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Fuzzy ideals of almost distributive lattices

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Abstract. The notion of an Almost Distributive Lattice (ADL) is a common abstraction of several lattice theoretic and ring theoretic generalization of Boolean algebra and Boolean rings. In this paper, we introduce the notion of L-fuzzy ideal of an ADL with truth values in a complete lattice L satisfying the infinite meet distributive law and prove certain properties of these. Mainly, it is proved that the class of L-fuzzy ideals of an ADL forms a complete distributive lattice.

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1. Introduction

L.A. Zadeh [12] introduced the notion of a fuzzy subset of a set X as a function from X into $I = [0,1]$. Several algebraists took interest in the study of fuzzy subalgebras of several algebraic structures. Rosenfeld [5] defined the concept of fuzzy subgroup of a group. In 1982, Liu [3] defined and studied fuzzy sub-rings and as well as fuzzy ideals in rings. Subsequently, U.M. Swamy and K. L. N. Swamy [10] fuzzified prime ideals of rings. Goguen [2] realized that the unit interval $[0,1]$ is not sufficient to take the truth values of general fuzzy statements. U.M. Swamy and K. L. N. Swamy initiated, Fuzzy prime ideals of rings with truth values in a complete lattice satisfying the infinite meet distributive law. More-generally, U. M. Swamy and D. V. Raju [7, 8, 9] introduced the concept of Algebraic Fuzzy systems with truth values in a complete lattice satisfying the infinite meet distributive law.

U. M. Swamy and G. C. Rao [6] have introduced the notion of an Almost Distributive Lattice (ADL) as a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebras (Boolean rings). An ADL $(A, \wedge, \vee, 0)$ satisfies all the axioms of a distributive lattice, except possibly the commutativity of the operations $\vee$ and $\wedge$. It is known that, in any ADL, the commutativity of $\vee$ is equivalent to that of $\wedge$ and also to the right distributivity of $\vee$ over $\wedge$.

In this paper, we introduce the notion of L-fuzzy ideal of a given ADL $(A, \wedge, \vee, 0)$ and we initiate the study of these. Even though most of the results proved here look like analagous to those of the lattices [1, 4], the proofs require different techniques in view of the absence of the commutativity of the operations and the right distributivity of $\vee$ over $\wedge$.

2. Preliminaries

In this section, we collect all the necessary preliminaries which will be used in the discussion in the main text of this paper.

We recall the concept of Almost Distributive Lattices (ADL's) from [6].

Definition 2.1. An algebra $A = (A, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions for all a, b and $c \in A$:
(i) $0 \wedge a = 0$,
(ii) $a \vee 0 = a$,
(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
(iv) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
(v) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
(vi) $(a \vee b) \wedge b = b$.

Any distributive lattice bounded below is an ADL, where 0 is the smallest element. Also, a commutative regular ring $(R, +, ., 0, 1)$ with unity can be made into an ADL by defining the operations $\wedge$ and $\vee$ on R by

$$a \wedge b = a_0 b \quad \text{and} \quad a \vee b = a + b - a_0 b,$$

where, for any $a \in R$, a_0 is the unique idempotent in R such that $aR = a_0 R$ and 0 is the additive identity in R. Further any nonempty set X can be made into an ADL by fixing an arbitrarily choosen element 0 in X and by defining the operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0. \end{cases}$$

This ADL $(X, \wedge, \vee, 0)$ is called a discrete ADL. An ADL A is said to be associative ADL if the operation $\vee$ on A is associative. Through out this paper, by an ADL we always mean an associative ADL only.

Definition 2.2. Let $A = (A, \vee, \wedge, 0)$ be an ADL. For any a and $b \in A$, define

$$a \leq b \;\text{ if }\; a = a \wedge b \;(\Leftrightarrow a \vee b = b).$$

Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

Theorem 2.3. *The following hold for any a, b and c in an ADL A:*

(1) $a \wedge 0 = 0 = 0 \wedge a$ *and* $a \vee 0 = a = 0 \vee a$,
(2) $a \wedge a = a = a \vee a$,
(3) $a \wedge b \leq b \leq b \vee a$,
(4) $a \wedge b = a \Leftrightarrow a \vee b = b$,
(5) $a \wedge b = b \Leftrightarrow a \vee b = a$,
(6) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *(i.e., $\wedge$ is associative),*
(7) $a \vee (b \vee a) = a \vee b$,
(8) $a \leq b \Rightarrow a \wedge b = a = b \wedge a \ \Leftrightarrow a \vee b = b = b \vee a$,
(9) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$,
(10) $(a \vee b) \wedge c = (b \vee a) \wedge c$,
(11) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$,
(12) $a \wedge b = \inf\{a, b\} \Leftrightarrow a \wedge b = b \wedge a \Leftrightarrow a \vee b = \sup\{a, b\}$.

An element $m \in A$ is said to be maximal if for any $x \in A$, $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$, for all $x \in A$.

Definition 2.4. Let I be a non empty subset of an ADL A. Then I is called an ideal of A if $a, b \in I \Rightarrow a \vee b \in I$ and $a \wedge x \in I$ for all $x \in A$.

As a consequence, for any ideal I of A, $x \wedge a \in I$ for all $a \in I$ and $x \in A$. For any $S \subseteq A$, the smallest ideal of A containing S is called the ideal generated by S in A and is denoted by $(S]$. It is known that

$$(S] = \Big\{ \big(\bigvee_{i=1}^{n} x_i \big) \wedge a \mid n \geq 0, x_i \in S \text{ and } a \in A \Big\}.$$

When $S = \{x\}$, we write $(x]$ for $(\{x\}]$. Note that $(x] = \{x \wedge a \mid a \in A\}$.

For any non-empty set X and for any algebraic system A, it is well known that the set A^X of all mappings of X into A can be made into an algebraic system of the same type as A by defining the fundamental operations point-wise. If A is an equationally definable algebra (like group or ring or a module or a lattice), then A^X is an algebra in the variety generated by A. In particular, if 2 is the two-element lattice $\{0, 1\}$ with $0 < 1$, then 2^X is a Boolean algebra for any non-empty set X, since 2 is a Boolean algebra. It is well known that 2^X is isomorphic with the Boolean algebra $P(X)$ of all subsets of X. Therefore, the usual (or crisp) subsets of X can be identified with mapping of X into 2. If we replace 2 by a complete lattice L, satisfying the infinite meet distributive law, then mapping of X into L are called L-fuzzy subsets of X. Formally, we have the following.

Definition 2.5. An L-fuzzy subset λ of X is a mapping from X into L, where L is a complete lattice satisfying the infinite meet distributive law. If L is the unit interval $[0, 1]$ of real numbers, then these are the usual fuzzy subsets of X.

Definition 2.6. Let $\lambda : X \longrightarrow L$ be a L-fuzzy subset of X. Then the set $\Big\{\lambda(x) : x \in X\Big\}$ is called the image of λ and it is denoted by $\lambda(X)$ or $Im(\lambda)$. For any $\alpha \in L$, the set $\lambda_\alpha = \lambda^{-1}\Big([\alpha, 1]\Big) = \Big\{x \in X : \alpha \leq \lambda(x)\Big\}$ is called the α-cut of λ.

3. L-fuzzy ideals

In this section, we introduce the notion of L-fuzzy ideal of an ADL $(A, \vee, \wedge, 0)$ Here after L stands a non-trivial complete lattice in which the infinite meet distributive law is satisfied. That is,

$$a \wedge \Big(\bigvee_{s \in S} s\Big) = \bigvee_{s \in S}(a \wedge s), \text{ for any } S \subseteq L \text{ and } a \in L.$$

Throughout this paper A stands for a non-trivial ADL $(A, \vee, \wedge, 0)$ with 0 as the zero element in A.

Definition 3.1. A L-fuzzy subset λ of A is said to be a L -fuzzy ideal of A, if

$$\lambda(0) = 1 \text{ and } \lambda(x \vee y) = \lambda(x) \wedge \lambda(y), \text{ for all } x, y \in A.$$

It can be easily seen that every L-fuzzy ideal λ of A is an antitone in the sense that for any $x, y \in A$,

$$x \leq y \Longrightarrow \lambda(y) \leq \lambda(x).$$

Before going to the main results, let us recall from [11], that two elements a and b in an ADL $(A, \vee, \wedge, 0)$ are said to be associates to each other if $a \wedge b = b$ and $b \wedge a = a$; in this case we write $a \sim b$.

Lemma 3.2. *Let λ be a L-fuzzy ideal of A, S a non-empty subset of A and $x, y \in A$. Then we have the following:*
(1) *if $x \sim y$, then $\lambda(x) = \lambda(y)$,*
(2) *$\lambda(x \wedge y) = \lambda(y \wedge x)$,*
(3) *if $x \in (S]$, then $\lambda(x) \geq \bigwedge_{i=1}^{n} \lambda(a_i)$ for some $a_1, a_2, ..., a_n \in S$,*
(4) *if $x \in (y]$, then $\lambda(x) \geq \lambda(y)$,*
(5) *if m is a maximal element in A, then $\lambda(m) \leq \lambda(x)$,*
(6) *$\lambda(m) = \lambda(n)$, for all maximal elements m and n in A.*

Proof. (1) Suppose $x \sim y$. Then $x \wedge y = y$ and $y \wedge x = x$, which is equivalent to saying that $x \vee y = x$ and $y \vee x = y$. Thus $\lambda(x) = \lambda(x \vee y) = \lambda(x) \wedge \lambda(y) = \lambda(y) \wedge \lambda(x) = \lambda(y \vee x) = \lambda(y)$.

(2) Clearly, $x \wedge y \sim y \wedge x$. Then, $\lambda\Big(x \wedge y\Big) = \lambda\Big(y \wedge x\Big)$. (By above (1))

(3) Let $x \in (S]$. Then $x = \Big(\bigvee_{i=1}^{n} a_i\Big) \wedge b, for some a_1, a_2, ..., a_n \in S$ and $b \in A$. Thus

$$\begin{aligned}\lambda(x) &= \lambda\Big(\Big(\bigvee_{i=1}^{n} a_i\Big) \wedge b\Big)\\ &= \lambda\Big(\bigvee_{i=1}^{n}(a_i \wedge b)\Big)\end{aligned}$$

$$= \bigwedge_{i=1}^{n} \lambda\Big(a_i \wedge b\Big) \Big(\text{Since,}\lambda(x \vee y) = \lambda(x) \wedge \lambda(y)\Big)$$
$$= \bigwedge_{i=1}^{n} \lambda\Big(b \wedge a_i\Big)(\text{ by(1)})$$
$$\geq \bigwedge_{i=1}^{n} \lambda\Big(a_i\Big). \Big(\text{Since } \lambda \text{ is an antitone}\Big)$$

(4) It follows from (3).

(5) Suppose m is a maximal element in A. Then $m \wedge x = x$. Since $x \wedge m \leq m$ and $x \wedge m \sim m \wedge x$, we get $\lambda(m) \leq \lambda(x \wedge m) = \lambda(m \wedge x) = \lambda(x)$.

(6) is a consequence of (5). □

For α-cuts, we have the following theorem.

Theorem 3.3. *A L-fuzzy subset λ of A is an L-fuzzy ideal of A if and only if the α-cut λ_α is an ideal of A, for all $\alpha \in L$.*

Proof. Let λ be a L -fuzzy ideal of A and $\alpha \in L$. Then $\lambda(0) = 1$ and hence $0 \in \lambda_\alpha$. Thus, $\lambda_\alpha \neq \varnothing$.

$$\begin{aligned} x \text{ and } y \in \lambda_\alpha &\Longrightarrow \alpha \leq \lambda(x) \text{ and } \alpha \leq \lambda(y) \\ &\Longrightarrow \alpha \leq \lambda(x) \wedge \lambda(y) = \lambda(x \vee y) \\ &\Longrightarrow x \vee y \in \lambda_\alpha \\ \text{and } x \in \lambda_\alpha \text{ and } a \in A &\Longrightarrow \alpha \leq \lambda(x) \leq \lambda(a \wedge x) = \lambda(x \wedge a) \\ &\Longrightarrow \alpha \leq \lambda(x \wedge a) \\ &\Longrightarrow x \wedge a \in \lambda_\alpha. \end{aligned}$$

So λ_α is an ideal of A, for all $\alpha \in L$.

Conversely, suppose that λ_α is an ideal of A, for all $\alpha \in L$. Particularly, λ_1, $0 \in \lambda_1$ and thus $\lambda(0) = 1$.

Let x and $y \in A$ and choose $t = \lambda(x \vee y) \in L$. Then $x \vee y \in \lambda_t$. Since $x \leq x \vee y$ and $(x \vee y) \wedge y = y$, we get x and $y \in \lambda_t$ and thus $t \leq \lambda(x)$ and $t \leq \lambda(y)$. So $t \leq \lambda(x) \wedge \lambda(y)$. Hence $\lambda(x \vee y) \leq \lambda(x) \wedge \lambda(y)$.

Let $\beta = \lambda(x) \wedge \lambda(y)$. Then $x, y \in \lambda_\beta$ and thus $x \vee y \in \lambda_\beta$. So, $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda(y)$. Hence $\lambda(x \vee y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in A$. Therefore λ is an L-fuzzy ideal of A. □

Definition 3.4. Let χ_S denote the characteristic function of any subset S of an ADL A, i.e.,

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Theorem 3.5. *A non-empty subset S of A is an ideal of A if and only if χ_S is a L-fuzzy ideal of A.*

Proof. Suppose that S is an ideal of A, then $\chi_S(0) = 1$, since $0 \in S$. For any $x, y \in A$,

$$\begin{aligned} x \vee y \in S &\Longrightarrow x \text{ and } y \in S \Big(\text{Since } x \leq x \vee y \text{ and } y = (x \vee y) \wedge y\Big) \\ &\Longrightarrow \chi_S(x \vee y) = 1 = 1 \wedge 1 = \chi_S(x) \wedge \chi_S(y) \end{aligned}$$

$$
\begin{aligned}
x \vee y \notin S &\Longrightarrow \text{either } x \notin S \text{ or } y \notin S \\
&\Longrightarrow \chi_S(x) = 0 \text{ or } \chi_S(y) = 0 \\
&\Longrightarrow \chi_S(x \vee y) = 0 = \chi_S(x) \wedge \chi_S(y).
\end{aligned}
$$

Thus, $\chi_S(x \vee y) = \chi_S(x) \wedge \chi_S(y)$, for all x and $y \in A$ and So χ_S is an L-fuzzy ideal of A.

Conversely suppose that χ_S is an L-fuzzy ideal of A. Then each α-cut of χ_S is an ideal of A, by the above Theorem. In particular, the 1-cut of χ_S, which is precisely S, is an ideal of A. □

For any L-fuzzy subsets λ and μ of A, we define

$$\lambda \leq \mu \Leftrightarrow \lambda(x) \leq \mu(x) \text{ in the ordering of } L, \text{ for all } x \in A.$$

The following is an important observation which supports the assumption that $\lambda(0) = 1$ for any L-fuzzy ideal λ of A.

Theorem 3.6. *Let λ be a $L-$fuzzy subset of A such that*

$$\lambda(x \vee y) = \lambda(x) \wedge \lambda(y), \textit{ for all } x, y \in A.$$

Define $\overline{\lambda} : A \to L$ by:

$$\overline{\lambda}(x) = \begin{cases} 1 & \textit{if } x = 0 \\ \lambda(x) & \textit{if } x \neq 0. \end{cases}$$

Then, $\overline{\lambda}$ is an $L-$fuzzy ideal of A. Moreover, $\overline{\lambda}$ is the smallest $L-$fuzzy ideal of A containing λ.

Proof. Clearly $\overline{\lambda}(0) = 1$. Let x and $y \in A$.

If $x \vee y = 0$, then $x = 0 = y$, since $x \leq x \vee y$ and $(x \vee y) \wedge y = y$. Thus, $\bar{\lambda}(x \vee y) = 1 = \bar{\lambda}(x) \wedge \bar{\lambda}(y)$.

If $x \vee y \neq 0$, then either $x \neq 0$ or $y \neq 0$, say $x \neq 0$, then, $\bar{\lambda}(x \vee y) = \lambda(x \vee y) = \lambda(x) \wedge \lambda(y) = \bar{\lambda}(x) \wedge \bar{\lambda}(y)$ (Note that $\lambda(x) \leq \lambda(0)$). Thus $\bar{\lambda}$ is an L-fuzzy ideal of A.

If $\{\lambda_i : i \in \Delta\}$ is a family of L-fuzzy ideals of A, then it can be easily verified

that $\bigwedge_{i \in \Delta} \lambda_i$ (the point-wise infimum of $\lambda_i's$) is an L-fuzzy ideal of A.

In particular. if λ is any L-fuzzy subset of A and, for any $x \in A$, if we define

$$\bar{\lambda}(x) = \bigwedge \left\{ \mu(x) : \mu \text{ is an } L\text{-fuzzy ideal of A and } \lambda \leq \mu \right\},$$

then $\bar{\lambda}$ is an L-fuzzy ideal A and becomes L-fuzzy ideal generated by λ.

In the following, we give a precise description of $\bar{\lambda}$. We write $F \subset\subset A$ to mean that F is a non-empty finite subset of A. □

Theorem 3.7. *Let λ be a L-fuzzy subset of A. Then the L-fuzzy ideal $\bar{\lambda}$ generated by λ is given by:*

$$\bar{\lambda}(0) = 1,$$

and

$$\bar{\lambda}(x) = \bigvee \left\{ \bigwedge_{a \in F} \lambda(a) \mid x \in (F], F \subset\subset A \right\}, \textit{ for any } 0 \neq x \in A.$$

Proof. Clearly $\lambda \leq \bar{\lambda}$. Let μ be a L-fuzzy ideal of A such that $\lambda \leq \mu$. Let $x \in A$ such that $x \in (F]$, where $F \subset\subset A$. Then, by Lemma 3.2,

$$\mu(x) \geq \bigwedge_{i=1}^{n} \mu(a_i) \quad \text{for some } a_1, a_2, ..., a_n \in F.$$

$$\text{Now, } \bigwedge_{a\in F} \lambda(a) \leq \bigwedge_{a\in F} \mu(a) \leq \bigwedge_{i=1}^{n} \mu(a_i) \leq \mu(x).$$

Thus, $\bar{\lambda}(x) \leq \mu(x)$, for all $x \in A$. So $\bar{\lambda} \leq \mu$.

Further, if $x, y \in A$ with $x \leq y$, then for any $F \subset\subset A$,

$$y \in (F] \Longrightarrow x \in (F] \Longrightarrow \bigwedge_{a\in F} \lambda(a) \leq \bar{\lambda}(x) \Longrightarrow \bar{\lambda}(y) \leq \bar{\lambda}(x).$$

Thus, $\bar{\lambda}$ is an antitone. So $\bar{\lambda}(x \vee y) \leq \bar{\lambda}(x)$. Also, we can observe that $\bar{\lambda}(x \vee y) = \bar{\lambda}(y \vee x) \leq \bar{\lambda}(y)$. This implies, $\bar{\lambda}(x \vee y) \leq \bar{\lambda}(x) \wedge \bar{\lambda}(y)$, for all $x, y \in A$.

On the other hand, by the infinite meet distributivity in L,

$$\begin{aligned}\bar{\lambda}(x)\wedge\bar{\lambda}(y) &= \Big[\bigvee\Big\{\bigwedge_{a\in F} \lambda(a) \mid x \in (F], F \subset\subset A\Big\}\Big]\wedge\Big[\bigvee\Big\{\bigwedge_{b\in G} \lambda(b) \mid y \in (G], G \subset\subset A\Big\}\Big] \\ &= \bigvee\Big\{\bigwedge_{a\in F} \lambda(a) \wedge \bigwedge_{b\in G} \lambda(b) : x \vee y \in (F] \vee (G] \subseteq (F \cup G], F \cup G \subset\subset A\Big\} \\ &\leq \bar{\lambda}(x \vee y).\end{aligned}$$

Then, $\bar{\lambda}(x \vee y) = \bar{\lambda}(x) \wedge \bar{\lambda}(y)$, for all $x, y \in A$. Thus, $\bar{\lambda}$ is the L-fuzzy ideal of A generated by λ. □

The above result yields the following.

Theorem 3.8. *The set of all L-fuzzy ideals of A is a complete lattice, in which the supremum $\bigvee_{i\in\Delta} \lambda_i$ and infimum $\bigwedge_{i\in\Delta} \lambda_i$ of any family $\{\lambda_i : i \in \Delta\}$ of L-fuzzy ideals of A are given by:*

$$\Big(\bigvee_{i\in\Delta} \lambda_i\Big)(x) = \bigvee\Big\{\bigwedge_{a\in F}\Big(\bigvee_{i\in\Delta} \lambda_i(a)\Big) : x \in (F], F \subset\subset A\Big\}$$

and

$$\Big(\bigwedge_{i\in\Delta} \lambda_i\Big)(x) = \bigwedge_{i\in\Delta} \lambda_i(x).$$

Corollary 3.9. *For any $L-$ fuzzy ideals λ and μ of A, the supremum $\lambda \vee \mu$ of λ and μ is given by:*

$$\Big(\lambda \vee \mu\Big)(x) = \vee\Big\{\bigwedge_{a\in F}\Big(\lambda(a) \vee \mu(a)\Big) : x \in (F], F \subset\subset A\Big\}$$

Theorem 3.10. *let I be an ideal of $A, I \neq A$ and $\alpha \in L$. Then the L-fuzzy subset α_I of A defined by:*

$$\alpha_I(x) = \begin{cases} 1 & \text{if } x \in I \\ \alpha & \text{if } x \notin I \end{cases}$$

is an L-fuzzy ideal of A. α_I defined above is called the α-level L-fuzzy ideal corresponding to I.

Recall that for any $x \in A, (x] = \Big\{y \in A : x \wedge y = y\Big\}$.
We have, $\alpha_{_A} = \alpha_{_{(A]}} = \chi_{_A}$, $\alpha_{_{\{0\}}} = \chi_{_{\{0\}}}$ and $0_{_I} = \chi_{_I}$. Moreover, for any $\alpha \in L$ and for any subset I of A, it can be verified that $\alpha_{_I}$ is an L-fuzzy ideal of A if and only if I is an ideal of A.

Theorem 3.11. *For a given $\alpha \in L$, the mapping $I \longmapsto \alpha_{_I}$ is an isomorphism of the lattice $\mathcal{I}(A)$ of all ideals of A onto the lattice of all α-level L-fuzzy ideals of A corresponding to ideals of A.*

We shall conclude this paper with the following theorem.

Theorem 3.12. *The lattice of all L-fuzzy ideals of A is distributive.*

Proof. Let λ, μ and ν be L-fuzzy ideals of A. Then clearly, $(\lambda\wedge\mu)\vee(\lambda\wedge\nu) \leq \lambda\wedge(\mu\vee\nu)$. Let $x \in A$ be fixed. Then,

$$\begin{aligned}
\Big(\lambda \wedge (\mu \vee \nu)\Big)(x) &= \lambda(x) \wedge \Big(\mu \vee \nu\Big)(x) \\
&= \lambda(x) \wedge \Big[\bigvee_{\substack{F \subset\subset A \\ x \in (F]}} \Big(\bigwedge_{a \in F} (\mu(a) \vee \nu(a))\Big)\Big] \\
&= \bigvee_{\substack{F \subset\subset A \\ x \in (F]}} \Big[\lambda(x) \wedge \bigwedge_{a \in F} \Big(\mu(a) \vee \nu(a)\Big)\Big] \\
&= \bigvee_{\substack{F \subset\subset A \\ x \in (F]}} \Big[\bigwedge_{a \in F} \Big[\Big(\lambda(x) \wedge \mu(a)\Big) \vee \Big(\lambda(x) \wedge \nu(a)\Big)\Big]\Big] \quad (*)
\end{aligned}$$

If $F = \{a_1, a_2, ..., a_m\}$ and $x \in (F]$, then

$$x = (\bigvee_{i=1}^{n} a_i)\wedge x, \text{ for some } a_1, a_2, ..., a_n \in F \text{ and } n \leq m.$$

Consider,

$$\begin{aligned}
\bigwedge_{a \in F} \Big[\Big(\lambda(x)\wedge\mu(a)\Big)\vee\Big(\lambda(x)\wedge\nu(a)\Big)\Big] &\leq \bigwedge_{i=1}^{n} \Big[\Big(\lambda(x)\wedge\mu(a_i)\Big)\vee\Big(\lambda(x)\wedge\nu(a_i)\Big)\Big] \\
&\leq \bigwedge_{i=1}^{n} \Big[\Big(\lambda(x\wedge a_i)\wedge\mu(x\wedge a_i))\vee(\lambda(x\wedge a_i)\wedge \nu(x\wedge a_i)\Big)\Big] \\
&\qquad \text{(Since } \lambda, \mu, \nu \text{ are antitones)} \\
&= \bigwedge_{i=1}^{n} \Big[\Big(\lambda \wedge \mu\Big)(x \wedge a_i) \vee \Big(\lambda \wedge \nu\Big)(x \wedge a_i)\Big] \\
&\leq \Big[\Big(\lambda\wedge\mu\Big) \vee \Big(\lambda\wedge\nu\Big)\Big](x) \quad \Big(\text{Since } x = \bigvee_{i=1}^{n} (a_i \wedge x)\Big)
\end{aligned}$$

Thus, by $(*)$, we get that

$$\Big[\lambda \wedge \Big(\mu \vee \nu\Big)\Big](x) \leq \Big[\Big(\lambda \wedge \mu\Big) \vee \Big(\lambda \wedge \nu\Big)\Big](x), \text{ for all } x \in A$$

This implies that, $\lambda \wedge \Big(\mu \vee \nu\Big) = \Big(\lambda \wedge \mu\Big) \vee \Big(\lambda \wedge \nu\Big)$.

□

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