
and
\[
(3.10) \quad \lim_{n \to \infty} \nu(x_n, z, t_n) \leq \nu(x, z, t + \delta).
\]
and

\( (3.10) \lim_{n \to \infty} \nu(x_n, z, t_n) \geq \nu(x, z, t - \delta). \)

Letting \( \delta \to 0^+ \), in (3.9) and (3.10) yields that

\[ \lim_{n \to \infty} \mu(x_n, z, t_n) \leq \mu(x, z, t) \quad \text{and} \quad \lim_{n \to \infty} \nu(x_n, z, t_n) \geq \nu(x, z, t). \]

It follows from (3.7) and (3.8) that

\[ \lim_{n \to \infty} \mu(x_n, z, t_n) \leq \mu(x, z, t) \quad \text{and} \quad \lim_{n \to \infty} \nu(x_n, z, t_n) \geq \nu(x, z, t). \]

Therefore, the mappings \( \mu, \nu : V \times V \times (0, \infty) \to [0, 1] \) are continuous. \( \square \)

**Definition 3.5.** A linear operator \( T : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *, \diamond) \) is said to be intuitionistic fuzzy 2-bounded (shortly, IF-2-B) if there exist constants \( h, k \in \mathbb{R} \) such that, \( \mu'(Tx, z, t) \geq \mu(hx, z, t) \) and \( \nu'(Tx, z, t) \leq \nu(kx, z, t) \) for every \( x, z(\text{nonzero}) \in V \) and for every \( t > 0 \).

**Theorem 3.6.** Suppose that \( (V, \mu, \nu, *, \diamond) \) and \( (V, \mu', \nu', *, \diamond) \) are intuitionistic fuzzy 2-normed spaces over \( \mathbb{F} \) with

(a) \( 1 \geq a \geq c \geq 0 \) and \( 1 \geq b \geq c \geq 0 \) implies \( a \ast b \geq c \)
(b) \( 0 \leq a \leq c \leq 1 \) and \( 0 \leq b \leq c \leq 1 \) implies \( a \ast b \leq c \).

If linear operators \( T, T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *, \diamond) \) are (IF-2-B) intuitionistic fuzzy 2-bounded, then \( T_1 + T_2 \) and \( cT(c \in \mathbb{F}) \) are also IF-2-B.

**Proof.** Since, linear operators \( T, T_1, T_2 : (V, \mu, \nu, *, \diamond) \to (V, \mu', \nu', *, \diamond) \) are IF-2-B there exists \( k_1, k_2, h_1, h_2 \in \mathbb{R}^+ \) such that,

\[ \mu'(T_1 x, z, t) \geq \mu(h_1 x, z, t) \quad \text{and} \quad \nu'(T_1 x, z, t) \geq \nu(k_1 x, z, t), \]

\[ \mu'(T_2 x, z, t) \geq \mu(h_2 x, z, t) \quad \text{and} \quad \nu'(T_2 x, z, t) \geq \nu(k_2 x, z, t), \]

for every \( x, z(\text{nonzero}) \in V \) and for every \( t > 0 \). Put \( h = \max\{h_1, h_2\} \) and \( k = \max\{k_1, k_2\} \) then for every \( x \in V \), we have

\[ \mu'((T_1 + T_2)x, z, t) = \mu'(T_1 x + T_2 x, z, t) \]

\[ \geq \mu'(T_1 x, z, \frac{t}{2}) \ast \mu'(T_2 x, z, \frac{t}{2}) \]

\[ \geq \mu(h_1 x, z, \frac{t}{2}) \ast \mu(h_2 x, z, \frac{t}{2}) \]

\[ = \mu(x, z, \frac{t}{2h_1}) \ast \mu(x, z, \frac{t}{2h_2}) \]

\[ \geq \mu(x, z, \frac{t}{2h}) \ast \mu(x, z, \frac{t}{2h}) \]

\[ \geq \mu(x, z, \frac{t}{3h}) \]

\[ = \mu(3hx, z, t). \]

Similarly, \( \nu'((T_1 + T_2)x, z, t) \leq \mu(3hx, z, t) \). Hence \( T_1 + T_2 \) is IF-2-B. Similarly, we can prove that \( cT \) is IF-2-B. \( \square \)
Definition 3.7. A linear operator \( T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond) \) is said to be intuitionistic fuzzy 2-continuous if it is intuitionistic fuzzy 2-continuous everywhere.

Definition 3.8. A map \( T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond) \) is said to be intuitionistic fuzzy 2-continuous if it is intuitionistic fuzzy 2-continuous for all scalars \( c_1, c_2 \in F \).

Proof. Let \( x_n \xrightarrow{\mu, \nu, *, \diamond} x \) as \( n \rightarrow \infty \) in \((V, \mu, \nu, *, \diamond)\). Since, linear operators \( T_1, T_2 : (V, \mu, \nu, *, \diamond) \rightarrow (V, \mu', \nu', *, \diamond) \) are IF-2-C, we get,

\[ T_1 x_n \xrightarrow{\mu', \nu', *, \diamond} T_1 x, \quad T_2 x_n \xrightarrow{\mu', \nu', *, \diamond} T_2 x \quad \text{as} \quad n \rightarrow \infty \]

in \((V, \mu', \nu', *, \diamond)\), by (3.1), (3.2), we get for all \( c_1, c_2 \in F \)

\[ c_1(T_1 x_n) + c_2(T_2 x_n) \xrightarrow{\mu, \nu, *, \diamond} c_1(T_1 x) + c_2(T_2 x) \quad \text{as} \quad n \rightarrow \infty \]

which gives

\[ (c_1 T_1 + c_2 T_2) x_n \xrightarrow{\mu', \nu', *, \diamond} (c_1 T_1 + c_2 T_2) x \quad \text{as} \quad n \rightarrow \infty. \]

Hence, \( c_1 T_1 + c_2 T_2 \) is intuitionistic fuzzy 2-continuous.

Definition 3.11. A linear operator \( T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond) \) is said to be strongly intuitionistic fuzzy 2-continuous at a point \( x \in V \) if for any given \( \epsilon > 0 \) there exists \( \delta(\epsilon) > 0 \)

\[ \mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \quad \text{and} \quad \nu'(Tx - Tx_0, z, \epsilon) < \nu(x - x_0, z, \delta). \]

A linear operator \( T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond) \) is strongly IF\(-2-C\), if \( T \) is strongly IF\(-2-C\) at each point of \( V \).

Theorem 3.12. A linear operator \( T : (V, \mu, \nu, *, \diamond) \rightarrow (V', \mu', \nu', *, \diamond) \) is strongly IF\(-2-C\), then it is IF\(-2-C\), but converse is not true.

Proof. Let a linear operator \( T : (V, \mu, \nu, *, \diamond) \rightarrow (V, \mu', \nu', *, \diamond) \) be strongly IF\(-2-C\). Let \( x_0 \in V \) if for any given \( \epsilon > 0 \) there exists \( \delta(\epsilon) > 0 \) such that for all \( x \in V \),

\[ \mu'(Tx - Tx_0, z, \epsilon) \geq \mu(x - x_0, z, \delta) \quad \text{and} \quad \nu'(Tx - Tx_0, z, \epsilon) < \nu(x - x_0, z, \delta). \]

Let \( \{x_n\} \) be a sequence in \( V \) such that \( \{x_n\} \rightarrow x_0 \) for all \( t > 0 \) then

\[ \lim_{n \rightarrow \infty} \mu(x - x_0, z, t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \nu(x - x_0, z, t) = 0. \]
Thus we see that
\[ \mu'(Tx_n - Tx_0, z, \epsilon) \geq \mu(x_n - x_0, z, \delta) \text{ and } \nu'(Tx_n - Tx_0, z, \epsilon) < \nu(x_n - x_0, z, \delta). \]
which implies that
\[ \lim_{n \to \infty} \mu'(Tx_n - Tx_0, z, \epsilon) = 1 \quad \text{and} \quad \lim_{n \to \infty} \nu'(Tx_n - Tx_0, z, \epsilon) = 0 \]
which gives \( Tx_n \to Tx_0 \) in \( (V, \mu', \nu', \ast, \circ) \). Hence, \( T \) is IF \(-2-C\). Conversely, we provide example, which is IF \(-2-C\) but not strongly IF \(-2-C\).

**Example 3.13.** Let \( (V = \mathbb{R}, ||\cdot, \cdot||) \) be 2-normed space over \( F \). Define \( a \ast b = \min\{a, b\} \) and \( a \circ b = \max\{a, b\} \), for all \( a, b \in [0,1] \). Let \( \mu, \nu, \mu', \nu' \) are fuzzy sets on \( V \times V \times (0, \infty) \) defined by \( \mu(x, z, t) = \frac{t}{t + \|x, z\|}, \nu(x, z, t) = \frac{\|x, z\|}{t + \|x, z\|} \) and
\[
\mu'(x, z, t) = \frac{t}{t + k\|x, z\|}, \\
\nu'(x, z, t) = \frac{k\|x, z\|}{t + k\|x, z\|},
\]
for all \( t \in \mathbb{R}^+ \) and \( k > 0 \).

In short,
\[
\mu(Y, t) = \frac{t}{t + \|Y\|}, \nu(Y, t) = \frac{\|Y\|}{t + \|Y\|} \quad \text{and} \quad \mu'(Y, t) = \frac{t}{t + k\|Y\|}, \nu'(Y, t) = \frac{k\|Y\|}{t + k\|Y\|}.
\]
Let us now define, \( T(Y) = \frac{Y^4}{1 + Y^2} \) for all \( Y \in V \). Let \( Y_0 \in V \) and \( \{Y_k\} \) be a sequence in \( V \) such that \( \{Y_k\} \to Y_0 \) in \( (V, \mu, \nu, \ast, \circ) \), i.e. for all \( t > 0 \), \( \lim_{k \to \infty} \mu(Y_k - Y_0, t) = 1 \) and \( \lim_{k \to \infty} \mu(Y_k - Y_0, t) = 0 \).

\[
\Rightarrow \lim_{k \to \infty} \frac{\|Y_k - Y_0\|}{t + \|Y_k - Y_0\|} = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{\|Y_k - Y_0\|}{t + \|Y_k - Y_0\|} = 0.
\]
Now for all \( t > 0 \),
\[
\mu'(TY_k - TY_0, t) = \frac{t}{t + k\|TY_k - TY_0\|} = \frac{t}{t + k\|\frac{Y_k^4}{1 + Y_k^2} - \frac{Y_0^4}{1 + Y_0^2}\|} \Rightarrow \lim_{n \to \infty} \mu'(TY_n - TY_0, t) = 1.
\]
Similarly, we get, \( \lim_{n \to \infty} \nu'(TY_n - TY_0, t) = 0 \) Thus \( T \) is IF \(-2-C\).

Let \( \epsilon > 0 \) be given. Then
\[
\mu'(TY - TY_0, \epsilon) \geq \mu(Y - Y_0, \delta) \Rightarrow \epsilon\|1 + Y^2\|\|1 + Y_0^2\| + k\|Y - Y_0\||\|Y + Y_0||Y^2 + Y_0^2\| + Y^2Y_0^2\|Y + Y_0\| \geq \delta + \|Y - Y_0\|
\]
and
\[
\nu'(TY - TY_0, \epsilon) \leq \nu(Y - Y_0, \delta) \Rightarrow \epsilon + \|1 + Y^2\|\|1 + Y_0^2\| + k\|Y - Y_0\||\|Y + Y_0||Y^2 + Y_0^2\| + Y^2Y_0^2\|Y + Y_0\| \leq \delta + \|Y - Y_0\|.
\]
So,
\begin{equation}
(3.11) \quad k\delta \|Y - Y_0\| + Y_0 \leq 1 + Y_2 \|1 + Y_2\| \leq \epsilon \|1 + Y_2\| Y - Y_0 \tag{3.11}
\end{equation}
\begin{equation}
\Rightarrow \delta \leq \frac{\epsilon \|1 + Y_2\|}{k\|Y - Y_0\| + Y_0} \quad \text{(for } Y \neq Y_0\).
\end{equation}

We see that \( T \) is IF-2-C at \( Y_0 \) if there exists \( \delta > 0 \) satisfying (3.11) for all \( Y \neq Y_0 \). Let
\[ \delta_1 = \inf_{Y \neq Y_0} \frac{\|Y - Y_0\| + Y_0}{\|1 + Y_2\|} \]
where the infimum is taken over all \( Y \), where, \( Y \neq Y_0 \). Then \( \delta = \frac{\delta_1}{k} \) satisfies (3.11). But \( \delta_1 = 0 \) which is impossible. Hence, \( T \) is not strongly IF-2-C.

\[ \square \]

**Theorem 3.14.** \( (V, \mu, \nu, *, \circ) \), \((V, \mu', \nu', *, \circ)\) are IF-2-NS and \( T : (V, \mu, \nu, *, \circ) \to (V, \mu', \nu', *, \circ) \) be a linear operators, then the following conditions are equivalent.

(a) \( T \) is intuitionistic fuzzy 2-bounded (IF-2-B).

(b) If there exist constants \( h, k \in \mathbb{R} - \{0\} \) such that \( \mu' (Tx, z, t) \geq \mu (hx, z, t) \) and \( \nu' (Tx, z, t) \leq \nu (kz, z, t) \) for every \( x, z \) (nonzero) \( \in V \) and for every \( t > 0 \).

(c) \( T \) is intuitionistic fuzzy 2-continuous at some point \( x_0 \in V \).

(d) \( T \) is intuitionistic fuzzy 2-continuous (IF-2-C).

**Proof.** (a) \( \Leftrightarrow \) (b) Obviously, result holds by definition (3.5).

(c) \( \Leftrightarrow \) (d) Suppose, \( T \) is intuitionistic fuzzy 2-continuous at some point \( x_0 \in V \). Let \( \{x_n\} \to (\mu, \nu) \) as \( n \to \infty \) in \((V, \mu, \nu, *, \circ)\). By theorems (3.1) and (3.2), we see that,
\[ (x_n - x) + x_0 \to (\mu, \nu) \]
\[ \Rightarrow \]
\[ T((x_n - x) + x_0) \to (\mu', \nu') \]
\[ \Rightarrow \]
\[ n \to \infty \]
\[ T((x_n - x) + x_0) \to (\mu', \nu') T x_0 \quad \text{as } n \to \infty. \]

Since \( T \) is a linear, we obtain that
\[ (Tx_n - Tx + Tx_0) \to (\mu', \nu') T x_0 \quad \text{as } n \to \infty \]
implies that \( Tx_n \to (\mu', \nu') T x_0 \). By definition (3.7), we conclude that, \( T \) is IF-2-C. Obviously, converse hold.

(a) \( \Leftrightarrow \) (d) It follows from [10] and converse holds by definitions (3.5), (3.8). \[ \square \]

**Definition 3.15.** Let \( (V, \mu, \nu, *, \circ) \) be an intuitionistic fuzzy 2-normed space. A subset \( D \) of \( V \) is said to compact if any sequence in \( D \) has a subsequence converging to an element of \( D \).

**Theorem 3.16.** Let \( T : (V_1, \mu_1, \nu_1, *, \circ) \to (V_2, \mu_2, \nu_2, *, \circ) \) be a mapping and \( D \) be a compact subset of \( V_1 \). If \( T \) is a IF-2-C on \( V_1 \) then \( T(D) \) is a compact subset of \( V_2 \).

**Proof.** Let \( y_n \) be a sequence in \( T(D) \) then for each \( n \) there exist \( x_n \in D \) such that \( T(x_n) = y_n \). Since \( D \) is a compact there exists \( \{x_{n_k}\} \) a subsequence of \( \{x_n\} \) and \( x_0 \in D \) such that \( \{x_{n_k}\} \to (\mu_1, \nu_1) \) \( x_0 \) in \( (V_1, \mu_1, \nu_1, *, \circ) \). Since \( T \) is an intuitionistic fuzzy 2-continuous at \( x_0 \). By definition (3.7)
\[ \{x_{n_k}\} \to (\mu_1, \nu_1) \]
\[ 0 \Rightarrow \]
\[ T\{x_{n_k}\} \to (\mu_2, \nu_2) \]
\[ T(x_0) \Rightarrow \]
\[ \{y_{n_k}\} \to (\mu_2, \nu_2) \]
\[ 0, \]
\[ 10 \]
for some \( y_0 \in T(D) \) such that \( T(x_0) = y_0 \) implies that \( T(D) \) is compact subset of \( V_2 \).

### 4. Intuitionistic fuzzy \( \psi \)-2-normed space

**Definition 4.1.** The five-tuple \((V, \mu, \nu, *, \diamond)\) is said to be an intuitionistic fuzzy \( \psi \)-2-normed space, if \( V \) is a vector space over \( F \in \{ \mathbb{R}, \mathbb{C} \} \), \( * \) is a continuous t-norm, \( \diamond \) is a continuous t-conorm, and \( \mu, \nu \) are fuzzy sets on \( V \times V \times (0, \infty) \) satisfying the following conditions. For every \( x, y, z \in V \) and \( s, t > 0 \),

\[
\begin{align*}
(a) & \mu(x, y, t) + \nu(x, y, t) \leq 1; \\
(b) & \mu(x, y, t) > 0; \\
(c) & \mu(x, y, t) = 1 \text{ if and only if } x \text{ and } y \text{ are linearly dependent}; \\
(d) & \mu(\alpha x, y, t) = \mu(x, y, \frac{1}{\alpha}) \text{ for each } \alpha \neq 0; \\
(e) & \mu(x, y, t) * \mu(x, z, s) \leq \mu(x, y + z, t + s); \\
(f) & \mu(x, y, \cdot) : (0, \infty) \to [0, 1] \text{ is continuous}; \\
(g) & \lim_{t \to \infty} \mu(x, y, t) = 1 \text{ and } \lim_{t \to 0} \mu(x, y, t) = 0; \\
(h) & \nu(x, y, t) = \mu(y, x, t); \\
(i) & \nu(x, y, t) < 1; \\
(j) & \nu(x, y, t) = 0 \text{ if and only if } x \text{ and } y \text{ are linearly dependent}; \\
(k) & \nu(\alpha x, y, t) = \nu(x, y, \frac{1}{\alpha}) \text{ for each } \alpha \neq 0; \\
(l) & \nu(x, y, t) \circ \nu(x, z, s) \geq \nu(x, y + z, t + s); \\
(m) & \nu(x, y, \cdot) : (0, \infty) \to [0, 1] \text{ is continuous}; \\
(n) & \lim_{t \to \infty} \nu(x, y, t) = 0 \text{ and } \lim_{t \to 0} \nu(x, y, t) = 1; \\
(o) & \nu(x, y, t) = \nu(y, x, t).
\end{align*}
\]

In this case \((\mu, \nu)_2\) is called an intuitionistic fuzzy \( \psi \)-2-norm on \( V \).

**Definition 4.2.** Let \((V, \mu, \nu, *, \diamond)\) be an intuitionistic fuzzy \( \psi \)-2-normed space. A sequence \( \{x_n\} \) is said to be convergent to \( x \in V \) with respect to the intuitionistic fuzzy \( \psi \)-2-norm \((\mu, \nu)_2\), if for every \( r > 0 \) and \( t > 0 \), \( r \in (0, 1) \) there exists \( n_0 \in \mathbb{N} \) such that \( \mu(x_n - x, z, t) > 1 - r \) and \( \nu(x_n - x, z, t) < r \) for all \( n \geq n_0 \) and for all \( z \in V \).

**Definition 4.3.** Let \((V, \mu, \nu, *, \diamond)\) be an intuitionistic fuzzy \( \psi \)-2-normed space. A sequence \( \{x_n\} \) in \( V \) is said to be Cauchy if for each \( r > 0 \) and each \( t > 0 \), \( r \in (0, 1) \) there exists \( n_0 \in \mathbb{N} \) such that \( \mu(x_n - x_m, z, t) > 1 - r \) and \( \nu(x_n - x_m, z, t) < r \) for all \( n, m \geq n_0 \) and for all \( z \in V \).

**Definition 4.4.** Let \((V, \mu, \nu, *, \diamond)\) be an intuitionistic fuzzy \( \psi \)-2-normed space and let \( r \in (0, 1), t > 0 \) and \( x \in X \). The set \( B(x, r, t) = \{ y \in V : \mu(y - x, z, t) > 1 - r, \nu(y - x, z, t) < r, \forall z \in V \} \) is called the open ball with center \( x \) and radius \( r \) with respect to \( t \).

**Definition 4.5.** Let \((V, \mu, \nu, *, \diamond)\) be an intuitionistic fuzzy \( \psi \)-2-normed space. A set \( U \subset V \) is said to be an open set if each of its points is the centre of some open ball contained in \( U \). The open set in an intuitionistic fuzzy \( \psi \)-2-normed space \((V, \mu, \nu, *, \diamond)\) is denoted by \( U \).

**Theorem 4.6.** In intuitionistic fuzzy \( \psi \)-2-normed space \((V, \mu, \nu, *, \diamond)\). A sequence \( \{x_n\} \) converges to \( x \) if and only if \( \mu(x_n - x, z, t) \to 1 \) and \( \nu(x_n - x, z, t) \to 0 \) as \( n \to \infty \).
Proof. Fix $t > 0$. Suppose $\{x_n\}$ converges to $x$ in IF $\psi$-2-normed space $(V, \mu, \nu, *, \diamond)$ then for a given $r, r \in (0, 1)$ there exists an integer $n_0 \in N$ such that $\mu(x_n - x, z, t) > 1 - r$ and $\nu(x_n - x, z, t) < r$. Thus $1 - \mu(x_n - x, z, t) > r$ and $\nu(x_n - x, z, t) < r$, hence, $\mu(x_n - x, z, t) \to 1$ and $\nu(x_n - x, z, t) \to 0$ as $n \to \infty$.

Conversely, if for each $t > 0$, $\mu(x_n - x, z, t) \to 1$ and $\nu(x_n - x, z, t) \to 0$ as $n \to \infty$ then for every $r \in (0, 1)$, there exists an integer $n_0$ such that $1 - \mu(x_n - x, z, t) > r$ and $\nu(x_n - x, z, t) < r, \forall n \geq n_0$. Hence, $\mu(x_n - x, z, t) > 1 - r$ and $\nu(x_n - x, z, t) < r$.

Thus, $\{x_n\}$ converges to $x$ in IF $\psi$-2-normed space $(V, \mu, \nu, *, \diamond)$. \hfill\Box

**Theorem 4.7.** The limit is unique for a convergent sequence $\{x_n\}$ in intuitionistic fuzzy $\psi$-2-normed space $(V, \mu, \nu, *, \diamond)$.

**Proof.** Let $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} x_n = y$.

\[
\begin{align*}
\lim_{n \to \infty} x_n &= x = \left\{ \begin{array}{ll}
\lim_{n \to \infty} \mu(x_n - x, z, t) = 1, \\
\lim_{n \to \infty} \nu(x_n - x, z, t) = 0.
\end{array} \right. \\
\lim_{n \to \infty} x_n &= y = \left\{ \begin{array}{ll}
\lim_{n \to \infty} \mu(x_n - y, z, t) = 1, \\
\lim_{n \to \infty} \nu(x_n - y, z, t) = 0.
\end{array} \right.
\end{align*}
\]

\[
\nu(x - y, z, s + t) = \nu(x - x + x_n - y, z, s + t) \\
\geq \nu(x - x_n, z, s) \circ \nu(x_n - y, z, t) \\
= \nu(x_n - x, z, s \frac{1}{\psi(-1)}) \circ \nu(x_n - y, z, t) \\
= \nu(x_n - x, z, s \frac{1}{\psi(1)}) \circ \nu(x_n - y, z, t) \\
= \nu(x_n - x, z, s) \circ \nu(x_n - y, z, t).
\]

As $n \to \infty$ we have, $\nu(x - y, z, s + t) = 0 \Rightarrow x = y$. Thus, The limit is unique for a convergent sequence $\{x_n\}$ in intuitionistic fuzzy $\psi$-2-normed space $(V, \mu, \nu, *, \diamond)$. \hfill\Box

**Theorem 4.8.** In IF $\psi$-2-NS $(V, \mu, \nu, *, \diamond)$. Every convergent sequence is a Cauchy sequence.

**Proof.** Let $\{x_n\}$ be a convergent sequence in IF $\psi$-2-NS $(V, \mu, \nu, *, \diamond)$ with $\lim_{n \to \infty} x_n = x$. Let $r \in (0, 1), t, s > 0$ then there exist an integer $n_0 \in N$ such that $\mu(x_n - x, z, s) > 1 - r$ and $\nu(x_n - x, z, s) < r$. For $n, p \in \mathbb{N}$

\[
\mu(x_{n+p} - x_n, z, s + t) = \mu(x_{n+p} - x + x_n - x_n, z, s + t) \\
\geq \mu(x_{n+p} - x, z, s) \ast \mu(x_n - x_n, z, t) \\
= \mu(x_{n+p} - x, z, s) \ast \mu(x_n - x, z, s \frac{1}{\psi(-1)}) \\
= \mu(x_{n+p} - x, z, s) \ast \mu(x_n - x, z, t) \\
> (1 - r) \ast (1 - r) \\
= (1 - r), \forall n \geq n_0.
\]
Similarly, 
\[
\nu(x_{n+p} - x_n, z, s + t) = \nu(x_{n+p} - x + x - x_n, z, s + t) \\
\leq \nu(x_{n+p} - x, z, s) \circ \nu(x - x_n, z, t) \\
= \nu(x_{n+p} - x, z, s) \circ \nu(x - x,\frac{t}{\psi(-1)}) \\
= \nu(x_{n+p} - x, z, s) \circ \nu(x - x, z, t) \\
< r \circ r \\
= r, \forall n \geq n_0.
\]

Hence, \(\{x_n\}\) is a Cauchy sequence in IF \(\psi\)-2-NS \((V, \mu, \nu, *, \circ)\). \(\square\)

**Theorem 4.9.** In IF \(\psi\)-2-NS \((V, \mu, \nu, *, \circ)\). A sequence \(\{x_n\}\) is a Cauchy sequence if and only if \(\mu(x_{n+p} - x, z, t) \to 1\) and \(\nu(x_{n+p} - x, z, t) \to 0\) as \(n \to \infty\).

**Proof.** Fix \(t > 0\). Suppose \(\{x_n\}\) is a Cauchy sequence in IF \(\psi\)-2-normed space \((V, \mu, \nu, *, \circ)\) then for a given \(r \in (0, 1)\) there exists an integer \(n_0 \in N\) such that \(\mu(x_{n+p} - x_n, z, t) > 1 - r\) and \(\nu(x_{n+p} - x_n, z, t) < r\). Thus \(1 - \mu(x_{n+p} - x_n, z, t) > r\) and \(\nu(x_{n+p} - x_n, z, t) < r\), hence, \(\mu(x_{n+p} - x_n, z, t) \to 1\) and \(\nu(x_{n+p} - x_n, z, t) \to 0\) as \(n \to \infty\).

Conversely, if for each \(t > 0\), \(\mu(x_{n+p} - x_n, z, t) \to 1\) and \(\nu(x_{n+p} - x_n, z, t) \to 0\) as \(n \to \infty\) then for every \(r, r \in (0, 1)\), there exists an integer \(n_0\) such that \(1 - \mu(x_{n+p} - x_n, z, t) > r\) and \(\nu(x_{n+p} - x_n, z, t) < r\), \(\forall n \geq n_0\). Hence, \(\mu(x_{n+p} - x_n, z, t) > 1 - r\) and \(\nu(x_{n+p} - x_n, z, t) < r\), \(\forall n \geq n_0\). Thus, \(\{x_n\}\) is a Cauchy sequence in IF \(\psi\)-2-normed space \((V, \mu, \nu, *, \circ)\). \(\square\)

**Definition 4.10.** An intuitionistic fuzzy \(\psi\)-2-normed space \((V, \mu, \nu, *, \circ)\) is said to be complete if every Cauchy sequence in IF \(\psi\)-2-NS \((V, \mu, \nu, *, \circ)\) is convergent.

**Theorem 4.11.** Let \((V, \mu, \nu, *, \circ)\) be a IF \(\psi\)-2-NS. A sufficient condition for the IF \(\psi\)-2-NS \((V, \mu, \nu, *, \circ)\) to be complete is that every Cauchy sequence in \((V, \mu, \nu, *, \circ)\) has a convergent subsequence.

**Proof.** Let \(\{x_n\}_n\) be a Cauchy sequence in \((V, \mu, \nu, *, \circ)\) and \(\{x_{n_k}\}_k\) be a subsequence of \(\{x_n\}_n\) that converges to \(x \in V\) and \(s, t, s + t > 0\). Since \(\{x_n\}_n\) is a Cauchy sequence in \((V, \mu, \nu, *, \circ)\), We have for \(r \in (0, 1)\) there exists an integer \(n_0 \in N\) such that \(\mu(x_n - x_k, z, s) > 1 - r\) and \(\nu(x_n - x_k, z, s) < r, \forall n, k \geq n_0\). Again, since \(\{x_{n_k}\}_k\) converges to \(x\). We have \(\mu(x_{n_k} - x, z, t) > 1 - r\) and \(\nu(x_{n_k} - x, z, t) < r, \forall n, k \geq n_0\)

\[
\mu(x - x, z, s + t) = \mu(x_n - x_n, z, s + t) \\
\geq \mu(x_n - x_{n_k}, z, s) * \mu(x_{n_k} - x, z, t) \\
> (1 - r) * (1 - r) \\
= (1 - r), \forall n \geq n_0.
\]
Similarly,
\[ \nu(x_n - x, z, s + t) = \nu(x_n - x_n^k + x_n^k, z, s + t) \leq \mu(x_n - x_n^k, z, s) \circ \mu(x_n^k - x, z, t) < r \circ r \]
\[ = r, \forall n \geq n_0. \]
Thus \( \{x_n\}_n \) converges to \( x \) in \( (V, \mu, \nu, \ast, \diamond) \). Hence IF \( \psi\)-2-NS \( (V, \mu, \nu, \ast, \diamond) \) is complete. \( \square \)

**Remark 4.12.** Straightforwardly, we get the results (3.1), (3.2), (3.3), (3.4), (3.6), (3.10), (3.12), (3.14), (3.16) are also holds in INF\( \psi\)-2-NS.

**Theorem 4.13.** Every intuitionistic fuzzy \( \psi\)-2-normed space is intuitionistic fuzzy 2-normed space, converse is not true.

**Proof.** Let \( (V, \mu, \nu, \ast, \diamond) \) be a IF \( \psi\)-2-NS. By definition (2.14), take \( \psi(\alpha) = |\alpha| \) then definition (2.8) implies \( (V, \mu, \nu, \ast, \diamond) \) be a IF 2-NS. Conversely, let \( (V, \mu, \nu, \ast, \diamond) \) be a IF 2-NS. If \( \psi(\alpha) \neq |\alpha| \) then definitions (4.1 and 2.14) implies \( (V, \mu, \nu, \ast, \diamond) \) is not a IF \( \psi\)-2-NS. \( \square \)

5. **Acknowledgments**

The authors are grateful to the referees for their valuable comments and suggestions.

**References**


1 S. G. Dapke (sadashivgdapke@gmail.com)
Assistant Professor, Department of Mathematics, Iqra’s H. J. Thim College, Mehrun, Jalgaon, India.
2 C. T. Aage (ctaage@gmail.com)
Assistant Professor, School of Mathematical Sciences, North Maharashtra University, Jalgaon, India.
3 J. N. Salunke (drjnsalunke@gmail.com)
Professor, School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded, India.