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Comprehensive assessment method for multi-attribute decision making

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Abstract. There are lots of methods for multi-attribute decision making problems. However, there is no standard to judge which is more reasonable. This paper proposes a comprehensive assessment method based on the existing methods for multi-attribute decision making problems. First, we introduce the ranking vector and the transformation function to make different methods in one dimension. Then, we give the reliability degree of different methods and discuss its several desirable properties. Next, we propose a comprehensive assessment method. Further more, an example is illustrated to demonstrate the validity and feasibility of the proposed method.

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1. Introduction

Since hesitant fuzzy set (HFSs) have been introduced as an effective tool to show the hesitancy, an increasing amount of attention has been attracted in different areas, especially in multi-attribute decision making (MADM) problems. Meanwhile, numerous studies [3, 4, 6, 12, 13, 17, 19, 20, 22, 27, 28] focused on solving MADM problems under hesitant fuzzy environment. Xu and Li [12, 20] proposed a series of distance and similarity measures of HFSs, which can be utilized to deal with MADM problems. A variety of aggregation operators were proposed for aggregating hesitant fuzzy information and applied to develop some models for hesitant fuzzy MADM problems [13, 17, 18, 23, 24, 25, 26]. Chen et al. [3] presented an approach to MADM problems based on interval-valued hesitant preference relation considering the difference of opinions between individual decision makers. To address MADM problems, Zhang and Wei [27] extended the concept of VIKOR and TOPSIS methods to develop a methodology for solving MADM problems. Xu and Zhang [19] introduced a novel approach based on TOPSIS and the maximizing deviation method for solving MADM problems. Zhang and Xu [28] extended the TODIM method, which is based on prospect theory to solve MADM problems under hesitant fuzzy environment. Dong et al. [5] proposed an optimization-based two-stage model based on consensus measure in the hesitant linguistic group decision making.

However, although numerous methods for MADM problems have been proposed, they always produce different results. There is no standard to judge which one is more reasonable. In this paper, we propose a comprehensive assessment method based on the existing methods for MADM problems. And we mainly address the following three questions:

1) How to make the existing methods in one dimension?

Different methods for MADM problems were discussed in different dimensions. Therefore, the first aim of this paper is to make different methods in one dimension. Actually, different methods rank alternatives by a set of data. So, we introduce the concept of ranking vector. And then we define the transformation function to make the existing methods in one dimension.

2) Which one is more reasonable?

As mentioned above, a great deal of methods can address MADM problems. Therefore, the challenge naturally becomes how to decide which one is more reasonable. So, the second aim of this paper is to propose reliability degree of different methods. In Section 3, we give two kinds of reliability degree, the intransitive reliability degree and the transitive reliability degree.

3) How to establish a comprehensive assessment method?

Now that there are too many methods for MADM problems, and they can't convince each other. So, the third aim of this paper is to establish a comprehensive assessment method based on the existing methods for MADM problems. In Section 3, we propose a comprehensive assessment method based on the existing methods for MADM problems, in which we use the reliability degree as the weight of different methods.

The remainder of this paper is organized as follows: In Section 2, we describe basic definitions of HFSs and review the traditional techniques used to rank hesitant fuzzy elements. In Section 3, we define the ranking vector and the transformation function to make existing MADM methods in one dimension. In addition, we define the reliability degree as well as discussing its properties. Then a comprehensive assessment method is put forward based on existing MADM methods. In Section 4, we apply the comprehensive assessment method to a numerical example. This paper is concluded in Section 5.

2. Preliminaries

In this section, we describe basic definitions of HFSs. In addition, we review the traditional ranking techniques of HFSs in MADM contexts.

Definition 2.1 ([15, 16])**.** Let X be a nonempty set. Then a hesitant fuzzy set (HFS) on X is in term of a function that when applied to X returns a subset of $[0, 1]$.

For conveniences, the HFS is often expressed simply by mathematical symbol in Xia and Xu [18]

$$H = \{\langle x, h(x)\rangle : x \in X\},$$

where $h(x)$ is a set of values in [0,1], denoting the possible membership degrees of element $x \in X$ to the set H. According to Rodríguez et al. [14] and Xia and Xu [18], we call $h(x)$ the hesitant fuzzy element (HFE).

Example 2.2. If $X = \{x_1, x_2, x_3\}$, $h(x_1) = \{0.3, 0.5\}$, $h(x_2) = \{0.1, 0.4\}$ and $h(x_3) = \{0.6, 0.7, 0.9\}$ are the HFEs of $x_i (i = 1, 2, 3)$ to a set H, respectively. Then H can be considered as a HFS, i.e.,

$$H = \{\langle x_1, \{0.3, 0.5\}\rangle, \langle x_2, \{0.1, 0.4\}\rangle, \langle x_3, \{0.6, 0.7, 0.9\}\rangle\}.$$

Clearly, a HFS H can be seen as a FS, if there is only one element in $h(x)$. In this situation, HFSs include FSs as a special case.

Hereafter, for notional convenience, h stands for HFE $h(x)$ for $x \in X$ and we assume that $|h| = n$, that is, $h = \cup_{\gamma\in h}\{\gamma\} = \{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(n)}\}$, where all elements in h are arranged in increasing order, i.e., $\gamma^{(1)} \leq \gamma^{(2)} \leq \cdots \leq \gamma^{(n)}$.

In what follows, we will describe briefly the existing techniques which are available for ranking HFEs. We group these techniques of ranking HFEs into three major categories: component-wise ordering technique, score function ordering technique, lexicographical ordering technique. Farhadinia [7] showed that a ranking function of HFSs is directly defined by the use of ranking function of its HFEs. Therefore, we mainly discuss here the ranking functions of HFEs and drop the discussion on the corresponding ranking functions of HFSs.

(i) **Component-wise ordering technique** [8]:

Let $h_1 = \cup_{\alpha\in h_1}\{\alpha\} = \{\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(n)}\}$ and $h_2 = \cup_{\beta\in h_2}\{\beta\} = \{\beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(n)}\}$ be two HFEs. The component-wise ordering law of HFEs is defined as:

$$h_1 \preceq h_2 \textit{ if and only if } \alpha^{(i)} \leq \beta^{(i)} \ 1 \leq i \leq n. \tag{2.1}$$

Note that the number of values in different HFEs may be different. We denote $l(h(x))$ as the number of elements in $h(x)$, that is, $l(h(x)) = |h(x)|$. To operate correctly, [20] and [1] gave the following regulation: If $l(h_1(x)) < l(h_2(x))$, then new elements h derived by $h = \xi h^+ + (1-\xi)h^-$ can be appended to $h_1(x)$, where h^+ and h^- are the maximum and minimum elements in $h_1(x)$,respectively. The parameter ξ can be seen as an index of risk. Here, we extent the shorter one by adding the maximum value.

Example 2.3. Continued Example 2.2, compare h_2 and h_3 with component-wise ordering technique. By above regulation, then $h_2 = \{0.1, 0.4, 0.4\}$. Since $0.1 < 0.6$, $0.4 < 0.7$, $0.4 < 0.9$, then $h_2 \preceq h_3$.

(ii) **Score function ordering technique** [7, 10, 18, 20]:

Denote $S(h)$ as score function of a HFE h, the score function ordering law between two HFEs h_1 and h_2 as follows:

If $S(h_1) > S(h_2)$, then $h_1 > h_2$; If $S(h_1) = S(h_2)$, then $h_1 = h_2$.

Let $h = \{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(n)}\}$. Xu and Xia [18] considered the arithmetic-mean as the score function of a HFE h, denoted here by

$$S_{AM}(h) = \frac{1}{n}\sum_{i=1}^{n}\gamma^{(i)}. \tag{2.2}$$

Farhadinia [10] proposed the novel score function $S_N(h)$ as follows:

$$S_N(h) = \frac{\Sigma_{i=1}^{n}\delta(i)\gamma^{(i)}}{\Sigma_{i=1}^{n}\delta(i)}, \tag{2.3}$$

where $\{\delta(1), \delta(2), \cdots, \delta(n)\}$ is a positive-valued monotonic increasing sequence of index i.

Another score function was introduced by Xu and Xia [20] based on the similarity measure between a HFE and the full HFE $\mathbf{1}_h = \{1\}$ by

$$S^{-s}(h) = s(h, \mathbf{1}_h), \tag{2.4}$$

where s is a similarity measure for HFEs. Among the score functions, there are two representative ones defined as:

· The hesitant normalized Hamming similarity score function [20]

$$S^{-s_{hnh}}(h) = 1 - s_{hnh}(h, \mathbf{1}_h) = 1 - \frac{1}{n}\sum_{i=1}^{n}|\gamma_i - 1|. \tag{2.5}$$

· The hesitant normalized Euclidean similarity score function [20]

$$S^{-s_{hne}}(h) = 1 - s_{hne}(h, \mathbf{1}_h) = 1 - \left(\frac{1}{n}\sum_{i=1}^{n}\left(\gamma^{(i)} - 1\right)^2\right)^{\frac{1}{2}}. \tag{2.6}$$

In the sequel, Farhadinia gave a series of score functions in [7] for ranking HFEs. Here we will not list one by one.

(iii) **Lexicographical ordering technique** [2, 9, 11]:

Definition 2.4 ([1])**.** For $X, Y \in R^n$, the lexicographical ordering on the Euclidean space R^n, denoted by $<_{lex}$, is defined by requiring $X = (x_1, x_2, \cdots, x_n) <_{lex} Y = (y_1, y_2, \cdots, y_n)$ if and only if there is $1 \le i \le n$ so that

$$x_j = y_j \text{ holds for } j < i \text{ and } x_i < y_i.$$

Let H be a HFS and $H = \{\langle x, h(x)\rangle : x \in X\}$. We denote the ranking vector associated with h by $R(h)$. The lexicographical ordering comparative law between two HFEs h_1 and h_2 as follows:

If $R(h_1) <_{lex} R(h_2)$, then $h_1 < h_2$; If $R(h_1) =_{lex} R(h_2)$, then $h_1 = h_2$.

Chen et al. [2] considered $R(h) = (S_{AM}(h), 1 - \sigma_c(h))$, where $S_{AM}(h)$ is the arithmetic-mean given by Equation (2.2), and $\sigma_c(h)$ is the deviation function defined by

$$\sigma_c(h) = \left(\frac{1}{n}\sum_{i=1}^{n}\left(\gamma^{(i)} - S_{AM}(h)\right)^2\right)^{\frac{1}{2}}. \tag{2.7}$$

Liao et al. [11] considered $R(h) = (S_{AM}(h), 1 - \sigma_l(h))$, where $\sigma_l(h)$ is the deviation function defined by

$$\sigma_l(h) = \left(\frac{1}{C_n^2} \sum_{i<j=1}^{n} \left(\gamma^{(j)} - \gamma^{(i)} \right)^2 \right)^{\frac{1}{2}}. \tag{2.8}$$

Farhadinia [9] considered $R(h) = (S_{AM}(h), v_\phi(h))$, where $v_\phi(h)$ is the successive deviation function defined by

$$v_\phi(h) = \sum_{i=1}^{n-1} \phi(\gamma^{(i+1)} - \gamma^{(i)}). \tag{2.9}$$

Here, $\phi : [0,1] \to [0,1]$ is an increasing real function with $\phi(0) = 0$.

3. Comprehensive assessment method based on the existing methods for MADM

There are many methods proposed to solve multi-attribute decision making problems. Each method has its advantage together with disadvantage. They can't convince each other. Now we establish a fuzzy comprehensive assessment method based on the results of the existing methods for multi-attribute making problems.

Actually, for any sequence corresponds to a ranking vector.

Example 3.1. In the illustrative example of Ref. [9], $S_{cxx}(HFS_{Y1}) = 0.5167$, $S_{cxx}(HFS_{Y2}) = 0.5275$, $S_{cxx}(HFS_{Y3}) = 0.4937$, $S_{cxx}(HFS_{Y4}) = 0.5708$. Then, $Y_3 \le Y_1 \le Y_2 \le Y_4$. Obviously, the ranking result of x_i is decided by the set of date $(0.5167, 0.5275, 0.4937, 0.5708)$. So we call this data ***ranking vector***.

Example 3.2. In the illustrative example of Ref. [9], the ranking result of Xu and Xia's method (via $S_{xux}^{-d_{hne}}$) is $x_3 > x_2 > x_4 > x_1$; The ranking result of $GHFWG_1(via\ S_{xix})$ is $x_4 > x_1 > x_3 > x_2$.

From Example 3.1 and Example 3.2, we can see, different methods have different ranking results for the same MADM problem. Let $X = \{x_1, x_2, x_3, x_4\}$. The raking result of score function method is $x_4 > x_2 > x_1 > x_3$, and the ranking result of $GHFWG_1(via\ S_{xix})$ is $x_4 > x_1 > x_3 > x_2$. For simplicity, we denote the first ranking result o_1, and the second ranking result o_2, that is, o_1 represents $x_4 > x_2 > x_1 > x_3$ and o_2 represents $x_4 > x_1 > x_3 > x_2$. For each order o_i, each alternative correspond a value. For example, the corresponding value of x_1 under o_1 is 0.5167.

Definition 3.3. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives, $O = \{o_1, o_2, \cdots, o_m\}$ be a set of existing orders, and a_{ij} is a corresponding value of x_j under order o_i. Then, we call vector $A_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ the ranking vector of order o_i.

Example 3.4. Let $X = \{x_1, x_2, x_3, x_4\}$. For the same MADM problem, the ranking vectors of different method are different. The ranking vector of o_1 is $(0.5024, 0.5366, 0.5456, 0.5101)$; The ranking vector of o_2 is $(1.234, 1.462, 1.466, 1.513)$; The ranking vector of o_3 is $(0.0867, 0.0875, 0.0937, 0.0908)$.

From Example 3.4, it is easy to see that the ranking vectors of different methods are established on different dimensions for the same MADM problem. Therefore, we give the following definition so that the data can be unified in one dimension.

Definition 3.5. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives, $O = \{o_1, o_2, \cdots, o_m\}$ be a set of existing orders, and $A_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ is the ranking vector of order o_i. Then we define transformation function: T: $\{A_1, A_2, \cdots, A_m\} \to R^n$ as follows:

$$T(A_i) = (\frac{a_{i1}}{\Sigma_{k=1}^n a_{ik}}, \cdots, \frac{a_{ij}}{\Sigma_{k=1}^n a_{ik}}, \cdots, \frac{a_{in}}{\Sigma_{k=1}^n a_{ik}}) \tag{3.1}$$

Write $T(A_i) = A_i^{'}$.

Remark 3.6. By the transformation function above, we can see, the ranking vector of order o_i transform A_i into $A_i^{'}$. In the process of transformation, it is easy to see the effect of dimension has been eliminated. Besides, it shows a nice property that the proportion of the interval between adjacent alternatives stay the same.

Example 3.7. Continued Example 3.4, by the transformation function, different ranking vectors in different methods can be normalized as follows: The ranking vector of o_1 becomes $(0.2398, 0.2562, 0.2605, 0.2435)$; The ranking vector of o_2 becomes $(0.2174, 0.2577, 0.2583, 0.2666)$; The ranking vector of o_3 becomes $(0.2417, 0.2439, 0.2612, 0.2531)$.

Different methods provide different ranking results. It is worth to study that which ranking result is more reasonable. To propose the reliability degree, we make parameter specification first.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives, and $O = \{o_1, o_2, \cdots, o_m\}$ be a set of existing orders. Assume $o_1 : x_1 > x_3 > x_2 > x_4$, then $2^{o_1}_{adj} = \{x_1 > x_3, x_3 > x_2, x_2 > x_4\}$ and $2^{o_1} = \{x_1 > x_3, x_1 > x_2, x_1 > x_4, x_3 > x_2, x_3 > x_4, x_2 > x_4\}$.

$$F^t(x_i > x_j) = \begin{cases} 1, & x_i > x_j \in 2^{o_t}_{adj}; \\ 0, & otherwise. \end{cases}, \quad F^{t'}(x_i > x_j) = \begin{cases} 1, & x_i > x_j \in 2^{o_t}; \\ 0, & otherwise. \end{cases}$$

For any $x_i > x_j \in 2^{o_k}_{adj}$, $F_k(x_i > x_j) = \sum_{t=1}^{m} F^t$ and for any $x_i > x_j \in 2^{o_k}$, $F_k^{'}(x_i > x_j) = \sum_{t=1}^{m} F^{t'}$.

Remark 3.8. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives, and $O = \{o_1, o_2, \cdots, o_m\}$ be a set of existing orders. Then for any i, j, there must be $F_k(x_i > x_j) \leq F_k^{'}(x_i > x_j)$.

Example 3.9. For the same MADM problem, assume the ranking result of this problem are $o_1 : x_4 < x_3 < x_1 < x_2$, $o_2 : x_3 < x_2 < x_4 < x_1$ and $o_3 : x_4 < x_1 < x_3 < x_2$ respectively. Then $2^{o_1}_{adj} = \{x_4 > x_3, x_3 > x_1, x_1 > x_2\}$, $2^{o_2}_{adj} = \{x_3 > x_2, x_2 > x_4, x_4 > x_1\}$ and $2^{o_3}_{adj} = \{x_4 > x_1, x_1 > x_3, x_3 > x_2\}$. $F^1(x_3 > x_2) = 0$, $F^2(x_3 > x_2) = 1$ and $F^3(x_3 > x_2) = 1$. So $F_2(x_3 > x_2) = 0 + 1 + 1 = 2$. Besides, $2^{o_1} = \{x_4 > x_3, x_4 > x_1, x_4 > x_2, x_3 > x_1, x_3 > x_2, x_1 > x_2\}$, $2^{o_2} = \{x_3 > x_2, x_3 > x_4, x_3 > x_1, x_2 > x_4, x_2 > x_1, x_4 > x_1\}$ and $2^{o_3} = \{x_4 > x_1, x_4 > x_3, x_4 > x_2, x_1 >$

$x_3, x_1 > x_2, x_3 > x_2\}$. $F^{1'}(x_3 > x_2) = 1$, $F^{2'}(x_3 > x_2) = 1$ and $F^{3'}(x_3 > x_2) = 1$. So $F_2^{'}(x_3 > x_2) = 1 + 1 + 1 = 3$.

Definition 3.10. *(Definition of intransitive reliability degree)* Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives, and $O = \{o_1, o_2, \cdots, o_m\}$ is a set of existing orders, then we define the intransitive reliability degree $\mathscr{R}_k(x_i > x_j)$ as follows:

$$\mathscr{R}_k(x_i > x_j) = \frac{\sum\limits_{i,j} F_k(x_i > x_j)}{\sum\limits_{k=1}^{m} \sum\limits_{i,j} F_k(x_i > x_j)}. \tag{3.2}$$

Proposition 3.11. *(Properties of the intransitive reliability degree $\mathscr{R}_k(x_i > x_j)$) Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives, and $O = \{o_1, o_2, \cdots, o_m\}$ is a set of existing orders. Then the intransitive reliability degree $\mathscr{R}_k(x_i > x_j)$ has the following properties:*

(1) $0 \leq \mathscr{R}_k(x_i > x_j) \leq 1$,
(2) $\sum\limits_{k=1}^{m} \mathscr{R}_k(x_i > x_j) = 1$,
(3) *if* $\mathscr{R}_k(x_i > x_j) = 1$, *then* $m = k = 1$.

Proof. The proof is straightforward. □

Definition 3.12. (Definition of transitive reliability degree) Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives, and $O = \{o_1, o_2, \cdots, o_m\}$ is a set of existing orders, then we define the transitive reliability degree $\mathscr{R}_k^{'}(x_i > x_j)$ as follows:

$$\mathscr{R}_k^{'}(x_i > x_j) = \frac{\sum\limits_{i,j} F_k^{'}(x_i > x_j)}{\sum\limits_{k=1}^{m} \sum\limits_{i,j} F_k^{'}(x_i > x_j)}. \tag{3.3}$$

Proposition 3.13. *(Properties of the transitive reliability degree $\mathscr{R}_k^{'}(x_i > x_j)$) Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives, and $O = \{o_1, o_2, \cdots, o_m\}$ is a set of existing orders. Then the transitive reliability degree $\mathscr{R}_k^{'}(x_i > x_j)$ has the following properties:*

(1) $0 \leq \mathscr{R}_k^{'}(x_i > x_j) \leq 1$,
(2) $\sum\limits_{k=1}^{m} \mathscr{R}_k^{'}(x_i > x_j) = 1$,
(3) *if* $\mathscr{R}_k^{'}(x_i > x_j) = 1$, *then* $m = k = 1$.

Proof. The proof is straightforward. □

Remark 3.14. The reliability degree shows that one ranking result obtains recognition degree among the others existing ranking results. The more reliability degree is, the bigger weight should be.

Next, we develop a new multi-attribute decision making method by assessing comprehensively the existing methods.

Comprehensive assessment method:

Step 1. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a discrete set of alternatives, and $O = \{o_1, o_2, \cdots, o_m\}$ be a set of existing orders about X, $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T$ be the weight vector of the orders with $\sum_{i=1}^{m} \omega_i = 1$ and $\omega_i \geq 0 (i = 1, 2, \cdots, m)$. $A_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ is the ranking vector of order o_i. We utilize transformation function to transform A_i into $A_i^{'}$. Then we unified different orders in one dimension.

Step 2. To make our method have larger feasibility and wider practicability, the weights of the orders are also properly incorporated into a multi-attribute decision making problems. If the weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T$ of all orders is known, then go to Step 3; Otherwise, the weight of each order needs to be accounted for. So we use the transitive reliability degree $\mathscr{R}_k^{'}(x_i > x_j)$ as the weight of o_k, i.e., $\omega_k = \mathscr{R}_k^{'}(x_i > x_j)$.

Step 3. Obtain a new hesitant fuzzy set. Based on Step 1 and Step 2, we can get a new hesitant fuzzy set $\{\langle x_i, h(x_i)\rangle : x_i \in X\}$ where $h(x_i) = \{\omega_k a_{ki}^{'} : k = 1, 2, \cdots, m\}$.

Step 4. Reorder x_i by techniques for ranking HFEs.

Comprehensive assessment method is based the existing method. We use scientific approach to obtain the weights of different orders. The core of the method is to select more recognizable alternative in each position. So, Comprehensive assessment method is more accurate. Besides, Comprehensive assessment method can be used to test whether a method is reasonable. However, it is difficult to choose all existing MADM methods.

4. Numerical comparisons

To illustrate the validity of the comprehensive assessment method, we apply an example to the method.

Example 4.1. (adapted from [21]) Due to the limited technology and capital, an enterprise itself may be unable to build the cloud platform and tries to seek a cloud service to realize its CRM. After the market research and preliminary screening, there are four potential cloud services for further evaluation, including SAP Sales on Demand (x_1), Salesforce Sales Cloud (x_2), Microsoft Dynamic CRM (x_3) and Oracle Cloud CRM (x_4). Five experts are invited to evaluate these cloud services on four indicators (attributes), including performance (a_1), payment (a_2), reputation (a_3) and security (a_4). The attribute weight is given by decision maker as $\omega = (0.2, 0.3, 0.15, 0.35)^T$. The four candidates x_i(i = 1,2,3,4) are to be evaluated in anonymity with hesitant fuzzy information by the decision makers under the above four attributes $a_j (j = 1, 2, 3, 4)$, as listed in Table 1.

Table 1. Hesitant fuzzy decision matrix

	a_1	a_2	a_3	a_4
x_1	$\{0.2, 0.4, 0.7\}$	$\{0.2, 0, 6, 0.8\}$	$\{0.2, 0.3, 0.6, 0.7, 0.9\}$	$\{0.3, 0.4, 0.5, 0.7, 0.8\}$
x_2	$\{0.2, 0.4, 0.7, 0.9\}$	$\{0.1, 0.2, 0.4, 0.5\}$	$\{0.3, 0.4, 0.6, 0.9\}$	$\{0.5, 0.6, 0.8, 0.9\}$
x_3	$\{0.3, 0.5, 0.6, 0.7\}$	$\{0.2, 0.4, 0.5, 0.6\}$	$\{0.3, 0.5, 0.7, 0.8\}$	$\{0.1, 0.5, 0.6, 0.8\}$
x_4	$\{0.3, 0.5, 0.6\}$	$\{0.2, 0.4\}$	$\{0.5, 0.6, 0.7\}$	$\{0.8, 0.9\}$

To get the comprehensive ranking result, the following steps are obtained as comprehensive assessment method:

Step 1. To solve this MADM problem with existing methods, normalized ranking vectors and ranking results based on existing methods are listed in Table 2.

Table 2. Different methods' ranking results

Techniques	*Normalized ranking vectors*	*Ranking results*
S_{AM}	$(0.2450, 0.2502, 0.2341, 0.2707)$	$o_1 : x_4 \succ x_2 \succ x_1 \succ x_3$
$S^{-s_{hnh}}$	$(0.2354, 0.2533, 0.2371, 0.2742)$	$o_2 : x_4 \succ x_2 \succ x_3 \succ x_1$
$S^{-s_{hne}}$	$(0.2612, 0.2432, 0.2385, 0.2571)$	$o_3 : x_1 \succ x_4 \succ x_2 \succ x_3$
$GHFWA_1$	$(0.2413, 0.2574, 0.2218, 0.2795)$	$o_4 : x_4 \succ x_2 \succ x_1 \succ x_3$
$GHFWG_1$	$(0.2491, 0.2409, 0.2428, 0.2672)$	$o_5 : x_4 \succ x_1 \succ x_3 \succ x_2$

Step 2. Calculate the weights of different methods using the transitive reliability degree as follows:

$$F^{'}(x_1 > x_2) = 2,\ F^{'}(x_1 > x_3) = 4,\ F^{'}(x_1 > x_4) = 1,\ F^{'}(x_2 > x_1) = 3,$$
$$F^{'}(x_2 > x_3) = 4,\ F^{'}(x_2 > x_4) = 0,\ F^{'}(x_3 > x_1) = 1,\ F^{'}(x_3 > x_2) = 1,$$
$$F^{'}(x_3 > x_4) = 0,\ F^{'}(x_4 > x_1) = 4,\ F^{'}(x_4 > x_2) = 5,\ F^{'}(x_4 > x_3) = 5.$$

Then $\sum\limits_{i,j} F_1^{'}(x_i > x_j) = F^{'}(x_4 > x_2) + F^{'}(x_4 > x_1) + F^{'}(x_4 > x_3)$

$$+F^{'}(x_2 > x_1) + F^{'}(x_2 > x_3) + F^{'}(x_1 > x_3)$$
$$= 5 + 4 + 5 + 3 + 4 + 4 = 25.$$

Similarly, $\sum\limits_{i,j} F_2^{'}(x_i > x_j) = 22$, $\sum\limits_{i,j} F_3^{'}(x_i > x_j) = 21$,

$\sum\limits_{i,j} F_4^{'}(x_i > x_j) = 25$, $\sum\limits_{i,j} F_5^{'}(x_i > x_j) = 21$.

Thus, $\mathscr{R}_1^{'}(x_i > x_j) = \frac{25}{114}$, $\mathscr{R}_2^{'}(x_i > x_j) = \frac{22}{114}$, $\mathscr{R}_3^{'}(x_i > x_j) = \frac{21}{114}$,

$\mathscr{R}_4^{'}(x_i > x_j) = \frac{21}{114}$, $\mathscr{R}_5^{'}(x_i > x_j) = \frac{21}{114}$.

So, the weights of the orders is $(\frac{25}{114}, \frac{22}{114}, \frac{21}{114}, \frac{25}{114}, \frac{21}{114})$, respectively.

Step 3. The new hesitant fuzzy set is obtained as $\{\langle x_i, h(x_i)\rangle : x_i \in X\}$, where

$h(x_1) = (0.0537, 0.0454, 0.0481, 0.0529, 0.0459)$,
$h(x_2) = (0.0549, 0.0489, 0.0448, 0.0564, 0.0444)$,
$h(x_3) = (0.0513, 0.0458, 0.0439, 0.0486, 0.0447)$,
$h(x_4) = (0.0594, 0.0529, 0.0474, 0.0613, 0.0492)$.

Step 4. Rank the new HFEs by score function.

$S(h(x_1)) = 0.0492$, $S(h(x_2)) = 0.0499$, $S(h(x_3)) = 0.0469$, $S(h(x_4)) = 0.0540$.

Then we can obtain a new order $x_4 \succ x_2 \succ x_1 \succ x_3$.

From the final result, it is accordant completely with score function and $GHFWA_1$ ranking results. Therefore, in this example, score function and $GHFWA_1$ method have higher accuracy relatively. The reliability degrees of score function and $GHFWA_1$ method are also highest, which keep consistent in final result.

5. Conclusion

Since HFS is introduced, many methods for MADM problems have been proposed. However, there is no unified standard. It is necessary to present a comprehensive assessment method based on the existing MADM methods under the environment of HFSs. Firstly, we proposed the ranking vector and the transformation function to make existing methods in one dimension. Subsequently, we introduced the reliability degree and discussed its properties. Finally, we presented a comprehensive assessment method in which we use the transitive reliability degree $\mathscr{R}_k^{'}(x_i > x_j)$ as the weight of o_k.

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