Notes on $\gamma$-soft operator and some counter examples on (supra) soft continuity


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Abstract. Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied and improved the soft set theory, which is initiated by Molodtsov [27] and easily applied to many problems having uncertainties from social life. The main purpose of this paper, is to introduce more properties of of $\gamma$-soft interior and $\gamma$-soft closure mentioned in [8, 12]. Also, counter examples of (supra) soft continuity, mentioned in [9, 12], are introduced.

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Corresponding Author: Name: Alaa Mohamed Abd El-latif
E-mail: (Alaa_8560@yahoo.com, dr.Alaa_daby@yahoo.com)

1. Introduction

The concept of soft sets was first introduced by Molodtsov [27] in 1999 as a general mathematical tool for dealing with uncertain objects. In [27, 28], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [25], the properties and applications of soft set theory have been studied increasingly [4, 22, 28, 30]. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [2, 3, 6, 11, 20, 23, 24, 25, 26, 28, 29, 33]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making
method was constructed by using these new operations [7]. Recently, in 2011, Shabir and Naz [31] initiated the study of soft topological spaces. They defined soft topology on the collection $\tau$ of soft sets over $X$. Consequently, they defined basic notions of soft topological spaces such as open and closed soft sets, soft subspace, soft closure, soft nbd of a point and soft separation axioms, which is extended in [5, 32]. In [12], Kandil et. al. introduced some soft operations such as semi open soft, pre open soft, $\alpha$-open soft and $\beta$-open soft and investigated their properties in detail. Kandil et al. [19] introduced the notion of soft semi separation axioms. In particular they study the properties of the soft semi regular spaces and soft semi normal spaces. The notion of soft ideal was initiated for the first time by Kandil et al. [15]. They also introduced the concept of soft local function. These concepts are discussed with a view to find new soft topologies from the original one.

In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

**Definition 2.1.** [27] Let $X$ be an initial universe and $E$ be a set of parameters. Let $P(X)$ denote the power set of $X$ and $A$ be a non-empty subset of $E$. A pair $(F, A)$ denoted by $F_A$ is called a soft set over $X$, where $F$ is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over $X$ is a parametrized family of subsets of the universe $X$. For a particular $e \in A$, $F(e)$ may be considered the set of $e$-approximate elements of the soft set $(F, A)$ and if $e \notin A$, then $F(e) = \phi$, i.e. $F_A = \{F(e) : e \in A \subseteq E, F : A \rightarrow P(X)\}$. The family of all these soft sets over $X$ denoted by $SS(X)_A$.

**Definition 2.2.** [32] The soft set $(F, E) \in SS(X)_E$ is called a soft point in $X_E$ if there exist $x \in X$ and $e \in E$ such that $F(e) = \{x\}$ and $F(e') = \phi$ for each $e' \in E - \{e\}$, and the soft point $(F, E)$ is denoted by $x_e$.

**Definition 2.3.** [31] Let $\tau$ be a collection of soft sets over a universe $X$ with a fixed set of parameters $E$, then $\tau \subseteq SS(X)_E$ is called a soft topology on $X$ if

1. $\tilde{X}, \tilde{\phi} \in \tau$, where $\tilde{\phi}(e) = \phi$ and $\tilde{X}(e) = X$, $\forall e \in E$,  
2. the union of any number of soft sets in $\tau$ belongs to $\tau$,  
3. the intersection of any two soft sets in $\tau$ belongs to $\tau$.

The triplet $(X, \tau, E)$ is called a soft topological space over $X$.

**Definition 2.4.** Let $(X, \tau, E)$ be a soft topological space. A mapping $\gamma : SS(X)_E \rightarrow SS(X)_E$ is said to be an operation on $OS(X)$ if $F_E \subseteq \gamma(F_E) \forall F_E \in OS(X)$. The collection of all $\gamma$-open soft sets is denoted by $OS(\gamma) = \{F_E : F_E \subseteq \gamma(F_E), F_E \in SS(X)_E\}$. Also, the complement of $\gamma$-open soft set is called $\gamma$-closed soft set, i.e. $CS(\gamma) = \{F_E : F_E \text{ is a } \gamma \text{-open soft set, } F_E \in SS(X)_E\}$ is the family of all $\gamma$-closed soft sets.
Definition 2.5. Let \((X, \tau, E)\) be a soft topological space. Different cases of \(\gamma\)-operations on \(SS(X)_E\) are as follows:

1. If \(\gamma = \text{int}(\text{cl})\), then \(\gamma\) is called pre open soft operator. We denote the set of all pre open soft sets by \(\text{POS}(X, \tau, E)\), or when there can be no confusion by \(\text{POS}(X)\) and the set of all pre closed soft sets by \(\text{PCS}(X, \tau, E)\), or \(\text{PCS}(X)\).

2. If \(\gamma = \text{int}(\text{cl}(\text{int}))\), then \(\gamma\) is called \(\alpha\)-open soft operator. We denote the set of all \(\alpha\)-open soft sets by \(\alpha\text{OS}(X, \tau, E)\), or \(\alpha\text{OS}(X)\) and the set of all \(\alpha\)-closed soft sets by \(\alpha\text{CS}(X, \tau, E)\), or \(\alpha\text{CS}(X)\).

3. If \(\gamma = \text{cl}(\text{int})\), then \(\gamma\) is called semi open soft operator. We denote the set of all semi open soft sets by \(\beta\text{OS}(X, \tau, E)\), or \(\beta\text{OS}(X)\) and the set of all semi closed soft sets by \(\beta\text{CS}(X, \tau, E)\), or \(\beta\text{CS}(X)\).

4. If \(\gamma = \text{cl}(\text{int}(\text{cl}))\), then \(\gamma\) is called \(\beta\)-open soft operator. We denote the set of all \(\beta\)-open soft sets by \(\beta\text{OS}(X, \tau, E)\), or \(\beta\text{OS}(X)\) and the set of all \(\beta\)-closed soft sets by \(\beta\text{CS}(X, \tau, E)\), or \(\beta\text{CS}(X)\).

Definition 2.6. [12] Let \((X, \tau, E)\) be a soft topological space, \((F, E) \in SS(X)_E\). Then, the pre soft interior (resp. semi soft interior, \(\alpha\)-soft interior, \(\beta\)-soft interior) of \((F, E)\) is denoted by \(\text{PSint}(F, E)\) (resp. \(\text{SSint}(F, E)\), \(\alpha\text{Sint}(F, E)\), \(\beta\text{Sint}(F, E)\) ), which is the soft union of all pre open (resp. semi open, \(\alpha\)-open, \(\beta\)-open) soft sets contained in \((F, E)\).

Definition 2.7. Let \((X, \tau, E)\) be a soft topological space, \((F, E) \in SS(X)_E\) and \(x_e \in SS(X)_E\). Then

1. \(x_e\) is called \(\gamma\)-interior soft point of \((F, E)\) if \(\exists (G, E) \in \text{OS}(\gamma)\) such that \(x_e \in (G, E) \subseteq (F, E)\), the set of all \(\gamma\)-interior soft points of \((F, E)\) is called the \(\gamma\)-soft interior of \((F, E)\) and is denoted by \(\gamma\text{Sint}(F, E)\) consequently, \(\gamma\text{Sint}(F, E) = \bigcup\{(G, E) : (G, E) \subseteq (F, E), (G, E) \in \text{OS}(\gamma)\}\).

2. \(x_e\) is called \(\gamma\)-closure soft point of \((F, E)\) if \((F, E) \cap (H, E) \neq \emptyset \ \forall \ \gamma\)-open soft set \((H, E)\) containing \(x_e\). The set of all \(\gamma\)-closure soft points of \((F, E)\) is \(\gamma\)-soft closure of \((F, E)\) and is denoted by \(\gamma\text{Scl}(F, E)\) consequently, \(\gamma\text{Scl}(F, E) = \bigcap\{(H, E) : (H, E) \in \text{CS}(\gamma), (F, E) \subseteq (H, E)\}\).

Theorem 2.1. [12] Let \((X, \tau, E)\) be a soft topological space and \((F, E) \in SS(X)_E\). Then

1. \((F, E) \in \text{SOS}(X)\) if and only if \(\text{cl}(F, E) = \text{cl}(\text{int}(F, E))\).
2. If \((G, E) \in \text{OS}(X)\), then \((G, E) \cap \text{cl}(F, E) \subseteq \text{cl}(\text{int}(F, E))\).
3. If \((H, E) \in \text{CS}(X)\), then \(\text{int}(\text{cl}(G, E) \cup (H, E)) \subseteq \text{int}(\text{cl}(G, E) \cup (H, E))\).

3. More properties of \(\gamma\)-soft interior and \(\gamma\)-soft closure

Theorem 3.1. [8] Let \((X, \tau, E)\) be a soft topological space. Then, the following properties are satisfied:

1. \(\text{PScI}(F, E) = (F, E) \cup \text{cl}(\text{int}(F, E))\).
2. \(\text{PSint}(F, E) = (F, E) \cap \text{cl}(\text{int}(F, E))\).
3. \(\text{aScI}(F, E) = (F, E) \cup \text{cl}(\text{int}(F, E))\).
4. \(\text{aSint}(F, E) = (F, E) \cap \text{cl}(\text{int}(F, E))\).
5. \(\text{SScl}(F, E) = (F, E) \cap \text{int}(\text{cl}(F, E))\).
(6): \( SSint(F, E) = (F, E) \cap \cl(\int(F, E)) \).
(7): \( \beta Scl(F, E) = (F, E) \cup int(\cl(F, E)) \).
(8): \( \beta Sint(F, E) = (F, E) \cap \cl(\int(F, E)) \).

Proof. We shall prove only the first statement, the other cases are similar. Since
\[ cl(\int([F, E] \cup cl(\int(F, E)])) \subseteq cl(\int(F, E)) \cup \cl(\int(F, E)) = cl(\int(F, E)) \subseteq (F, E) \cup \cl(\int(F, E)) \] from Theorem 2.1 (3). This means that, \( (F, E) \cup \cl(\int(F, E)) \) is a pre closed soft set containing \( (F, E) \). So, \( PScl(F, E) \subseteq (F, E) \cup \cl(\int(F, E)) \). On the other hand, \( PScl(F, E) \) is pre closed soft. So, we have \( cl(\int(F, E)) \subseteq cl(\int(PScl(F, E))) \subseteq PScl(F, E) \). Hence, \( (F, E) \cup \cl(\int(F, E)) \subseteq PScl(F, E) \). Therefore, \( PScl(F, E) = (F, E) \cup \cl(\int(F, E)) \). The rest of the proof by a similar way.

Theorem 3.2. Let \( (X, \tau, E) \) be a soft topological space. Then, the following properties are satisfied:

(1): \( PScl(PSInt(F, E)) = PSInt(F, E) \cup \cl(\int(F, E)) \).
(2): \( SScl(SInt(F, E)) = SSInt(F, E) \cup \cl(\int(F, E)) \).

Proof.
(1): Since \( cl(\int[PSInt(F, E) \cup \cl(\int(F, E)])) \subseteq cl(\int(F, E)) \cup \cl(\int(F, E)) = cl(\int(F, E)) \subseteq (F, E) \cup \cl(\int(F, E)) \) from Theorem 2.1 (3). This means that, \( PSInt(F, E) \cup \cl(\int(F, E)) \) is a pre closed soft set containing \( PSInt(F, E) \). So, \( PScl(PSInt(F, E)) \subseteq PScl(F, E) \cup \cl(\int(F, E)) \). On the other hand, \( PScl(PSInt(F, E)) \) is the largest pre closed soft set containing \( PSInt(F, E) \). Hence, \( PSInt(F, E) \cup \cl(\int(F, E)) \) \( \subseteq PScl(PSInt(F, E)) \). Therefore, \( PScl(PSInt(F, E)) = PSInt(F, E) \cup \cl(\int(F, E)) \).
(2): By a similar way.

Definition 3.1. \cite{8} Let \( (X, \tau, E) \) be a soft topological space and \( (F, E) \in SS(X)_E \). Then \( (F, E) \) is called a b-open soft set if \( (F, E) \subseteq cl(\int(F, E)) \cup \int(\cl(F, E)) \). The set of all b-open soft sets is denoted by \( BOS(X, \tau, E) \), or \( BOS(X) \) and the set of all b-closed soft sets is denoted by \( BCS(X, \tau, E) \), or \( BCS(X) \).

Remark 3.1. \cite{8} It is obvious that, \( POS(X) \cup POS(X) \subseteq BOS(X) \subseteq BCS(X) \).

Theorem 3.3. Let \( (X, \tau, E) \) be a soft topological space. Then, the following are equivalent:

(1): \( (F, E) \) is a b-open soft set.
(2): \( (F, E) = PSInt(F, E) \cup SInt(F, E) \).
(3): \( (F, E) \subseteq PScl(PSInt(F, E)) \).

Proof.
(1) \( \Rightarrow \) (2): Let \( (F, E) \) be a b-open soft set. Then, \( (F, E) \subseteq cl(\int(F, E)) \cup \int(\cl(F, E)) \). By Theorem 3.1, \( PSInt(F, E) \cup SInt(F, E) = ([F, E] \cap \int(\cl(F, E))] \cup ([F, E] \cap \int(\cl(F, E))] = (F, E) \cap [\int(\cl(F, E))] \cup [\int(\cl(F, E))] = (F, E) \).
(2) \( \Rightarrow \) (3): \( (F, E) = PSInt(F, E) \cup SInt(F, E) = PSInt(F, E) \cup ([F, E] \cap \int(\cl(F, E))] \subseteq PSInt(F, E) \cap \cl(\int(F, E)) = PScl(PSInt(F, E)), \) from Theorem 3.1 (3) and Theorem 3.2 (1).
Theorem 3.4. Let the following properties are satisfied:

\[ \text{(4)} \] Counterexamples on soft continuity in (supra) soft topological spaces

Proof.

(1): Since \( bScl(F,E) \) is a \( b \)-closed soft set. Then, \( cl(int(bScl(F,E))) \subseteq \tilde{bScl}(F,E) \). It follows that, \( cl(int(cl(F,E))) \subseteq \tilde{bScl}(F,E) \). So, \( (F,E) \) is a \( \tilde{b} \)-closed soft function, but not \( \tilde{b} \)-continuous.

(2): By a similar way.

4. Counterexamples on soft continuity in (supra) soft topological spaces

Remark 4.1. The converse of [12], Theorem 5.3 is not true in general, as shown in the following examples.

Examples 4.1. (1): Let \( X = \{a,b,c,d\}, Y = \{x,y,z,w\}, A = \{e_1,e_2\} \) and \( B = \{k_1,k_2\} \). Define \( u : X \rightarrow Y \) and \( p : A \rightarrow B \) as follows:

\( u(a) = x, \ u(b) = z, \ u(c) = y, \ u(d) = w, \)

\( p(e_1) = k_1, \ p(e_2) = k_2. \)

Let \( (X,\tau_1,A) \) be a soft topological space over \( X \) where,

\( \tau_1 = \{\tilde{X}, \tilde{\phi}, (F_1,A),(F_2,A),(F_3,A)\}, \)

where \((F_1,A),(F_2,A),(F_3,A)\) are soft sets over \( X \) defined as follows:

\( F_1(e_1) = \{d\}, \ F_1(e_2) = \{a,b\}, \)

\( F_2(e_1) = \{a,b\}, \ F_2(e_2) = \{d\}, \)

\( F_3(e_1) = \{a,b,d\}, \ F_3(e_2) = \{a,b,d\}. \)

Let \( (Y,\tau_2,B) \) be a soft topological space over \( Y \) where,

\( \tau_2 = \{\tilde{Y}, \tilde{\phi}, (G,B)\}, \)

where \((G,B)\) is a soft set over \( Y \) defined by:

\( G(k_1) = \{x\}, \ G(k_2) = \{z\}. \)

Let \( f_{pu} : (X,\tau_1,A) \rightarrow (Y,\tau_2,B) \) be a soft function. Then, \( f_{pu}^{-1}((G,B)) = \{(e_1,\{a\}), (e_2,\{b\})\} \) is a pre (resp. \( \beta \)) open soft set, but not open soft. Hence, \( f_{pu} \) is a \( \beta \)-continuous soft function, but not \( \beta \)-continuous.

(2): Let \( X = \{a,b,c\}, Y = \{x,y,z\}, A = \{e_1,e_2\} \) and \( B = \{k_1,k_2\} \). Define \( u : X \rightarrow Y \) and \( p : A \rightarrow B \) as follows:

\( u(a) = x, \ u(b) = z, \ u(c) = y, \)

\( p(e_1) = k_2, \ p(e_2) = k_1. \)

Let \( (X,\tau_1,A) \) be a soft topological space over \( X \) where,

\( \tau_1 = \{\tilde{X}, \tilde{\phi}, (F,A)\}, \)

where \((F,A)\) is a soft set over \( X \) defined as follows:

\( F(e_1) = \{b\}, \ F(e_2) = \{a\}. \)
Let \((Y, \tau_2, B)\) be a soft topological space over \(Y\) where, 
\[\tau_2 = \{Y, \hat{\phi}, (G, B)\}\], where \((G, B)\) is a soft set over \(Y\) defined by: 
\[G(k_1) = \{x\}\]  
\[G(k_2) = \{x, z\}\]. 
Let \(f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B)\) be a soft function. Then, \(f^{-1}_{pu}((G, B)) = \{(e_1, \{a, b\}), (e_2, \{b\})\}\) is a semi (resp. \(\alpha\)-) open soft set, but not open soft set. Hence, \(f_{pu}\) is a semi- (resp. \(\alpha\)-) continuous soft function, but not continuous soft.

**Remark 4.2.** The converse of [12, Theorem 5.4] is not true in general, as shown in the following examples.

**Examples 4.2.**

1. Let \(X = \{a, b, c\}\), \(Y = \{x, y, z\}\), \(A = \{e_1, e_2\}\) and \(B = \{k_1, k_2\}\). Define \(u : X \rightarrow Y\) and \(p : A \rightarrow B\) as follows: 
   \[u(a) = x, \ u(b) = z, \ u(c) = z,\]  
   \[p(e_1) = k_1, \ p(e_2) = k_2.\] 
Let \((X, \tau_1, A)\) be a soft topological space over \(X\) where, 
\[\tau_1 = \{\tilde{X}, \tilde{\phi}, (F_1, A), (F_2, A), (F_3, A)\}\], where \((F_1, A), (F_2, A), (F_3, A)\) are soft sets over \(X\) defined as follows: 
\[F_1(e_1) = \{a\}\]  
\[F_2(e_1) = \{b\}\]  
\[F_3(e_1) = \{a, b\}\]. 
Let \((Y, \tau_2, B)\) be a soft topological space over \(Y\) where, 
\[\tau_2 = \{Y, \hat{\phi}, (G, B)\}\], where \((G, B)\) is a soft set over \(Y\) defined by: 
\[G(k_1) = \{z\}\]  
\[G(k_2) = \{z\}\]. 
Let \(f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B)\) be a soft function. Then, \(f^{-1}_{pu}((G, B)) = \{(e_1, \{b, c\}), (e_2, \{b, c\})\}\) is a semi open soft set, but not \(\alpha\)-open soft. Hence, \(f_{pu}\) is a semi-continuous soft function, but not \(\alpha\)-continuous soft.

2. In Examples 4.1 (1), let \((Y, \tau_2, B)\) be a soft topological space over \(Y\) where, 
\[\tau_2 = \{Y, \hat{\phi}, (G, B)\}\], where \((G, B)\) is a soft set over \(Y\) defined by: 
\[G(k_1) = \{x\}\]  
\[G(k_2) = \{y\}\]. 
Let \(f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B)\) be a soft function. Then, \(f^{-1}_{pu}((G, B)) = \{(e_1, \{a\}), (e_2, \{c\})\}\) is a \(\beta\)-open soft set, but not semi open soft. Hence, \(f_{pu}\) is a \(\beta\)-continuous soft function, but not semi-continuous soft.

3. Let \(X = \{a, b, c, d\}\), \(Y = \{x, y, z, w\}\), \(A = \{e_1, e_2\}\) and \(B = \{k_1, k_2\}\). Define \(u : X \rightarrow Y\) and \(p : A \rightarrow B\) as follows: 
   \[u(a) = x, \ u(b) = z, \ u(c) = w, \ u(d) = z,\]  
   \[p(e_1) = k_1, \ p(e_2) = k_2.\] 
Let \((X, \tau_1, A)\) be a soft topological space over \(X\) where, 
\[\tau_1 = \{\tilde{X}, \tilde{\phi}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}\], where \((F_1, A), (F_2, A), (F_3, A), (F_4, A)\) are soft sets over \(X\) defined as follows: 
\[F_1(e_1) = \{a\}\]  
\[F_2(e_1) = \{b\}\]  
\[F_3(e_1) = \{a, b\}\]  
\[F_4(e_1) = \{a, b, d\}\]. 
Let \((Y, \tau_2, B)\) be a soft topological space over \(Y\) where, 
\[\tau_2 = \{Y, \hat{\phi}, (G, B)\}\], where \((G, B)\) is a soft set over \(Y\) defined by: 
\[G(k_1) = \{x, z\}\]  
\[G(k_2) = \{y, z\}\].
Let \( f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B) \) be a soft function. Then, \( f_{pu}^{-1}((G, B)) = \{(e_1, \{a, b\}), (e_2, \{b, d\})\} \) is a \( \beta \)-open soft set, but not pre open soft. Hence, \( f_{pu} \) is a \( \beta \)-continuous soft function, but not pre-continuous soft.

(4): In (1), let \((Y, \tau_2, B)\) be a soft topological space over \( Y \) where, \( \tau_2 = \{(\tilde{Y}, \tilde{\phi}, (G, B))\} \), where \((G, B)\) is a soft set over \( Y \) defined by:
\[
G(k_1) = \{z\}, \quad G(k_2) = \{x\}.
\]
Let \( f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B) \) be a soft function. Then, \( f_{pu}^{-1}((G, B)) = \{(e_1, \{b, c\}), (e_2, \{a\})\} \) is a pre open soft set, but not \( \alpha \)-open soft set. Hence, \( f_{pu} \) is a pre-continuous soft function, but not \( \alpha \)-continuous soft.

**Remark 4.3.** The converse of [9, Theorem 6.2] is not true in general, as shown in the following examples.

**Examples 4.3.**

(1): Let \( X = \{a, b, c\}, Y = \{x, y, z\}, \ A = \{e_1, e_2\} \) and \( B = \{k_1, k_2\} \). Define \( u : X \rightarrow Y \) and \( p : A \rightarrow B \) as follows:

\[
u(a) = x, \quad u(b) = z, \quad u(c) = y,

p(e_1) = k_2, \quad p(e_2) = k_1.
\]
Let \((X, \tau_1, A)\) be a soft topological space over \( X \) where,
\[
\tau_1 = \{\tilde{X}, \tilde{\phi}, (F_1, A)\}, \quad \text{where} \quad \{F_1, A\} \text{ is a soft set over } X \text{ defined as follows:}
\]
\[
F_1(e_1) = \{a\}, \quad F_1(e_2) = \{a\}.
\]
\[
F_2(e_1) = \{a, b\}, \quad F_2(e_2) = \{a, b\}.
\]
\[
F_3(e_1) = \{b, c\}, \quad F_2(e_2) = \{b, c\}.
\]
Let \((Y, \tau_2, B)\) be a soft topological space over \( Y \) where,
\[
\tau_2 = \{\tilde{Y}, \tilde{\phi}, (G, B)\}, \quad \text{where} \quad \{G, B\} \text{ is a soft set over } Y \text{ defined by:}
\]
\[
G(k_1) = \{x, y\}, \quad G(k_2) = \{x, y\}.
\]
Let \( f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B) \) be a soft function. Then, \( f_{pu}^{-1}((G, B)) = \{(e_1, \{a, c\}), (e_2, \{a, c\})\} \) is a supra pre (resp. \( \beta \)-) open soft set, but not supra open soft set. Hence, \( f_{pu} \) is a supra pre- (resp. \( \beta \)-) continuous soft function, but not supra continuous soft.

(2): Let \( X = \{a, b, c\}, Y = \{x, y, z\}, \ A = \{e_1, e_2\} \) and \( B = \{k_1, k_2\} \). Define \( u : X \rightarrow Y \) and \( p : A \rightarrow B \) as follows:

\[
u(a) = x, \quad u(b) = z, \quad u(c) = y,

p(e_1) = k_2, \quad p(e_2) = k_1.
\]
Let \((X, \tau_1, A)\) be a soft topological space over \( X \) where,
\[
\tau_1 = \{\tilde{X}, \tilde{\phi}, (F_1, A)\}, \quad \text{where} \quad \{F_1, A\} \text{ is a soft set over } X \text{ defined as follows:}
\]
\[
F_1(e_1) = \{a, b\}, \quad F_1(e_2) = \{a\}.
\]
\[
F_2(e_1) = \{a, c\}, \quad F_2(e_2) = \{a, c\}.
\]
\[
F_3(e_1) = \{b, c\}, \quad F_2(e_2) = \{b, c\}.
\]
Let \((Y, \tau_2, B)\) be a soft topological space over \( Y \) where,
\[
\tau_2 = \{\tilde{Y}, \tilde{\phi}, (G, B)\}, \quad \text{where} \quad \{G, B\} \text{ is a soft set over } Y \text{ defined by:}
\]
\[
G(k_1) = \{x, y\}, \quad G(k_2) = \{x, z\}.
\]
Let \( f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B) \) be a soft function. Then, \( f_{pu}^{-1}((G, B)) = \{(e_1, \{a, b\}), (e_2, \{a, c\})\} \) is a supra semi (resp. \( \alpha \)-) open soft set, but not supra open soft set. Hence, \( f_{pu} \) is a supra semi- (resp. \( \alpha \)-) continuous soft function, but not supra continuous soft.

**Remark 4.4.** The converse of [9], Theorem 6.3] is not true in general, as shown in the following examples.

**Examples 4.4.**

(1): Let \( X = \{a, b, c, d\} \), \( Y = \{x, y, z, w\} \), \( A = \{e_1, e_2\} \) and \( B = \{k_1, k_2\} \). Define \( u : X \rightarrow Y \) and \( p : A \rightarrow B \) as follows:

\[ u(a) = y, \quad u(b) = x, \quad u(c) = x, \quad u(d) = x, \]

\[ p(e_1) = k_1, \quad p(e_2) = k_2. \]

Let \( (X, \tau_1, A) \) be a soft topological space over \( X \) where,

\[ \tau_1 = \{\tilde{X}, \tilde{\phi}, (F_1, A)\}, \]

where \( (F_1, A) \) is a soft set over \( X \) defined as follows:

\[ F_1(e_1) = \{\alpha, \beta\}, \quad F_1(e_2) = \{\alpha, \gamma\}. \]

The supra soft topology \( \mu_1 \) is defined as follows, \( \mu_1 = \{\tilde{X}, \tilde{\phi}, (F_1, A), (F_2, A), (F_3, A), (F_4, A)\} \), where \( (F_1, A), (F_2, A), (F_3, A), (F_4, A) \) are soft sets over \( X \) defined as follows:

\[ F_1(e_1) = \{\alpha, \beta\}, \quad F_1(e_2) = \{\alpha, \gamma\}. \]

\[ F_2(e_1) = \{\alpha, \beta\}, \quad F_2(e_2) = \{\alpha, \beta\}. \]

\[ F_3(e_1) = \{\alpha, \beta\}, \quad F_3(e_2) = \{\alpha, \beta\}. \]

\[ F_4(e_1) = \{\alpha, \beta\}, \quad F_4(e_2) = \{\alpha, \beta\}. \]

Let \( (Y, \tau_2, B) \) be a soft topological space over \( Y \) where,

\[ \tau_2 = \{\tilde{Y}, \tilde{\phi}, (G, B)\}, \]

where \( (G, B) \) is a soft set over \( Y \) defined by:

\[ G(k_1) = \{x, y\}, \quad G(k_2) = \{x, w\}. \]

Let \( f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B) \) be a soft function. Then, \( f_{pu}^{-1}((G, B)) = \{(x, \{a, b\}), (y, \{a, c\})\} \) is a supra semi open soft set, but not supra \( \alpha \)-open soft set. Hence, \( f_{pu} \) is a supra semi-continuous soft function, but not supra \( \alpha \)-continuous soft.

(2): In (1), let \( u : X \rightarrow Y \) defined as follows:

\[ u(a) = y, \quad u(b) = x, \quad u(c) = y, \quad u(d) = y. \]

Let \( (Y, \tau_2, B) \) be a soft topological space over \( Y \) where,

\[ \tau_2 = \{\tilde{Y}, \tilde{\phi}, (G, B)\}, \]

where \( (G, B) \) is a soft set over \( Y \) defined by:

\[ G(k_1) = \{x, y\}, \quad G(k_2) = \{x, w\}. \]

Let \( f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B) \) be a soft function. Then, \( f_{pu}^{-1}((G, B)) = \{(x, \{b\}), (y, \{b\})\} \) is a supra \( \beta \)-open soft set, but not supra semi open soft set. Hence, \( f_{pu} \) is a supra \( \beta \)-continuous soft function, but not supra semi-continuous soft.

(3): In (1), let \( u : X \rightarrow Y \) defined as follows:

\[ u(a) = x, \quad u(b) = y, \quad u(c) = y, \quad u(d) = x. \]

Let \( (Y, \tau_2, B) \) be a soft topological space over \( Y \) where,

\[ \tau_2 = \{\tilde{Y}, \tilde{\phi}, (G, B)\}, \]

where \( (G, B) \) is a soft set over \( Y \) defined by:

\[ G(k_1) = \{x, z\}, \quad G(k_2) = \{x, w\}. \]

Let \( f_{pu} : (X, \tau_1, A) \rightarrow (Y, \tau_2, B) \) be a soft function. Then, \( f_{pu}^{-1}((G, B)) = \{(x, \{a, d\}), (x, \{a, d\})\} \) is a supra \( \beta \)-open soft set, but not supra pre open soft set. Hence, \( f_{pu} \) is a supra \( \beta \)-continuous soft function, but not supra pre-continuous soft.
In (1), let \( u : X \to Y \) defined as follows:
\[
u(a) = x, \quad u(b) = y, \quad u(c) = x, \quad u(d) = y.
\]
Let \((Y, \tau_2, B)\) be a soft topological space over \( Y \) where,
\[
\tau_2 = \{ \tilde{Y}, \tilde{\phi}, (G, B) \}, \quad \text{where} \quad (G, B)\text{ is a soft set over } Y \text{ defined by:}
\]
\[
G(k_1) = \{ x, z \}, \quad G(k_2) = \{ x, w \}.
\]
Let \( f_{pu} : (X, \tau_1, A) \to (Y, \tau_2, B) \) be a soft function. Then,
\[
f_{pu}^{-1}((G, B)) = \{(e_1, \{a, c\}), (e_2, \{a, c\})\} \text{ is a supra pre open soft set, but not supra } \alpha\text{-open soft set. Hence, } f_{pu}\text{ is a supra pre-continuous soft function, but not supra } \alpha\text{-continuous soft.}
\]

5. Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied and improved the soft set theory, which is initiated by Molodtsov [27] and easily applied to many problems having uncertainties from social life. The main purpose of this paper, is to introduce more properties of of \( \gamma \)-soft interior and \( \gamma \)-soft closure mentioned in [8, 12]. Also, counter examples of (supra) soft continuity, mentioned in [9, 12], are introduced. In the next study, we extend the notion of b-open soft sets to supra soft topological spaces and other topological properties. Also, we will use some topological tools in soft set application, like rough sets.

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A. KANDIL (dr.ali_kandil@yahoo.com)
Department of Mathematics, Faculty of Science, Helwan University, Helwan, Egypt.
O. A. E. Tantawy (drosamat@yahoo.com)
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt.

S. A. El-Sheikh (sobhyelsheikh@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt.

A. M. Abd El-Latif (Alaa–8560@yahoo.com)
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt.