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Some hypergroups on grneral complex fuzzy automata

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Abstract. In this paper, we first define the notion of a complex fuzzy subset and the notion of a general complex fuzzy automaton and construct some H_ν- groups on the set of states of a general complex fuzzy automaton. We then construct some commutative hypergroups on the set of states of a complex fuzzy automaton.

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1. Introduction and Preliminaries

Zadeh [20] introduced the theory of fuzzy sets and, soon after, Wee [18] introduced the concept of fuzzy automata. Automata have a long history both in theory and application [1, 2] and are the prime examples of general computational systems over discrete spaces [8]. In the conventional spectrum of automata (i.e. deterministic finite-state automata, non-deterministic finite-state automata, probabilistic automata and fuzzy finite-state automata), deterministic finite-state automata have found the most application in different areas [3, 11, 12, 16]. Fuzzy automata not only provide a systematic approach for handling uncertainty in such systems, but are can also be used in continuous spaces [9, 13, 14, 15, 17]. Moreover, they are able to create capabilities which are not easily achievable by other mathematical tools [19].

In 2004, M. Doostfatemeh and S. C. Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [7].

In this paper, by using [5, 6, 7], we introduce several new concepts and derive related results.

Definition 1.1. Let $C^* = \{c + di : c, d \in [0, 1], i = \sqrt{-1}\}$. A complex fuzzy subset μ of X is a function of X into C^*. So if μ be a complex fuzzy subset of X, then $|\mu|$ is a fuzzy subset of X. If $\mu(x) = c + di$, then $\mu(x) = r\exp(i\theta)$, which θ is argument of $\mu(x)$ and $r = |\mu(x)| = \sqrt{c^2 + d^2}$. For a nonempty set X, $\tilde{P}(X)$ denotes the set of all complex fuzzy subsets on X.

Definition 1.2 ([10]). Let Σ be a set. A word of Σ is the product of a finite sequence of elements in Σ, Λ denotes the empty word and Σ^* is the set of all words on Σ. In fact, Σ^* is the free monoid on Σ. The length $\ell(x)$ of word $x \in \sum^*$ is the number of its letters, so $\ell(\Lambda) = 0$.

Definition 1.3. A complex fuzzy finite-state automaton (CFFA) is a six-tuple denoted as $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to C^*$ is the complex fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol and $\omega : Q \to Z$ is the output function. The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$.
Associated with each $|\delta(q_i, a_k, q_j)|$, there is a membership value in $[0, 1]$ called the weight of the transition. The set of all transitions of $\tilde{F}$ is denoted as Δ.

Definition 1.4. A general complex fuzzy automaton (GCFA) $\tilde{F}$ is an eight-tuple machine denoted as (i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R}$ is the set of fuzzy start states,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \to Z$ is the output function,
(vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to C^*$ is the augmented transition function,
(vii) $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ is the membership assignment function,
(viii) $F_2 : [0, 1]^* \to [0, 1]$ is called the multi-membership resolution function.

We note that the function $F_1(\mu, |\delta|)$ has two parameters, μ and $|\delta|$, where μ is the membership value of a predecessor and $|\delta|$ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = |\tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j)| = F_1(\mu^t(q_i), |\delta(q_i, a_k, q_j)|).$$

So $\tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = \mu^{t+1}(q_j)\exp(i\theta)$ such that θ is the argument of $\delta(q_i, a_k, q_j)$. This means that the membership value of the state q_j at time $t + 1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.
If $\tilde{\delta}((q_j, \mu^{t_j}(q_j)), a_j, q_{j+1}) = r_j \exp(i\theta_j)$, $j = 1, 2, ..., n$, then we define

$$\bigvee_{j=1}^{n} \tilde{\delta}((q_j \mu^{t_j}(q_j)), a_j, q_{j+1}) = r\exp(i\theta),$$

where $r = max\{r_1, r_2, ..., r_n\}$ and $\theta = max\{\theta_1, \theta_2, ..., \theta_n\}$.
Also we define

$$\bigwedge_{j=1}^{n} \tilde{\delta}((q_j, \mu^{t_j}(q_j)), a_j, q_{j+1}) = r\exp(i\theta),$$

where $r = min\{r_1, r_2, ..., r_n\}$ and $\theta = min\{\theta_1, \theta_2, ..., \theta_n\}$.
The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the set of all active states at time $t_i, \forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$, $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1$. Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write:

$$q \in Domain(Q_{act}(t_i)) \text{ and } T \subset Domain(Q_{act}(t_i)).$$

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j will lead to the multi-membership resolution algorithm.

Algorithm 1.5. *(Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to that:*

(1) *each transition weight $|\delta(q_i, a_k, q_j)|$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i,*

$$v_i = |\tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j)| = F_1(\mu^t(q_i), |\delta(q_i, a_k, q_j)|),$$

(2) *these membership values are not necessarily equal. Hence, they will be processed by another function F_2, called the multi-membership resolution function,*

(3) *the result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,*

$$\mu^{t+1}(q_j) = \underset{i=1}{\overset{n}{F_2}}[v_i] = \underset{i=1}{\overset{n}{F_2}}[F_1(\mu^t(q_i), |\delta(q_i, a_k, q_j)|)].$$

Where

- *n: is the number of simultaneous transitions to the active state q_j at time $t+1$,*
- *$|\delta(q_i, a_k, q_j)|$: is the weight of a transition from q_i to q_j upon input a_k.,*
- *$\mu^t(q_i)$: is the membership value of q_i at time t,*
- *$\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t+1$.*

Definition 1.6. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general complex fuzzy automaton. We define max-min general complex fuzzy automata of the form:

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$$

such that

$$\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to C^*,$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and let for every i, $i \geq 0$,

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p, \\ 0, & otherwise \end{cases}$$

and for every i, $i \geq 1$,

$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p) = r\exp(i\theta)$,

$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p))$

and recursively

$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \ldots u_n, p)$

$= \vee\{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \ldots$

$\wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \ldots, p_{n-1} \in Q_{act}(t_{n-1})\}$,

in which $u_i \in \Sigma$, $\forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be u_i, $\forall 1 \leq i \leq n-1$.

Definition 1.7. Let $\tilde{F}^*$ be a max-min general complex fuzzy automaton. The response function $r^{\tilde{F}^*} : \Sigma^* \times Q \to C^*$ of $\tilde{F}^*$ is define by

$$r^{\tilde{F}^*}(x, q) = \bigvee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, q),$$

for any $x \in \Sigma^*$, $q \in Q$.

Definition 1.8 ([4])**.** A nonempty set H endowed with a hyperoperation $\circ : H^2 \to P^*(H)$ is called a hypergroupoid, where $P^*(H)$ is the set of all nonempty subsets of H. The image of the pair $(a, b) \in H^2$ is denoted by $a \circ b$ and called the hyperproduct of a and b. If A and B are nonempty subsets of H, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$.

Definition 1.9 ([4])**.** The hypergroupoid $\langle H, \circ \rangle$ is called semihypergroup, if the hyperoperation "$\circ$" is associative. A semihypergroup $\langle H, \circ \rangle$ is called hypergroup if

$$H \circ a = a \circ H = H, \quad \forall a \in H.$$

Definition 1.10 ([4])**.** Let $\langle H, \circ \rangle$ and $\langle K, * \rangle$ be hypergroupoids and $f : H \to K$ be a function. We say that

(i) f is a homomorphism, if for all $(a, b) \in H^2$, $f(a \circ b) \subset f(a) * f(b)$,

(ii) f is a good homomorphism if for all $(a, b) \in H^2$, $f(a \circ b) = f(a) * f(b)$.

Definition 1.11 ([4])**.** Let $\langle H, \circ \rangle$ be a hypergroupoid and let R be an equivalence relation on H. We say that R is regular to the right, if the following implication holds:

$$aRb \Rightarrow \forall u \in H, a \circ u \overline{R} b \circ u$$

$$(\text{ i.e. } \forall x \in a \circ u, \exists y \in b \circ u : xRy \text{ and } \forall y \in b \circ u, \exists x \in a \circ u : xRy.)$$

Regularity to the left is defined similarly. We say that R is regular if it is regular both to the right and to the left.

Definition 1.12 ([4])**.** Let H be a semihypergroup and R be an equivalence on H.

(i) If R is regular, then H/R is a semihyper group,with respect to the following hyperoperation:

$$\overline{x} \otimes \overline{y} = \{\overline{z} : z \in x \circ y\}, \forall (\overline{x}, \overline{y}) \in (H/R)^2.$$

(ii) In the above-mentioned hypothesis, the canonical projection $\prod : H \to H/R$ is a good epimorphism and if $\langle H, \circ \rangle$ is a hypergroup, then $\langle H/R, \otimes \rangle$ is also a hypergroup.

Definition 1.13 ([4]). The hypergroupoid $\langle H, \circ \rangle$ is called an H_ν-group, if
(i) weak associativity is satisfied:

$$x \circ (y \circ z) \cap (x \circ y) \circ z \neq \varnothing, \forall (x, y, z) \in H^3,$$

and
(ii) the reproductive axiom holds:

$$H \circ x = x \circ H = H, \ \forall x \in H.$$

2. Hypergroups and general complex fuzzy automata

In this section, we construct some H_ν- groups on the set of states of a general complex fuzzy automaton. We then construct some commutative hypergroups on the set of states of a complex fuzzy automaton.

Theorem 2.1. *Let* $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a general complex fuzzy automaton. We define on* Q *the following hyperoperation for all* $x \in \Sigma^*$ *and for all* α, $0 < \alpha < 90$ *:*

$$p \underset{x}{\circ} q = \begin{cases} \{p_1, q_1\}, & \text{if } 0 \leq \theta_1 < \alpha, 0 \leq \theta_2 < \alpha \\ \{p_1\}, & \text{if } 0 \leq \theta_1 < \alpha, \alpha \leq \theta_2 \leq 90 \\ \{q_1\}, & \text{if } 0 \leq \theta_2 < \alpha, \alpha \leq \theta_1 \leq 90 \\ \varnothing, & \text{otherwise,} \end{cases}$$

where θ_1 *is argument of* $\tilde{\delta}^*((p, \mu^{t_p}(p)), x, p_1)$ *and* θ_2 *is argument of* $\tilde{\delta}^*((q, \mu^{t_q}(q)), x, q_1)$. *Now let*

$$p \circ q = (\bigcup_{x \in \Sigma^* \setminus \{\Lambda\}} p \underset{x}{\circ} q) \cup (p \underset{\Lambda}{\circ} q).$$

Then $\langle Q, \circ \rangle$ *is a commutative* H_ν*-group.*

Proof. We first show that the hyperoperation "$\circ$" is week associative. Since we have $\tilde{\delta}^*((p, \mu^{t_p}(p)), \Lambda, p) = 1$, $\theta_1 = 0$ and since $\tilde{\delta}^*((q, \mu^{t_q}(q)), \Lambda, q) = 1$, $\theta_2 = 0$. Then we have $p \underset{\Lambda}{\circ} q = \{p, q\}, \forall p, q \in Q$.
Thus we have

$$\begin{aligned}(p \circ q) \circ r &= [(\bigcup_{x \in \Sigma^* \setminus \{\Lambda\}} p \underset{x}{\circ} q) \cup (p \underset{\Lambda}{\circ} q)] \circ r = [(\bigcup_{x \in \Sigma^* \setminus \{\Lambda\}} p \underset{x}{\circ} q) \circ r] \cup [(p \underset{\Lambda}{\circ} q) \circ r] \\ &= [\bigcup_{t \in \bigcup_{x \in \Sigma^* \setminus \{\Lambda\}} p \underset{x}{\circ} q} (t \circ r)] \cup [\bigcup_{s \in p \underset{\Lambda}{\circ} q} (s \circ r)] \supseteq (p \circ r) \cup (q \circ r) \supseteq \{p, q, r\}.\end{aligned}$$

Similarly,

$$p \circ (q \circ r) \supseteq \{p, q, r\}.$$

So $p \circ (q \circ r) \cap (p \circ q) \circ r \neq \varnothing$, $\forall (p, q, r) \in Q^3$. Hence the hyperoperation "$\circ$" is week associative.

We claim that

$$Q \circ q = q \circ Q = Q, \forall q \in Q.$$

It is clear that $Q \circ q \subseteq Q$. For the reverse inclusion, let $p \in Q$. Since $p \underset{\Lambda}{\circ} q = \{p, q\}$, we have $p \in p \underset{\Lambda}{\circ} q \subseteq p \circ q \subseteq Q \circ q$. Therefore $Q \subseteq Q \circ q$. □

Example 2.2. In Theorem 2.1, let $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a\}$, $Q_{act}(t_0) = \tilde{R} = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\}$, $F_1(\mu, |\delta|) = Min(\mu, |\delta|)$, $Z = \varnothing$, ω and F_2 are not applicable, $\delta(q_0, a, q_1) = 0.4 + 0.2i$, $\delta(q_1, a, q_2) = 0.3 + 0.2i$, $\delta(q_2, a, q_2) = 0.1 + 0.2i$ and $\alpha = 45$.

If we choose the input string $x = aa \dots a$, then $Q_{act}(t_1) = \{(q_1, \mu^{t_1}(q_1))\}$, $Q_{act}(t_i) = \{(q_2, \mu^{t_i}(q_2))\}, \forall i \geq 2$,

$\mu^{t_1}(q_1) = |\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1)| = F_1(\mu^{t_0}(q_0), |\delta(q_0, a, q_1)|) = F_1(1, 0.4) = 0.4$,
$\mu^{t_2}(q_2) = |\tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_2)| = F_1(\mu^{t_1}(q_1), |\delta(q_1, a, q_2)|) = F_1(0.4, 0.4) = 0.4$,
$\mu^{t_3}(q_2) = |\tilde{\delta}((q_2, \mu^{t_2}(q_2)), a, q_2)| = F_1(\mu^{t_2}(q_2), |\delta(q_2, a, q_2)|) = F_1(0.4, 0.2) = 0.2$,
$\mu^{t_i}(q_2) = 0.2, \forall i \geq 4$,
$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a, q_1) = 0.4\exp(26.5i)$,
$\tilde{\delta}^*((q_1, \mu^{t_1}(q_1)), a, q_2) = 0.4\exp(33.6i)$,
$\tilde{\delta}^*((q_2, \mu^{t_2}(q_2)), a, q_2) = 0.2\exp(63.4i)$,
$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aa, q_2) = 0.4\exp(26.5i) \wedge 0.4\exp(33.6i) = 0.4\exp(26.5i)$,
$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aaa, q_2) = 0.2\exp(26.5i), \dots$.

Thus we have

$$q_0 \underset{a}{\circ} q_0 = \{q_1\},\ q_0 \underset{aa}{\circ} q_0 = \{q_2\},\ q_0 \underset{a}{\circ} q_1 = \{q_1, q_2\},$$
$$q_1 \underset{a}{\circ} q_1 = \{q_2\},\ q_1 \underset{a}{\circ} q_2 = \{q_2\} \text{ and } q_2 \underset{a}{\circ} q_2 = \varnothing.$$

Thus we have

$\circ$	q_0	q_1	q_2
q_0	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$
q_1	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_1, q_2\}$
q_2	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$

Theorem 2.3. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a general complex fuzzy automaton. We define on Q the following hyperoperation for all $x \in \Sigma^*$ and for all β, $0 < \beta < 1$:*

$$p \underset{x}{o} q = \begin{cases} \{p_1, q_1\}, & \text{if } \beta < r_1 \leq 1, \beta < r_2 \leq 1 \\ \{p_1\}, & \text{if } \beta < r_1 \leq 1, \text{ and } 0 \leq r_2 \leq \beta \\ \{q_1\}, & \text{if } \beta < r_2 \leq 1, \text{ and } 0 \leq r_1 \leq \beta \\ \varnothing, & \text{otherwise,} \end{cases}$$

where $r_1 = |\tilde{\delta}^*((p, \mu^{t_p}(p)), x, p_1)|$ and $r_2 = |\tilde{\delta}^*((q, \mu^{t_q}(q)), x, q_1)|$. Now let

$$p \circ q = (\bigcup_{x \in \Sigma^* \setminus \{\Lambda\}} p \underset{x}{\circ} q) \cup (p \underset{\Lambda}{\circ} q).$$

Then $\langle Q, \circ \rangle$ is a commutative H_ν-group.

Proof. We first show that the hyperoperation "$\circ$" is week associative. Since we have $\tilde{\delta}^*((p, \mu^{t_p}(p)), \Lambda, p) = 1$, $r_1 = 1$ and since $\tilde{\delta}^*((q, \mu^{t_q}(q)), \Lambda, q) = 1$, $r_2 = 1$. Then we have $p \underset{\Lambda}{\circ} q = \{p, q\}, \forall p, q \in Q$.

Thus we have

$$(p \circ q) \circ r = [(\bigcup_{x \in \Sigma^* \setminus \{\Lambda\}} p \underset{x}{\circ} q) \cup (p \underset{\Lambda}{\circ} q)] \circ r = [(\bigcup_{x \in \Sigma^* \setminus \{\Lambda\}} p \underset{x}{\circ} q) \circ r] \cup [(p \underset{\Lambda}{\circ} q) \circ r]$$
$$= [\bigcup_{t \in \bigcup_{x \in \Sigma^* \setminus \{\Lambda\}} p \underset{x}{\circ} q} (t \circ r)] \cup [\bigcup_{s \in p \underset{\Lambda}{\circ} q} (s \circ r)] \supseteq (p \circ r) \cup (q \circ r) \supseteq \{p, q, r\}.$$

Similarly,

$$p \circ (q \circ r) \supseteq \{p, q, r\}.$$

So $p \circ (q \circ r) \cap (p \circ q) \circ r \neq \varnothing$, $\forall (p, q, r) \in Q^3$. Hence the hyperoperation " $\circ$ " is week associative.

We claim that

$$Q \circ q = q \circ Q = Q, \forall q \in Q.$$

It is clear that $Q \circ q \subseteq Q$. For the reverse inclusion, let $p \in Q$. Since $p \underset{\Lambda}{\circ} q = \{p, q\}$, we have $p \in p \underset{\Lambda}{\circ} q \subseteq p \circ q \subseteq Q \circ q$. Therefore $Q \subseteq Q \circ q$. □

Example 2.4. In Theorem 2.3, let $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a\}$, $Q_{act}(t_0) = \tilde{R} = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\}$, $F_1(\mu, |\delta|) = Min(\mu, |\delta|)$, $Z = \varnothing$, ω and F_2 are not applicable, $\delta(q_0, a, q_1) = 0.4 + 0.2i$, $\delta(q_1, a, q_2) = 0.3 + 0.2i$, $\delta(q_2, a, q_2) = 0.1 + 0.2i$ and $\beta = 0.3$.

If we choose the input string $x = aa \dots a$, then $Q_{act}(t_1) = \{(q_1, \mu^{t_1}(q_1))\}$, $Q_{act}(t_i) = \{(q_2, \mu^{t_i}(q_2))\}, \forall i \geq 2$,
$\mu^{t_1}(q_1) = |\tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1)| = F_1(\mu^{t_0}(q_0), |\delta(q_0, a, q_1)|) = F_1(1, 0.4) = 0.4$,
$\mu^{t_2}(q_2) = |\tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_2)| = F_1(\mu^{t_1}(q_1), |\delta(q_1, a, q_2)|) = F_1(0.4, 0.4) = 0.4$,
$\mu^{t_3}(q_2) = |\tilde{\delta}((q_2, \mu^{t_2}(q_2)), a, q_2)| = F_1(\mu^{t_2}(q_2), |\delta(q_2, a, q_2)|) = F_1(0.4, 0.2) = 0.2$,
$\mu^{t_i}(q_2) = 0.2, \forall i \geq 4$,
$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), a, q_1) = 0.4 \exp(26.5i)$,
$\tilde{\delta}^*((q_1, \mu^{t_1}(q_1)), a, q_2) = 0.4 \exp(33.6i)$,
$\tilde{\delta}^*((q_2, \mu^{t_2}(q_2)), a, q_2) = 0.2 \exp(63.4i)$,
$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aa, q_2) = 0.4 \exp(26.5i) \wedge 0.4 \exp(33.6i) = 0.4 \exp(26.5i)$,
$\tilde{\delta}^*((q_0, \mu^{t_0}(q_0)), aaa, q_2) = 0.2 \exp(26.5i), \dots$.

Thus we have

$q_0 \underset{a}{\circ} q_0 = \{q_1\}$, $q_0 \underset{aa}{\circ} q_0 = \{q_1\}$,$q_0 \underset{aaa}{\circ} q_0 = \varnothing$, $q_0 \underset{a}{\circ} q_1 = \{q_1, q_2\}$,
$q_1 \underset{a}{\circ} q_1 = \{q_2\}$, $q_1 \underset{a}{\circ} q_2 = \{q_2\}$ and $q_2 \underset{a}{\circ} q_2 = \varnothing$.

Thus we have

$\circ$	q_0	q_1	q_2
q_0	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$
q_1	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_1, q_2\}$
q_2	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$

Theorem 2.5. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton, $x \in \Sigma^*$, $p_0 \in Q_{act}(t_0)$, $r_1 = |\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, p)|$, $r_2 = |\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, q)|$ and define the equivalence relation R_x on Q by pR_xq if and only if $r_1 = r_2$. We define on Q the following commutative hyper operation*

$$p \underset{x}{o} q = \begin{cases} \{p, q\}, & \text{if } r_1 \neq r_2 \\ \bigcup\limits_{t \leq r_1} = s, & \text{if } r_1 = r_2 \\ p_0, & \text{if } p = q = p_0 \end{cases}$$

where $t = |\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, s)|$ and $\bar{s} = \{s' \in Q : s'R_xs\}$. Then $\langle Q, \underset{x}{o} \rangle$ is a hypergroup.

Proof. It is clear that the hyperoperation "$\underset{x}{o}$" is associative. We claim that

$$Q\underset{x}{o}q = q\underset{x}{o}Q = Q, \quad \forall q \in Q.$$

It is clear that $Q\underset{x}{o}q \subseteq Q$. For the reverse inclusion, let $p \in Q$. Then we have

$$p \in p\underset{x}{o}q \subseteq Q\underset{x}{o}q.$$

Theus $Q \subseteq Q\underset{x}{o}q$. □

Theorem 2.6. *In Theorem 2.5, the equivalence relation R_x on Q is regular, where*

$$pR_xq \Leftrightarrow r_1 = r_2,$$

$r_1 = |\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, p)|$ *and* $r_2 = |\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, q)|$.

Proof. It is easy to see that R_x is an equivalence relation. Now, let $s \in Q$ and pR_xq. Then it is clear that

$$(p\underset{x}{o}s)\overline{R_x}(q\underset{x}{o}s).$$

Thus R_x is regular on Q. □

Theorem 2.7. *In Theorem 2.5, $\langle Q/R_x, \otimes\rangle$ is a hypergroup, where*

$$\overline{p} \otimes \overline{q} = \{\overline{r} : r \in p\underset{x}{o}q\}, \forall(\overline{p}, \overline{q}) \in (Q/R_x)^2.$$

Proof. By Theorem 1.12, since $\langle Q, \underset{x}{o}\rangle$ is a hypergroup and the equivalence relation R_x on Q is regular, then we conclude that $\langle Q/R_x, \otimes\rangle$ is a hypergroup and the canonical projection $\prod : Q \to Q/R_x$ is a good epimorphism. □

Theorem 2.8. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton, $x \in \Sigma^*$, $p_0 \in Q_{act}(t_0)$, θ_1 is argument of $\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, p)$ and θ_2 is argument of $\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, q)$ and define the equivalence relation R_x on Q by pR_xq if and only if $\theta_1 = \theta_2$. We define on Q the following commutative hyper operation*

$$p\underset{x}{o}q = \begin{cases} \{p, q\}, & \text{if } \theta_1 \neq \theta_2 \\ \bigcup\limits_{\theta \leq \theta_1} = s, & \text{if } \theta_1 = \theta_2 \\ p_0, & \text{if } p = q = p_0 \end{cases}$$

where θ is argument of $\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, s)$ and $\bar{s} = \{s' \in Q : s'R_xs\}$. Then $\langle Q, \underset{x}{o}\rangle$ is a hypergroup.

Proof. It is clear that the hyperoperation "$\underset{x}{o}$" is associative. We claim that

$$Q\underset{x}{o}q = q\underset{x}{o}Q = Q, \quad \forall q \in Q.$$

It is clear that $Q\underset{x}{o}q \subseteq Q$. For the reverse inclusion, let $p \in Q$. Then we have

$$p \in p\underset{x}{o}q \subseteq Q\underset{x}{o}q.$$

Thus $Q \subseteq Q\underset{x}{o}q$. □

Theorem 2.9. *In Theorem 2.8, the equivalence relation R_x on Q is regular, where*

$$pR_xq \Leftrightarrow \theta_1 = \theta_2,$$

θ_1 is argument of $\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, p)$ and θ_2 is argument of $\tilde{\delta}^*((p_0, \mu^{t_{p_0}}(p_0)), x, q)$.

Proof. It is easy to see that R_x is an equivalence relation. Now, let $s \in Q$ and pR_xq. Then it is clear that

$$(p\underset{x}{o}s)\overline{R_x}(q\underset{x}{o}s).$$

Thus R_x is regular on Q. □

Theorem 2.10. *In Theorem 2.8, $\langle Q/R_x, \otimes\rangle$ is a hypergroup, where*

$$\overline{p} \otimes \overline{q} = \{\overline{r} : r \in p\underset{x}{o}q\}, \forall(\overline{p}, \overline{q}) \in (Q/R_x)^2.$$

Proof. By Theorem 1.12, since $\langle Q, \underset{x}{o}\rangle$ is a hypergroup and the equivalence relation R_x on Q is regular, we conclude that $\langle Q/R_x, \otimes\rangle$ is a hypergroup and the canonical projection $\prod : Q \to Q/R_x$ is a good epimorphism. □

Theorem 2.11. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general complex fuzzy automaton, λ be a complex fuzzy subset on Q, $\overline{D}(\lambda)(p) = \vee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$ and $p \in Q$. We define on Q the following hyperoperation*

$$p \circ p = \{r \in Q : |\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(r)|\},$$

and

$$p \circ q = (p \circ p) \cup (q \circ q), \text{ where } p \neq q.$$

Then $\langle Q, \circ\rangle$ is a commutative hypergroup.

Proof. We first show that the hyperoperation "$\circ$" is associative. We have

$$\begin{aligned}
&(p \circ q) \circ s \\
&= \{r_1 \in Q : |\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(r_1)|\} \cup \{r_2 \in Q : |\overline{D}(\lambda)(q)| \geq |\overline{D}(\lambda)(r_2)|\}] \circ s \\
&= \{r_1' \in Q : |\overline{D}(\lambda)(r_1)| \geq |\overline{D}(\lambda)(r_1')|, |\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(r_1)|\} \\
&\quad \cup \{r_2' \in Q : |\overline{D}(\lambda)(r_2)| \geq |\overline{D}(\lambda)(r_2')|, |\overline{D}(\lambda)(q)| \geq |\overline{D}(\lambda)(r_2)|\} \\
&\quad \cup \{r_3 \in Q : |\overline{D}(\lambda)(s)| \geq |\overline{D}(\lambda)(r_3)|\} \\
&\subseteq \{r_1' \in Q : |\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(r_1')|\} \cup \{r_2' \in Q : |\overline{D}(\lambda)(q)| \geq |\overline{D}(\lambda)(r_2')|\} \\
&\quad \cup \{r_3 \in Q : |\overline{D}(\lambda)(s)| \geq |\overline{D}(\lambda)(r_3)|\}.
\end{aligned}$$

Let

$$\begin{aligned}
T = &\{r_1 \in Q : |\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(r_1)|\} \\
&\cup \{r_2 \in Q : |\overline{D}(\lambda)(q)| \geq |\overline{D}(\lambda)(r_2)|\} \\
&\cup \{r_3 \in Q : |\overline{D}(\lambda)(s)| \geq |\overline{D}(\lambda)(r_3)|\}.
\end{aligned}$$

Then $(p \circ q) \circ s \subseteq T$.

Now, let $r \in \{r_1 \in Q : |\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(r_1)|\}$. Then $|\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(r)|$. Thus $r \in p \circ p \subseteq (p \circ q) \circ s$. So $(p \circ q) \circ s \supseteq T$. Hence

$$(p \circ q) \circ s = T.$$

Similarly,

$$p \circ (q \circ s) = T.$$

Therefore the hyperoperation "$\circ$" is associative.

We claim that

$$Q \circ q = q \circ Q = Q, \quad \forall q \in Q.$$

It is clear that $Q \circ q \subseteq Q$.
For the reverse inclusion, let $p \in Q$. Since $|\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(p)|$,

$$p \in \{r_1 \in Q : |\overline{D}(\lambda)(p)| \geq |\overline{D}(\lambda)(r_1)|\} \cup \{r_2 \in Q : |\overline{D}(\lambda)(q)| \geq |\overline{D}(\lambda)(r_2)|\} = p \circ q.$$

Then $p \in Q \circ q$. Thus $Q \subseteq Q \circ q$. □

3. Conclusions

In this paper, we have defined the notion of a complex fuzzy subset and the notion of a general complex fuzzy automaton.Then we have constructed some H_ν-groups and some commutative hypergroups on the set of states of a complex fuzzy automaton

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