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The inclusion measure and information energy for hesitant fuzzy sets and their application in decision making

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Abstract. Inclusion measure of hesitant fuzzy sets is an important subject in the theory of hesitant fuzzy set, and it has been studied and widely used in clustering analysis and decision making. In this paper, we aim at constructing inclusion measure for hesitant fuzzy sets based on fuzzy sets. The axiomatical definition of inclusion measure for hesitant fuzzy sets is firstly proposed, and we construct a new inclusion relation for hesitant fuzzy sets. Based on this new inclusion relation, different inclusion measures based on the inclusion measures of fuzzy sets for hesitant fuzzy sets are further constructed. In addition, in order to describe the fuzziness degree of HFSs, information energy for HFSs is introduced. Finally, a real example about hesitant fuzzy multi-attribute decision making is used to illustrate the validity and applicability of the proposed inclusion measures and information energies.

2010 AMS Classification: Insert the 2010 AMS Classification

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1. Introduction

Fuzzy set (FS) was originally introduced by Zadeh [27] in 1965, it has achieved a great success in various fields to handle this kind of uncertainty. With more and more imprecise, uncertain and hesitant information in the real life. Hesitant fuzzy set (HFS), as an extension of FS, is proposed by Torra [18, 19]. It allows the membership degree of an element to a given set having a few different values. While the membership degrees of an element to a given set provided by decision makers are not only by crisp values, so different extensions of HFS have been introduced, such as, dual hesitant fuzzy sets [33], interval-valued hesitant fuzzy set [2], generalized hesitant fuzzy sets [14], hesitant fuzzy linguistic term sets [15], hesitant fuzzy rough sets [4] and interval-valued intuitionistic hesitant fuzzy soft sets [13]. The relative information measure of HFS and its extension are introduced and applied to cluster analysis, decision making and other fields.

After proposing HFS, Xia and Xu [22] proposed a series of aggregation operators for hesitant fuzzy elements (HFEs) to solve decision making problem. Afterwards, Xu and Xia [23] applied the proposed distance measures, similarity measures for HFSs to decision making. Farhadinia [6] took further studied the relationship among distance measure, similarity measure and entropy for HFSs and applied similarity measure to clustering analysis. Later, Li et al. [11] pointed out some drawbacks of the existing distance measures for HFSs and improved distance measures for HFSs which contain hesitant degrees. Li et al. [11] also applied the proposed new distance measures to decision making. In addition, other decision making methods associated with distance measure for HFSs under the environment of hesitant fuzzy information were constructed, such as the TOPSIS method [24], VIKOR method [12], TODIM method [29], ELECTRE I method [3], ELECTRE II method [1] and the satisfaction degree based interactive decision making method [28]. All these studies show that aggregation operators, distance measures and similarity measures for HFSs play an important role in dealing decision making, while few researches study decision making by inclusion measures for HFSs. In this paper, a decision making method associated with inclusion measures is proposed.

Inclusion measures are constructed in term of inclusion relation. The inclusion relation of two sets is include or exclude in set theory. As a generalization, inclusion relation for FSs has proposed by Zadeh [27]. It is a binary relation and describes whether a FS is completely contained within another FS or not. But in practice, it is too strict, so Kosko [10] argued to consider the degree of contained. The axiomatical definition and constructive approach of inclusion measure for FSs have been studied in [16, 31], and applied these inclusion measure of FSs to clustering analysis [5, 26]. The constructive approaches of inclusion measure for FSs mainly divide into two methods: one is based on the cardinality of FSs, the other is based on fuzzy implicators. For the inclusion relation of hesitant fuzzy elements (HFEs), Zhang and Yang [25, 30] proposed a new partial order for HFEs by means of inclusion relation of FSs in [31], based on which, inclusion measures for HFEs were constructed by the cardinality of HFEs and fuzzy implicators. And inclusion measures for HFSs were further studied by means of inclusion measures for HFEs and applied to decision making. In some cases, inclusion measures proposed by Zhang and Yang [25, 30] are counterintuitive, for example, let $h_1 = \{0.03, 0.51, 0.52, 0.53\}$ and $h_2 = \{0.3, 0.45, 0.46\}$. Obviously, the degree of h_1 containing h_2 is bigger than that of h_2 containing h_1, while the degree of h_1 containing h_2 is smaller than that of h_2 containing h_1 by inclusion measures in [25, 30]. In this paper, several inclusion measures for HFSs are obtained by a new inclusion relation for HFSs and apply to decision making. In addition, entropy plays an important role in fuzzy set theory, while information energy has a closer connection with it. The larger information energy is, the smaller entropy is, that is, it is less fuzziness. Information energy for FSs is a measure of useful information obtained from a FS and the relative information energies were introduced in [7, 17, 20, 21], but information energy for HFSs has not researched.

In this paper, we introduce the concept of information energy for HFSs, and apply it to decision making.

The rest of the paper is organized as follows. In section 2, we review some basic concepts for FSs and HFSs. And we propose the axiomatical definition of inclusion measure for HFSs. In section 3, we introduce a method for transforming a HFS into a collection of FSs. Based on this method, a new inclusion relation for HFSs is proposed and several inclusion measures for HFSs are constructed in terms of inclusion measures for FSs. In section 4, we introduce information energy for HFSs and give several formulas of information energies. In section 5, a real example is provided to illustrate the application and validity of the proposed inclusion measures and information energies for HFSs in hesitant fuzzy decision making. The conclusion is given in the last section.

2. Preliminaries

To facilitate the presentation. In this section, we briefly recall some preliminary definitions and results. They will be necessary along the other sections of the paper.

In general, the class of objects encountered in the practical world have not be precisely defined criteria of membership, such as tall and short, young and old. In order to better described it, Zadeh introduced the concept of FS in [27].

Definition 2.1 ([27], Fuzzy set)**.** Let X be a nonempty set, a fuzzy set (FS) A on X is defined in terms of a function μ_A when applied to X returns a real number of $[0,1]$, that is,

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\},$$

where the real value $\mu_A(x)$ represents the degree of membership of x in A. For convenience, the collection of all FSs on X is denoted as $\mathcal{F}(X)$.

Definition 2.2 ([27], Inclusion relation for FSs)**.** Let A and B be two FSs on a nonempty set X. Then an inclusion relation for FSs is defined as follows:

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \forall x \in X. \tag{2.1}$$

Definition 2.3 ([18, 19], Hesitant fuzzy set)**.** Let X be a nonempty set. A hesitant fuzzy set (HFS) M on X is defined in terms of a function h_M when applied to X returns a subset of $[0,1]$, that is,

$$M = \{\langle x, h_M(x)\rangle \mid x \in X\},$$

where $h_M(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$ to the set M. For the sake of simplicity, Xia and Xu [22] called $h_M(x)$ a hesitant fuzzy element (HFE).

Note that $\mathcal{H}(X)$ denotes the collection of all HFSs on X. Empty HFS: $h_M(x) = 0$ for all $x \in X$, $M = \{\bar{0}\}$ for short. Full HFS: $h_M(x) = 1$ for all $x \in X$, $M = \{\bar{1}\}$ for short.

As we known, different HFEs have different cardinality. The concept of cardinality for HFEs has been introduced in [6, 23].

Definition 2.4 ([6, 23])**.** Let $M = \{\langle x, h_M(x)\rangle \mid x \in X\}$ be a HFS on a nonempty set X, we defined $l(h_M(x))$ as the cardinality of the HFE $h_M(x)$, where $l(h_M(x))$ stands for the number of values in $h_M(x)$. In symbols: $l(h_M(x)) = |h_M(x)|$.

In hesitant fuzzy multi-attribute decision making, the form of evaluation values to alternatives under attributes provided by experts are considered as HFEs, different alternatives have different HFEs under attributes. The cardinality of HFEs are usually different and HFEs are out of order, which cause difficulties to compare them. In order to better compare two HFEs, we need to arrange them in any order for convenience and add values to the smaller cardinality of HFE until it has the same cardinality to the bigger HFE.

In [22, 24], the authors proposed that all the elements in each HFE $h_M(x)$ can be arranged in increasing order or in decreasing order, supposing that HFEs are arranged in increasing order and $h_M^{\sigma(j)}(x)$ is referred to the jth smallest value in $h_M(x)$ in this paper. In most cases, for $x \in X$, $l(h_M(x)) \neq l(h_N(x))$, for convenience, let $l_x = \max\{l(h_M(x)), l(h_N(x))\}$. In order to operate correctly, we should extend the smaller one until both of them have the same cardinality when we compare them. The best way to extend the smaller one is to add the same values several times in it and the selection of adding value mainly depends on the decision makers' risk preference, optimists choose to add the maximum value, while pessimists may choose to add the minimum value. In this paper, we consider experts are optimists and HFEs are arranged in increasing order.

Based on the concept of cardinality and the method of adding values for HFEs, Farhadinia [6] proposed the following inclusion relation for HFSs.

Definition 2.5 ([6])**.** Let M and N be two HFSs on a nonempty set X and $X = \{x_1, x_2, \cdots, x_n\}$. Then the inclusion relation for HFSs is defined as follows:

$$M \sqsubseteq_1 N \;\; iff \;\; h_M^{\sigma(j)}(x_i) \leq h_N^{\sigma(j)}(x_i), 1 \leq i \leq n, 1 \leq j \leq l_{x_i}, \tag{2.2}$$

where $l_{x_i} = \max\{|h_M(x_i)|, |h_N(x_i)|\}$.

Farhadinia [6] also gave another inclusion relation based on the score function of HFSs as follows:

Definition 2.6 ([6])**.** Let M and N be two HFSs on a nonempty set X and $X = \{x_1, x_2, \cdots, x_n\}$. Then the inclusion relation for HFSs is defined as follows:

$$M \sqsubseteq_2 N \;\; iff \;\; Score(M) \leq Score(N), \tag{2.3}$$

where $Score(\cdot)$ [23] represents the score function of a HFS given by

$$Score(M) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}} h_M^{\sigma(j)}(x_i)\right), \; l_{x_i} = \max\{|h_M(x_i)|, |h_N(x_i)|\}.$$

In general, $\mu_A(x) \leq \mu_B(x)$ holds for just a few x in the formula (2.1). So we need to consider the degree of A containing B or B containing A. We know that inclusion measure for FSs has been established in fuzzy set theory and the axiomatic definition can be described as follows [32].

Definition 2.7 ([32])**.** (Inclusion measure for FSs) Let X be a nonempty set. A real function $IF : \mathcal{F}(X) \times \mathcal{F}(X) \to [0, 1]$ is called an inclusion measure for FSs if IF satisfies the following properties:

(IF1) for all A, $B \in \mathcal{F}(X)$, if $A \subseteq B$, then $IF(A, B) = 1$,

(IF2) $IF(X, \phi) = 0$, where $X = \{\overline{1}\}$, $\phi = \{\overline{0}\}$, (IF3) for all $M, N, O \in \mathcal{F}(X)$, if $M \subseteq N \subseteq O$, then $IF(O, M) \leq min\{IF(N, M), IF(O, N)\}$.

Based on the above definition of inclusion measure for FSs, we give the axiomatic definition of inclusion measure for HFSs.

Definition 2.8. (Axiomatic definition of inclusion measure for HFSs) Let $\mathcal{H}(X)$ be the collection of all HFSs on a nonempty set X. A function $IH : \mathcal{H}(X) \times \mathcal{H}(X) \to [0, 1]$ is called an inclusion measure if IH satisfies the following properties:
(IH1) for all $M, N \in \mathcal{H}(X)$, if $M \subseteq N$, then $IH(M, N) = 1$,
(IH2) $IH(X, \phi) = 0$ where $X = \{\overline{1}\}$, $\phi = \{\overline{0}\}$,
(IH3) For all $M, N, O \in \mathcal{H}(X)$, if $M \subseteq N \subseteq O$, then $IH(O, M) \leq min\{IH(N, M), IH(O, N)\}$.

Note that $(IH3) \Leftrightarrow (IH3)^{'}$: If $M \subseteq N \subseteq O \subseteq P$, then $IH(P, M) \leq IH(O, N)$.

In fuzzy set theory, several widely used inclusion measures in [5, 26, 31] are listed as follows:

$$\mathrm{IF}_1(\mathrm{A}, \mathrm{B}) = \begin{cases} \frac{|A \cap B|}{|A|}, & A \neq \phi, \\ 0, & A = \phi, \end{cases} \tag{2.4}$$

where $| A |= \sum_{i=1}^{|X|} A(x_i)$.

$$\mathrm{IF}_2(\mathrm{A}, \mathrm{B}) = \begin{cases} 0, & A = B = \phi, \\ \frac{|B|}{|A \cup B|}, & otherwise, \end{cases} \tag{2.5}$$

where $| A |= \sum_{i=1}^{|X|} A(x_i)$.

$$IF_3(A, B) = \sum_{i=1}^{|X|} \lambda_i T(A(x_i), B(x_i)), \tag{2.6}$$

where λ_i is a positive real number with $\sum_{i=1}^{|X|} \lambda_i = 1$, $T(\alpha, \beta) = \min\{\alpha, \beta\}$ is an implicator.

Throughout this paper, we denotes the HFS $M = \{\langle x, k\rangle \mid x \in X\}$ as $M = \{\overline{k}\}$, where k is a real number.

3. Inclusion measures for HFSs

In this section, we first introduce the relationship between HFSs and collections of FSs and propose a new inclusion relation for HFSs. Further, we construct some inclusion measures for HFSs based on the new inclusion relation.

As is known, given a collection of FSs, a HFS could be defined in terms of this collection of FSs.

Definition 3.1 ([18])**.** Let $\mathcal{M} = \{\mu_1, \mu_2, \cdots, \mu_n\}$ be a collection of n FSs, where each μ_i $(i = 1, 2, \cdots, n)$ is a membership function on a nonempty set X. Then the

HFS associated with $\mathcal{M}$, that is, $h_{\mathcal{M}}$ is defined as follows:

$$h_{\mathcal{M}}(x) = \bigcup_{\mu\in\mathcal{M}} \{\mu(x)\}.$$

Note that for the repeated memberships in $h_{\mathcal{M}}(x)$, we only calculate once. For the memberships in $h_{\mathcal{M}}(x)$ being zero, we omit it.

From Definition 3.1, it is clear that given a collection of FSs, the corresponding HFS can be obtained. Conversely, given a HFS, the corresponding collection of FSs could be derived by the following definition.

Definition 3.2. Let $M = \{\langle x, h_M(x)\rangle \mid x \in X\}$ be a HFS on a nonempty set $X = \{x_1, x_2, \cdots, x_n\}$ and $l_M = \max_{x\in X} |h_M(x)|$. If $|h_M(x)| \neq l_M$, for $x \in X$, we extend $h_M(x)$ by adding the maximum value in it until it has the same cardinality with l_M. Then the collection of FSs associated with M is defined as follows:

$$A_i^M = \{\langle x_j, h_M^{\sigma(i)}(x_j)\rangle \mid j = 1, 2, \cdots, n\}, \quad i = 1, 2, \cdots, l_M.$$

For convenience, we denote the collection of FSs by $F(M)$, that is,

$$F(M) = \{A_1^M, A_2^M, \cdots, A_{l_M}^M\}.$$

It is obvious that the cardinality of $F(M)$ is l_M. Especially, if the HFS $M = \{\bar{k}\}$, (k is a real number), that is to say, M is a FS, then $F(M) = \{M\}$.

Remark 3.3. From Definition 3.1, we know that a HFS can be obtained by a collection of FSs. Conversely, Definition 3.2 explains that a collection of FSs can be decided by a HFS. So we may study a HFS by means of a collection of FSs.

In the following, we give two examples.

Example 3.4. Let $X = \{x_1, x_2, x_3\}$ and $\mathcal{M} = \{\mu_1, \mu_2, \mu_3\}$, where $\mu_1 = \{\langle x_1, \{0.1\}\rangle, \langle x_2, \{0.4\}\rangle, \langle x_3, \{0.5\}\rangle\}$, $\mu_2 = \{\langle x_1, \{0.7\}\rangle, \langle x_2, \{0.6\}\rangle, \langle x_3, \{0.4\}\rangle\}$ and $\mu_3 = \{\langle x_1, \{0.4\}\rangle, \langle x_2, \{0.4\}\rangle\}$. Then the HFS associated with $\mathcal{M}$ is:

$$h_{\mathcal{M}}(x_1) = \{\mu_1(x_1)\} \cup \{\mu_2(x_1)\} \cup \{\mu_3(x_1)\} = \{0.1, 0.7, 0.4\},$$
$$h_{\mathcal{M}}(x_2) = \{\mu_1(x_2)\} \cup \{\mu_2(x_2)\} \cup \{\mu_3(x_2)\} = \{0.4, 0.6\},$$
$$h_{\mathcal{M}}(x_1) = \{\mu_1(x_3)\} \cup \{\mu_2(x_3)\} \cup \{\mu_3(x_3)\} = \{0.5, 0.4\}.$$

Thus, $H_{\mathcal{M}} = \{\langle x_1, \{0.1, 0.7, 0.4\}\rangle, \langle x_2, \{0.4, 0.6\}\rangle, \langle x_3, \{0.5, 0.4\}\rangle\}$.

Example 3.5. Let $M = \{\langle x_1, \{0.2\}\rangle, \langle x_2, \{0.4, 0.5\}\rangle, \langle x_3, \{0.5, 0.7, 0.8\}\rangle\}$ on $X = \{x_1, x_2, x_3\}$. Then the collection of FSs associated with M are:

$$A_1^M = \{\langle x_1, \{0.2\}\rangle, \langle x_2, \{0.4\}\rangle, \langle x_3, \{0.5\}\rangle\},$$
$$A_2^M = \{\langle x_1, \{0.2\}\rangle, \langle x_2, \{0.5\}\rangle, \langle x_3, \{0.7\}\rangle\},$$
$$A_3^M = \{\langle x_1, \{0.2\}\rangle, \langle x_2, \{0.5\}\rangle, \langle x_3, \{0.8\}\rangle\}.$$

Thus, $F(M) = \{A_1^M, A_2^M, A_3^M\}$.

Remark 3.6. According to Definition 3.2, we have $A_i^M(x) = h_M^{\sigma(i)}(x)$, $\forall A_i^M \in F(M)$ and $x \in X$.

Based on Remark 3.3, we can give the inclusion relation for HFSs by means of the inclusion relation for FSs.

Definition 3.7. Let M and N be two HFSs on a nonempty set X. Then the inclusion relation for HFSs is defined as follows:

$$M \sqsubseteq_3 N \;\; iff \;\; A_i^M \subseteq A_j^N, \forall A_i^M \in F(M), \forall A_j^N \in F(N), \tag{3.1}$$

where $F(M)$ and $F(N)$ are two collections of FSs associated with M and N, respectively.

Clearly, if M and N are FSs in Definition 3.7, then we can get that $M \sqsubseteq_3 N$ iff $\mu_M(x_i) \leq \mu_N(x_j)$, for $x_i, x_j \in X$. This is consistent with Definition 2.2.

By the formula (2.2) and the formula (3.1), we present the relationship between $\sqsubseteq_1$ and $\sqsubseteq_3$ in the following proposition.

Proposition 3.8. *Let M and N be two HFSs on a nonempty set X. If $M \sqsubseteq_3 N$, then $M \sqsubseteq_1 N$.*

Proof. According to the formula (3.1), we know if $M \sqsubseteq_3 N$, then $A_i^M \subseteq A_j^N$, for $\forall A_i^M \in F(M)$ and $\forall A_j^N \in F(N)$. Thus, $A_i^M \subseteq A_i^N$, for $\forall A_i^M \in F(M)$ and $\forall A_i^N \in F(N)$. By Definition 3.2 and Remark 3.6, we have $A_i^M(x) = h_M^{\sigma(i)}(x)$ and $A_i^N(x) = h_N^{\sigma(i)}(x)$, $\forall x \in X$. So $h_M^{\sigma(i)}(x) \leq h_N^{\sigma(i)}(x)$, $\forall x \in X$. This implies $M \sqsubseteq_1 N$ by Definition 2.5. □

In the following, we give an example to show that inclusion relation $\sqsubseteq_3$ is stricter than inclusion relation $\sqsubseteq_1$, while inclusion relations $\sqsubseteq_1$ and $\sqsubseteq_3$ are stricter than $\sqsubseteq_2$.

Example 3.9. Let $X = \{x_1, x_2\}$, and $M = \{\langle x_1, \{0.2, 0.3\}\rangle, \langle x_2, \{0.3, 0.4\}\rangle, \langle x_3, \{0.6\}\rangle\}$, $N = \{\langle x_1, \{0.3, 0.5\}\rangle, \langle x_2, \{0.5, 0.6\}\rangle, \langle x_3, \{0.7, 0.9\}\rangle\}$ be two HFSs on X. Then $F(M) = \{A_1^M, A_2^M\}$, $F(N) = \{A_1^N, A_2^N\}$, where

$A_1^M = \{\langle x_1, \{0.2\}\rangle, \langle x_2, \{0.3\}\rangle, \langle x_3, \{0.6\}\rangle\}$,
$A_2^M = \{\langle x_1, \{0.3\}\rangle, \langle x_2, \{0.4\}\rangle, \langle x_3, \{0.6\}\rangle\}$,
$A_1^N = \{\langle x_1, \{0.3\}\rangle, \langle x_2, \{0.5\}\rangle, \langle x_3, \{0.7\}\rangle\}$,
$A_2^N = \{\langle x_1, \{0.5\}\rangle, \langle x_2, \{0.6\}\rangle, \langle x_3, \{0.9\}\rangle\}$.

According to Definition 2.2, we have $A_i^M \subseteq A_j^N$, $i, j = 1, 2$. Therefore $M \sqsubseteq_3 N$ by Definition 3.7, consequently, $M \sqsubseteq_1 N$.

If we change M into $M' = \{\langle x_1, \{0.2, 0.4\}\rangle, \langle x_2, \{0.3, 0.4\}\rangle, \langle x_3, \{0.6\}\rangle\}$, then $M' \sqsubseteq_1 N$ and $M' \not\sqsubseteq_3 N$; if we change M into $M'' = \{\langle x_1, \{0.2, 0.3\}\rangle, \langle x_2, \{0.5, 0.7\}\rangle, \langle x_3, \{0.6\}\rangle\}$, then $M'' \not\sqsubseteq_1 N$, $M'' \not\sqsubseteq_3 N$ and $M'' \sqsubseteq_2 N$.

From Example 3.9, we can see that for any two HFSs, inclusion relations $\sqsubseteq_1$ and $\sqsubseteq_3$ sometimes can not be confirmed.

As we know, inclusion measure for FSs has been studied and applied to decision making and clustering analysis, while few articles study inclusion measures for HFSs. In the following, inclusion measures for HFSs will be introduced.

In the following, we denote $\bigvee$ and $\bigwedge$ as the maximum and minimum operations, respectively.

Theorem 3.10. *Let M and N be two HFSs on a nonempty set X. Then*

$$IH_1(M, N) = \bigvee_{i=1}^{l_M} \bigwedge_{j=1}^{l_N} IF(A_i^M, A_j^N) \tag{3.2}$$

is an inclusion measure for HFSs, where IF is an inclusion measure for FSs.

Proof. We show that $IH_1(M,N)$ satisfies three properties of Definition 2.8 as follows.

(IH1) If $M \sqsubseteq_3 N$, then $A_i^M \subseteq A_j^N$ for $\forall A_i^M \in F(M)$ and $A_j^N \in F(N)$ by Definition 3.7. Since IF is an inclusion measure for FSs, it follows from (IF1) of Definition 2.7 that $IF(A_i^M, A_j^N) = 1$, for $\forall A_i^M \in F(M)$ and $A_j^N \in F(N)$. Thus,

$$IH_1(M,N) = \bigvee_{i=1}^{l_M} \bigwedge_{j=1}^{l_N} IF(A_i^M, A_j^N) = 1.$$

(IH2) When $M = X$, $N = \phi$, we have $F(M) = \{\{\bar{1}\}\}$, $F(N) = \{\{\bar{0}\}\}$. By (IF2) of Definition 2.7, we can conclude that $IF(\{\bar{1}\}, \{\bar{0}\}) = 0$, which implies $IH_1(X, \phi) = 0$.

(IH3) If $M_1 \sqsubseteq_3 M_2 \sqsubseteq_3 M_3$, that is, $A_i^{M_1} \subseteq A_j^{M_2} \subseteq A_k^{M_3}$, for $\forall A_i^{M_1} \in F(M_1)$, $A_j^{M_2} \in F(M_2)$ and $A_k^{M_3} \in F(M_3)$. Then we can obtain that $IF(A_k^{M_3}, A_i^{M_1}) \leq IF(A_k^{M_3}, A_j^{M_2})$ and $IF(A_k^{M_3}, A_i^{M_1}) \leq I(A_j^{M_2}, A_i^{M_1})$, by (IF3) of Definition 2.7. Thus

$$\bigvee_{i=1}^{l_{M_3}} \bigwedge_{j=1}^{l_{M_1}} IF(A_k^{M_3}, A_i^{M_1}) \leq \bigvee_{i=1}^{l_{M_3}} \bigwedge_{j=1}^{l_{M_2}} IF(A_k^{M_3}, A_j^{M_2}),$$

$$\bigvee_{i=1}^{l_{M_3}} \bigwedge_{j=1}^{l_{M_1}} IF(A_k^{M_3}, A_i^{M_1}) \leq \bigvee_{i=1}^{l_{M_2}} \bigwedge_{j=1}^{l_{M_1}} IF(A_j^{M_2}, A_i^{M_1}).$$

That is, $IH_1(M_3, M_1) \leq \max\{IH_1(M_3, M_2), IH_1(M_2, M_1)\}$.

In summary, $IH_1(M,N)$ is an inclusion measure. This completes the proof. □

Theorem 3.11. *Let M and N be two HFSs on a nonempty set X, Then the following functions*

$$IH_2(M,N) = \bigwedge_{i=1}^{l_M} \bigvee_{j=1}^{l_N} IF(A_i^M, A_j^N), \tag{3.3}$$

$$IH_3(M,N) = \bigvee_{i=1}^{l_M} \bigvee_{j=1}^{l_N} IF(A_i^M, A_j^N), \tag{3.4}$$

$$IH_4(M,N) = \bigwedge_{i=1}^{l_M} \bigwedge_{j=1}^{l_N} IF(A_i^M, A_j^N) \tag{3.5}$$

are inclusion measures for HFSs, where IF is an inclusion measure for FSs.

Proof. Firstly, we show that $IH_2(M,N)$ satisfies three properties of Definition 2.8 as follows.

(IH1) If $M \sqsubseteq_3 N$, then $A_i^M \subseteq A_j^N$, for $\forall A_i^M \in F(M)$ and $A_j^N \in F(N)$, by Definition 3.7. Since IF is an inclusion measure for FSs, it follows from (IF1) of Definition 2.7 that $IF(A_i^M, A_j^N) = 1$, for $\forall A_i^M \in F(M)$ and $A_j^N \in F(N)$. Thus, $IH_2(M,N) = \bigwedge_{i=1}^{l_M} \bigvee_{j=1}^{l_N} IF(A_i^M, A_j^N) = 1$.

(IH2) When $M = X$, $N = \phi$, we have $F(M) = \{\{\bar{1}\}\}$, $F(N) = \{\{\bar{0}\}\}$. By (IF2) of Definition 2.7, we can conclude that $IF(\{\bar{1}\}, \{\bar{0}\}) = 0$, which implies $IH_2(X, \phi) = 0$.

(IH3) If $M_1 \sqsubseteq_3 M_2 \sqsubseteq_3 M_3$, that is, $A_i^{M_1} \subseteq A_j^{M_2} \subseteq A_k^{M_3}$, for $\forall A_i^{M_1} \in F(M_1)$, $A_j^{M_2} \in F(M_2)$ and $A_k^{M_3} \in F(M_3)$. Then we can obtain that $IF(A_k^{M_3}, A_i^{M_1}) \leq IF(A_k^{M_3}, A_j^{M_2})$ and $IF(A_k^{M_3}, A_i^{M_1}) \leq IF(A_j^{M_2}, A_i^{M_1})$, by (IF3) of Definition 2.7. Thus

$$\bigwedge_{i=1}^{l_{M_3}} \bigvee_{j=1}^{l_{M_1}} IF(A_k^{M_3}, A_i^{M_1}) \leq \bigwedge_{i=1}^{l_{M_3}} \bigvee_{j=1}^{l_{M_2}} IF(A_k^{M_3}, A_j^{M_2}),$$
$$\bigwedge_{i=1}^{l_{M_3}} \bigvee_{j=1}^{l_{M_1}} IF(A_k^{M_3}, A_i^{M_1}) \leq \bigwedge_{i=1}^{l_{M_2}} \bigvee_{j=1}^{l_{M_1}} IF(A_j^{M_2}, A_i^{M_1}).$$

So, $IH_2(M_3, M_1) \leq \max\{IH_2(M_3, M_2), IH_2(M_2, M_1)\}$.

Similar to the proof of $IH_2(M, N)$, it is sufficient to show that $IH_3(M, N)$ and $IH_4(M, N)$ satisfy three properties of Definition 2.8. □

We can also get the following theorem easily.

Theorem 3.12. *Let IH' and IH'' be two inclusion measures for HFSs. Then the following assertions hold:*

(1) $IH'(M, N)IH''(N^c, M^c)$ *is an inclusion measure for HFSs,*
(2) $IH'(M, N) \wedge IH''(N^c, M^c)$ *is an inclusion measure for HFSs,*
(3) $IH'(M, N) \vee IH''(N^c, M^c)$ *is an inclusion measure for HFSs,*
(4) $\lambda_1 IH'(M, N) + \lambda_2 IH''(N^c, M^c)]$ *is an inclusion measure for HFSs, where* $\lambda_1, \lambda_2 \in [0, 1]$ *and* $\lambda_1 + \lambda_2 = 1$,
(5) $0 \vee [IH'(M, N) + IH''(N^c, M^c) - 1]$ *is an inclusion measure for HFSs,*
(6) $1 \wedge [IH'(M, N) + IH''(N^c, M^c)]$ *is an inclusion measure for HFSs.*

Remark 3.13. If we replace similarity measures or distance measures for inclusion measures in Theorem 3.10 and Theorem 3.11 in above, we could obtain some new similarity measures or distance measures for HFSs. Furthermore, Theorem 3.12 also holds for similarity measures or distance measures of HFSs.

4. Information energy for HFSs

Information energy for FSs and interval-valued fuzzy numbers have been discussed in [7, 17, 20, 21], it is used to reflect the fuzziness degree of a set. The larger of the information energy, the smaller of the fuzziness degree. Information energy for HFSs has not been studied. In this section, the information energy for HFSs will be introduced.

The information energy for FSs has been proposed in [20] and described as follows.

Definition 4.1 ([20])**.** Let X be a nonempty set. Then a real function $E : \mathcal{F}(X) \to [0, 1]$ is called an information energy for FSs, if it satisfies the following properties:

(i) for all $A \in \mathcal{F}(X)$, $E(A) = 0$, if $\mu_A(x) = \frac{1}{2}$ $\forall x \in X$,
(ii) for all $A \in \mathcal{F}(X)$, $E(A) = 1$, if $\mu_A(x) = 0$ or $\mu_A(x) = 1$ $\forall x \in X$,
(iii) for all $A \in \mathcal{F}(X)$, $E(A) = E(A^c)$,
(iv) for all $A, B \in \mathcal{F}(X)$, if $A \subseteq B \subseteq \{\overline{\frac{1}{2}}\}$ or $A \supseteq B \supseteq \{\overline{\frac{1}{2}}\}$, then $E(A) \geq E(B)$.

From Definition 4.1, it is easy to conclude that information energy gets minimum 0 when $\mu_A(x) = \frac{1}{2}$ $\forall x \in X$, that is, it is fuzziest. While information energy gets maximum 1 when $\mu_A(x) = 0$ or $\mu_A(x) = 1$ $\forall x \in X$, that is, it is crisp.

Give a HFS M, we can get the only collection of $F(M)$ by Definition 3.2. Therefore, the information energy for HFSs can be expressed by the information energy of the collection of FSs. Based on these, the information energy for HFSs is defined as follows.

Definition 4.2. (Information energy for HFSs) Let X be a nonempty set and M be a HFS on X. Then information energy for HFSs is defined as follows:

$$IE(M) = \frac{1}{l_M} \sum_{i=1}^{l_M} E(A_i^M).$$

In Definition 4.2, information energy $IE(M)$ gets maximum 1 if $M = \{\overline{0}\}$ or $M = \{\overline{1}\}$; while information energy get minimum 0 if $M = \{\overline{\frac{1}{2}}\}$. If $M = \{\overline{k}\}$, information energy can be expressed as $IE(M) = E(M)$.

According to Definition 4.1 and Definition 4.2, the following theorem can be obtained.

Theorem 4.3. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a nonempty set and $\mathcal{H}(X)$ be the collection of all HFSs on X, It is easy to check that the functions IE_1 and IE_2 are two information energies for HFS:*

$$IE_1(M) = \frac{1}{nl_M} \sum_{j=1}^{l_M} \sum_{i=1}^{n} [4(A_j^M(x_i))^2 - 4A_j^M(x_i) + 1], \tag{4.1}$$

$$IE_2(M) = \frac{1}{nl_M} \sum_{j=1}^{l_M} \sum_{i=1}^{n} [1 - \sin A_j^M(x_i)\pi], \tag{4.2}$$

where $M \in \mathcal{H}(X)$ and $A_j^M \in F(M)$.

From Theorem 4.3, we can obtain that a lot of information energies can be constructed, the selection of E in Definition 4.2 is different, the information energy IE will different.

Remark 4.4. In Definition 4.2, we can find that the closer membership degrees of HFS M to $\{\overline{\frac{1}{2}}\}$, the bigger fuzziness of HFS M.

5. An application in hesitant fuzzy multi-attribute decision making

In this part, we apply the proposed inclusion measures and information energies to multi-attribute decision making problem under the environment of hesitant fuzzy information.

For a multi-attribute decision making problem. Let $H = \{H_1, H_2, \cdots, H_m\}$ be a collection of alternatives, $X = \{x_1, x_2, \cdots, x_n\}$ be a collection of attributes and $w = \{w_1, w_2, \cdots, w_n\}$ be the weight vector of attributes, where $w_j \in [0.1]$ and $\sum\limits_{j=1}^{n} w_j = 1$.

Based on the above statement, we give the following decision making method.

Step 1. The decision makers provide their evaluations about the alternative H_i under the attribute x_j, denoted by the HFE h_{ij} $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$.

Step 2. Identify the positive ideal solution H^+ [8, 24],

$$\begin{aligned} H^+ &= \{< x_j, \max_i\{h_{ij}^{\sigma(\lambda)}\} >| \ j = 1, 2, \cdots, n\} \\ &= \{\langle x_1, \{(h_1^1)^+, (h_1^2)^+, \cdots, (h_1^l)^+\}\rangle, \langle x_2, \{(h_2^1)^+, (h_2^2)^+, \cdots, (h_2^l)^+\}\rangle, \cdots, \\ &\qquad \langle x_n, \{(h_n^1)^+, (h_n^2)^+, \cdots, (h_n^l)^+\}\rangle\}, \end{aligned}$$

where $l = \max_{i=1,2,\cdots,m; j=1,2,\cdots,n}\{|h_{ij}|\}$, $|h_{ij}|$ denotes the cardinality of the HFE h_{ij} and $(h_i^j)^+ = \max_{1\le k\le m} h_{H_k}^{\sigma(j)}(x_i)$, $1 \le i \le n$, $1 \le j \le l$.

Step 3. Calculate the degree of H_i $(i = 1, 2, \cdots, m)$ containing H^+ by $IH_1(H^+, H_i)$ and the information energy $IE_1(H_i)$ provided by H_i $(i = 1, 2, \cdots, m)$, where $IF(\cdot, \cdot) = IF_1(\cdot, \cdot)$ in $IH_1(H^+, H_i)$.

Step 4. Calculate the sum of $IH_1(H^+, H_i)$ and $IE_1(H_i)$, and denote it as $S(H_i)$ $(i = 1, 2, \cdots, m)$.

Step 5. Get the priority of the alternatives H_i $(i = 1, 2, \cdots, m)$ by ranking $S(H_i)$ $(i = 1, 2, \cdots, m)$.

Now, a real example adapted from [9, 23] is employed to illustrate the proposed decision making method.

Example 5.1 ([9, 23])**.** Energy plays an important role in the socio-economic development of societies. Therefore, the most appropriate energy policy selection is of great importance. Now, suppose that there are five alternatives (energy projects) H_i $(i = 1, 2, 3, 4, 5)$ to be invested, and four attributes to be considered: x_1: technological; x_2: environmental; x_3: social political; x_4: economic. The attribute weight vector is $w = (0.15, 0.3, 0.2, 0.35)^T$. Several decision makers are invited to evaluate the performances of the five alternatives. To get a more reasonable result, it is better that the decision makers give their evaluations anonymously. Thus, each providing value only means that it is a possible value, and its importance is unknown. So, it is reasonable to allow these values repeated many times to appear only once, and all possible evaluations for an alternative under attributes can be considered as a HFS on X. The results evaluated by the decision makers are contained in a hesitant fuzzy decision matrix, show in Table 1.

It is clear that different HFEs of different HFSs have different numbers of values, so the cardinality of HFEs may be different. In order to identify the positive ideal solution H^+ of HFSs, we need to add values in the smaller cardinality of HFE. According to the regulation of adding value in HFE, we consider that decision makers are optimistic and add maximum values in the smaller cardinality HFEs. The result lists in Table 2.

In the following, we use the developed method to get the most desirable alternative.

Step 1. The evaluation values provided by decision makers are presented in Table 1, and all possible evaluation values for an alternative under the attributes can be considered as a HFS.

Step 2. According to Table 2, we have

$$\begin{aligned} H^+ = \{&< x_1, \{0.6, 0.7, 0.7, 0.8, 0.9\} >, < x_2, \{0.6, 0.9, 0.9, 0.9, 0.9\} >, \\ &< x_3, \{0.7, 0.8, 0.9, 0.9, 0.9\} >, < x_4, \{0.6, 0.8, 0.9, 0.9, 0.9\} >\}. \end{aligned}$$

Step 3. Calculate $IH_1(H^+, H_i)$ and $IE_1(H_i)$ by the formula (3.3) and the formula (4.1) in Theorem 3.10 and Theorem 4.3, respectively. we have

$IH_1(H^+, H_1) = 0.36$, $IH_1(H^+, H_2) = 0.36$, $IH_1(H^+, H_3) = 0.76$,
$IH_1(H^+, H_4) = 0.48$, $IH_1(H^+, H_5) = 0.6$.
$IE_1(H_1) = 0.2025$, $IE_1(H_2) = 0.172$, $IE_1(H_3) = 0.1767$,
$IE_1(H_4) = 0.3025$, $IE_1(H_5) = 0.334$.

Step 4. According to Step 3, we can conclude

$S(H_1) = 0.5625$, $S(H_2) = 0.532$, $S(H_3) = 0.9367$,
$S(H_4) = 0.7825$, $S(H_5) = 0.934$.

Step 5. According to $S(H_i)$ $(i = 1, 2, 3, 4, 5)$ in step 4, alternatives H_i $(i = 1, 2, 3, 4, 5)$ are ranked as

$$H_3 \geq H_5 \geq H_4 \geq H_2 \geq H_1.$$

This implies that H_3 is the most desirable energy project.

Table 1. Hesitant fuzzy information.

	x_1	x_2	x_3	x_4
H_1	$\{0.3, 0.4, 0.5\}$	$\{0.1, 0, 7, 0.8, 0.9\}$	$\{0.2, 0.4, 0.5\}$	$\{0.3, 0.5, 0.6, 0.9\}$
H_2	$\{0.3, 0.5\}$	$\{0.2, 0.5, 0.6, 0.7, 0.9\}$	$\{0.1, 0.5, 0.6, 0.8\}$	$\{0.3, 0.4, 0.7\}$
H_3	$\{0.6, 0.7\}$	$\{0.6, 0.9\}$	$\{0.3, 0.5, 0.7\}$	$\{0.4, 0.6\}$
H_4	$\{0.3, 0.4, 0.7, 0.8\}$	$\{0.2, 0.4, 0.7\}$	$\{0.1, 0.8\}$	$\{0.6, 0.8, 0.9\}$
H_5	$\{0.1, 0.3, 0.6, 0.7, 0.9\}$	$\{0.4, 0.6, 0.7, 0.8\}$	$\{0.7, 0.8, 0.9\}$	$\{0.3, 0.6, 0.7, 0.9\}$

Table 2. The extended hesitant fuzzy information.

	x_1	x_2	x_3	x_4
H_1	$\{0.3, 0.4, 0.5, 0.5, 0.5\}$	$\{0.1, 0, 7, 0.8, 0.9, 0.9\}$	$\{0.2, 0.4, 0.5, 0.5, 0.5\}$	$\{0.3, 0.5, 0.6, 0.9, 0.9\}$
H_2	$\{0.3, 0.5, 0.5, 0.5, 0.5\}$	$\{0.2, 0.5, 0.6, 0.7, 0.9\}$	$\{0.1, 0.5, 0.6, 0.8, 0.8\}$	$\{0.3, 0.4, 0.7, 0.7, 0.7\}$
H_3	$\{0.6, 0.7, 0.7, 0.7, 0.7\}$	$\{0.6, 0.9, 0.9, 0.9, 0.9\}$	$\{0.3, 0.5, 0.7, 0.7, 0.7\}$	$\{0.4, 0.6, 0.6, 0.6, 0.6\}$
H_4	$\{0.3, 0.4, 0.7, 0.8, 0.8\}$	$\{0.2, 0.4, 0.7, 0.7, 0.7\}$	$\{0.1, 0.8, 0.8, 0.8, 0.8\}$	$\{0.6, 0.8, 0.9, 0.9, 0.9\}$
H_5	$\{0.1, 0.3, 0.6, 0.7, 0.9\}$	$\{0.4, 0.6, 0.7, 0.8, 0.8\}$	$\{0.7, 0.8, 0.9, 0.9, 0.9\}$	$\{0.3, 0.6, 0.7, 0.9, 0.9\}$

If we use $IH_2(H^+, H_i)$, $IH_3(H^+, H_i)$ and $IH_4(H^+, H_i)$ $(i = 1, 2, 3, 4, 5)$ to calculate the degree of H_i containing H^+ respectively in step 3 and keep IE_1 and IF_1 unchanged, we can obtained the following results, listed as Table 3.

Table 3. Results obtained by inclusion measures IH_2, IH_3, IH_4 and IE_1.

	$S(H_1)$	$S(H_2)$	$S(H_3)$	$S(H_4)$	$S(H_5)$	*Ranking*
IH_2	0.9803	0.9776	0.9823	1.1914	1.3028	$H_5 \geq H_4 \geq H_3 \geq H_2 \geq H_1$
IH_3	1.0825	1.132	1.1767	1.3025	1.334	$H_5 \geq H_3 \geq H_4 \geq H_2 \geq H_1$
IH_4	0.4525	0.422	0.7045	0.6358	0.7507	$H_5 \geq H_3 \geq H_4 \geq H_2 \geq H_1$

From Table 3, we can see that the most desirable alternative derived by inclusion measures IH_2, IH_3 and IH_4 is H_5. Therefore, H_5 is the most desirable alternative.

In order to better describe the decision making method, we choose IE_2 to calculate the information energy of HFSs and keep IH_i $(i = 1, 2, 3, 4)$ and IF_1 unchanged, the results are listed in Table 4.

Table 4. Results obtained by inclusion measures IH_1, IH_2, IH_3, IH_4 and IE_2.

	$S(H_1)$	$S(H_2)$	$S(H_3)$	$S(H_4)$	$S(H_5)$	*Ranking*
IH_1	0.5861	0.5556	0.9556	0.8211	0.9708	$H_5 \geq H_3 \geq H_4 \geq H_2 \geq H_1$
IH_2	1.0039	1.0011	1.0048	1.2300	1.3395	$H_5 \geq H_3 \geq H_4 \geq H_1 \geq H_2$
IH_3	1.1061	1.1556	1.1992	1.3410	1.3708	$H_5 \geq H_4 \geq H_3 \geq H_2 \geq H_1$
IH_4	0.4761	0.4456	0.7270	0.6744	0.7874	$H_5 \geq H_4 \geq H_3 \geq H_2 \geq H_1$

From Table 4, we can conclude that H_5 is the most desirable alternative. Comparing the Table 3 with Table 4, the ranking results obtained by Table 3 and Table

4 are similar, H_5 is the most desirable alternative. The difference of the Table 3 and Table 4 is the ranking of H_1 and H_2, H_3 and H_4. It is caused by the information energy $IE_i(\cdot)$, implying the information energy IE_i is also an important factor in decision making. Note that, the selection of IH_i depends on decision makers. In general, we choose IH_1 and IH_2 to calculate the inclusion measure.

Comparing the above ranking results with the papers Li [11] and Xu [23], in which the ranking results obtained by distance measures for HFSs. It is clear that different distance measures lead to different ranking results and different values of the parameter in distance measures also lead to different ranking results [11, 23]. So, the ranking results are closed related to distance measures and the selection of their parameter. In Zhang and Yang [30], the ranking result is $H_1 \geq H_2 \geq H_4 \geq H_3 \geq H_5$, so H_1 is the most desirable alternative. However, $IH_1(H^+, H_1) = 0.36 < IH_1 (H^+, H_5) = 0.6$, $IH_i(H^+, H_1) < IH_1(H^+, H_5)$ $(i = 2, 3, 4)$ and $IE_1(H_1) = 0.2025 < IE_1(H_5) = 0.334$. Obviously, H_1 is not the most desirable alternative. Therefore, the method proposed in this paper is more reasonable. In hesitant fuzzy TODIM method or ELECTRE II method, more parameter value need to be provided in ranking, it will lead to the ranking result having deviation.

6. Conclusions

Inclusion measure is an important subject in hesitant fuzzy set theory, inclusion measure of FSs and other extension of FSs have already been studied and applied to some fields. In this paper, we mainly construct some new inclusion measures for HFSs by inclusion measures for FSs. We first proposed the axiomatical definition of inclusion measure for HFSs and discussed the relationship between HFSs and collections of FSs. Based on these, a new inclusion relation of HFSs was established and the relationships with existing inclusion relations of HFSs was also discussed. Further, several new inclusion measures for HFSs developed by collections of FSs in view of the relationship between HFSs and collections of FSs and the new inclusion relation. In addition, we introduced information energy for HFSs, which is used to reflect the fuzziness degree of a set. Finally, the validity and efficiency of the proposed decision making method associated with inclusion measures and information energies for HFSs has been illustrated by an energy police selection example. In the future, we hope that the inclusion measures can be applied to other aspects, such as pattern recognition, cluster analysis and image processing.

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