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On $L-$generalized filters and nets

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Abstract. In this paper, we consider the $L-$generalized filters and $L-$generalized nets on the non empty universal set X to be a natural generalization of the crisp generalized filters and the crisp generalized nets respectively. In each the crisp and $L-$generalized cases, the extended and restricted filters (nets) will be obtained respectively. The construction of convergence to limit points of the crisp generalized and $L-$generalized filters (nets) will be studied respectively. We also show that any crisp generalized net on the set X associates a unique $L-$generalized net on X, and in general the converse is not true. In addition to that we also show that for any given $L-$generalized filter there exists the $L-$generalized net, and the vice-versa is true. Finally, the relations between the convergence of the $L-$generalized filters and the $L-$generalized nets will be outlined. Moreover, to support these concepts and relations, the construction of some examples in all sections of paper will be studied.

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1. Introduction

Zadeh in his famous paper [23] introduced the concept of a fuzzy set A of the non-empty universal set X as a mapping $A : X \longrightarrow [0, 1]$. Chang in [4] had first tried to fuzzify the concept of a topology on a non-empty set X as a subfamily of $[0, 1]^X$, satisfying the same axioms of ordinary topology. Then Goguen in [10] replaced the real interval $[0, 1]$ with an arbitrary complete infinitely distributive lattice L, and introduced the concept of $L-$topology. After that Höhle in [11] introduced the concept of an $L-$fuzzy topology as a mapping $\tau : 2^X \longrightarrow L$, before replacing its name with a fuzzifying topology, which was introduced by Ying Mingsheng in [22]. Then Alexander Šostak in [21] introduced the concept of $L-$fuzzy topology as a mapping $\tau : L^X \longrightarrow L$ with considering $L = [0, 1]$ in the same time which

Tomasz Kubiak in [13, 14], introduced the concept of $(L, M)-$fuzzy topology as a mapping $\tau : L^X \longrightarrow M$, where L and M being complete lattices. Then in [20], Ramadan introduced the concept of smooth topological space as a mapping $\tau : [0,1]^X \longrightarrow [0,1]$. *Á* Cs*á*sz*á*r in [6] introduced the concept of generalized topology on the set X as a subfamily τ of 2^X, containing ϕ and closed under arbitrary union. Many of concepts in mathematics are defined or characterized in terms of a limit. In topological spaces the mathematicians introduced the limit concept in a most general way for filters and nets. The definition and properties of filters can be found in [1, 2, 5]. Filters in topological spaces help in studying many properties, as the closure of a set A can be characterized by using convergent filters containing A, and the continuity of a function on topological spaces can be characterized using convergent filters. Lowen in [15] introduced the notions of prefilter and fuzzy uniform space as extensions of filter and uniform space. In [7, 8, 9, 12, 19], the notion of a fuzzy filter is introduced and studied for lattice $I = [0,1]$. The concepts of fuzzy nets and fuzzification of filters were studied by Muthukumari in [16] and [17], respectively. After *Á*. Cs*á*sz*á*r in [6], introduced the concept of generalized topology on the set X, the need for studying the limit concept in generalized topology was studied in [18] by introducing generalized filters and in [3] by introducing generalized nets. The aim of this paper is to consider the $L-$generalized filters and the $L-$generalized nets on the non empty universal set X to be a natural generalization of the crisp generalized filters and the crisp generalized nets respectively. Moreover, in each the crisp and $L-$generalized cases, the extended and restricted generalized filters (nets) will be obtained. Also, the construction of convergence to limit points of the crisp generalized and $L-$generalized filters (nets) will be studied. We also show that any crisp generalized net on the set X associates a unique $L-$generalized net on X, and in general the converse is not true. Then, we show that for any given $L-$generalized filter there exists the $L-$generalized net, and the vice-versa is true. And then, the relations between the convergence of the $L-$generalized filters and the $L-$generalized nets will be outlined.

This paper is organized as follows. In section 2, some basic concepts and notions which will be used throughout this paper are listed. In section 3, some basic notions of ordinary generalized filters are recalled and the definition of their convergence are presented. In section 4, some basic notions of an $L-$ generalized filters are introduced and the definition of their convergence are also presented. In section 5, some basic notions of an $L-$ generalized nets are introduced and the relations between the convergence of crisp generalized nets and $L-$generalized nets are studied. Finally, in section 6, the relations between the convergence of $L-$generalized filters and $L-$generalized nets are studied.

2. Preliminaries

Let X be a given non-empty universal set, and let L be a given lattice. Then the family L^X is the set of all $L-$fuzzy subsets of X. Moreover, the family 2^X is the set of all ordinary subsets of X. Denote the smallest element of L by 0_L and the greatest element of L by 1_L. Also denote the smallest $L-$fuzzy subset in L^X by $\mathbf{0}_X$ and the greatest $L-$fuzzy subset in L^X by $\mathbf{1}_X$.

The support of any element A in L^X is defined as $supp(A) = \{x \in X : A(x) \neq 0_L\}$.

The fuzzy point $p(x_0, r)$ in X, where $x_0 \in X, r \in L - \{0_L\}$ is defined as:

$$p(x_0, r)(x) = \begin{cases} r, & x = x_0, \\ 0_L, & x \neq x_0 \end{cases}$$

The fuzzy point $p(x_0, r)$ is contained in A, denoted by $A \in L^X$, if $A(x_0) \geq r$.

Definition 2.1 ([18]). Let X be a non-empty universal set. Then we call $\mathbb{F} \subseteq 2^X$ a generalized filter on X, if it satisfies the following conditions:
(i) $\phi \notin \mathbb{F}$,
(ii) $\forall A \in \mathbb{F}, A \subseteq B \Rightarrow B \in \mathbb{F}$.
A generalized filter on X is called a filter on X, if $\forall A, B \in \mathbb{F}, A \cap B \in \mathbb{F}$.

Definition 2.2 ([18]). Let X be a non-empty universal set. Then we call $\mathbb{B} \subset 2^X$ a generalized filter base of the generalized filter $\mathbb{F}$ on X, if $\phi \notin \mathbb{B}$ and for all $B \in \mathbb{F}$: $\exists A \in \mathbb{B}, A \subseteq B$.

It is clear that any non-empty subfamily $\mathbb{B} \subseteq 2^X$ such that $\phi \notin \mathbb{B}$ generated a generalized filter $\mathbb{F}_{\mathbb{B}} =< \mathbb{B} >= \{B \subseteq X : \exists A \in \mathbb{B}, A \subseteq B\}$.

Definition 2.3 ([6]). Let X be a non-empty universal set. Then we call $\tau \subseteq 2^X$ a generalized topology on X, if it satisfies the following conditions:
(i) $\phi \in \tau$,
(ii) $\forall\ A_i \subseteq X, i \in \Delta$, if $A_i \in \tau$, then $\bigcup_{i\in\Delta} A_i \in \tau$.

Definition 2.4 ([6]). Let (X, τ) be a generalized topological space and let $x \in X$. Then the neighbourhood generalized filter of x, denoted by $\mathbb{N}_x$, is defined by:

$$\mathbb{N}_x =< \{B \subseteq X : B \in \tau, x \in B\} > .$$

Definition 2.5 ([18]). The generalized filter $\mathbb{F}$ in the generalized topological space (X, τ) converges to a point $x \in X$, if $\mathbb{N}_x \subseteq \mathbb{F}$ and we write $\mathbb{F} \to x$.

A generalized filter base $\mathbb{B}$ converges to x, if $< \mathbb{B} >\to x$.

Definition 2.6 ([3]). Let D be a partially ordered set under the relation $\leq$ and let X be a non-empty set. Then any mapping $\mathcal{A} : D \longrightarrow X$ is called a generalized net or a crisp generalized net on the set X.

If D is a directed set under the relation $\leq$, then $\mathcal{A} : D \longrightarrow X$ is called a net or a crisp net on the set X.

Definition 2.7 ([3]). Let (X, τ) be a generalized topological space. Then the generalized crisp net $\mathcal{A} : D \longrightarrow X$ on X converges to $x_0 \in X$, if for each neighbouhood U of x_0 in (X, τ), there exists $\lambda_0 \in D$ such that $\mathcal{A}(\lambda) \in U$, for all $\lambda \geq \lambda_0$.

Let X, Y be two non-empty ordinary sets, where $Y \subseteq X$ and L be a complete and completely distributive lattice.

Definition 2.8. For every $U \in L^X$, the restriction $U_{\downarrow Y}$ of U on L^Y is defined by:

$$U_{\downarrow Y}(x) = U(x); \quad x \in Y$$

Definition 2.9. For every $A \in L^Y$ the extension $A_{\uparrow X}$ of A on L^X is defined by:

$$A_{\uparrow X}(x) = \begin{cases} A(x), & x \in Y, \\ 0_L, & x \in X - Y. \end{cases}$$

One can show that for every $\phi \neq Y \subseteq X$, every fuzzy point $p(y_0, r)$ in Y is a fuzzy point in X, since $p(y_0, r)_{\uparrow X} = p(y_0, r)$. Moreover, every fuzzy point $p(x_0, r)$ in X is a fuzzy point in Y if $x_0 \in Y$, since

$$p(x_0, r)_{\downarrow Y} = \begin{cases} p(x_0, r), & x_0 \in Y, \\ \mathbf{0}_Y, & x_0 \in X - Y. \end{cases}$$

Proposition 2.10. *Let $V, U \in L^X, A, B \in L^Y$ and $\phi \neq Y \subseteq X$, then we have the following operations:*

(1) $V \leq U \Rightarrow V_{\downarrow Y} \leq U_{\downarrow Y}, A \leq B \Rightarrow A_{\uparrow X} \leq B_{\uparrow X}$,

(2) $(\mathbf{0}_X)_{\downarrow Y} = \mathbf{0}_Y, (\mathbf{1}_X)_{\downarrow Y} = \mathbf{1}_Y, (\mathbf{0}_Y)_{\uparrow X} = \mathbf{0}_X$ *and* $(\mathbf{1}_Y)_{\uparrow X} = \chi_Y$, *where* χ_Y *is the characteristic function of the set* Y,

(3) $U_{\downarrow Y} \wedge V_{\downarrow Y} = (U \wedge V)_{\downarrow Y}, U_{\downarrow Y} \vee V_{\downarrow Y} = (U \vee V)_{\downarrow Y}$,

(4) $A_{\uparrow X} \wedge B_{\uparrow X} = (A \wedge B)_{\uparrow X}, A_{\uparrow X} \vee B_{\uparrow X} = (A \vee B)_{\uparrow X}$,

(5) $(V_{\downarrow Y})_{\uparrow X} \leq V, (A_{\uparrow X})_{\downarrow Y} = A$,

(6) $(V_{\downarrow Y}) = \mathbf{0}_Y$ *if and only if* $supp(V) \cap Y = \phi$.

The generalization of (3) and (4) of the above proposition is true, since if

$\{U_i : i \in \Delta\} \subseteq L^X, \{A_i : i \in \Delta\} \subseteq L^Y$, then

(3)' $\bigvee_{i \in \Delta}((U_i)_{\downarrow Y}) = (\bigvee_{i \in \Delta} U_i)_{\downarrow Y}, \ \bigwedge_{i \in \Delta}((U_i)_{\downarrow Y}) = (\bigwedge_{i \in \Delta} U_i)_{\downarrow Y}$,

(4)' $\bigvee_{i \in \Delta}((A_i)_{\uparrow X}) = (\bigvee_{i \in \Delta} A_i)_{\uparrow X}, \ \bigwedge_{i \in \Delta}((A_i)_{\uparrow X}) = (\bigwedge_{i \in \Delta} A_i)_{\uparrow X}$.

From now to on in this article, in some times, we use the lattice of the form

$$P^*([0,1]) = \{M \subset [0,1] : 0 \in M\}.$$

The algebraic structure $(P^*([0,1]), \cup, \cap, ')$ forms a complemented, completely distributive and complete lattice with $0_{P^*([0,1))} = \{0\}$ being the smallest element and $1_{P^*([0,1])} = [0,1]$ being the greatest element.

The complementary operation is defined by:

$$\prime : P^*(L) \longrightarrow P^*(L),$$

where $M' = ([0,1] - M) \bigcup \{0\}$.

3. Ordinary generalized filters

It is known that for every topological space (X, τ), and $Y \subseteq X$, there exists a generalized subspace (Y, τ_Y), where τ_Y is defined by: $\tau_Y = \{A \cap Y : A \in \tau\}$.

Now, one can show that there exists another generalized topological subspace $(Y, \tau_{\downarrow Y})$, weaker than (Y, τ_Y), and is defined by:

$$\tau_{\downarrow Y} = \{Y, A \subseteq Y : A \in \tau\}.$$

Conversely, for every generalized topological space $(Y, \omega), Y \subseteq X$, there exists a super generalized space $(X, \omega_{\uparrow X})$, and $\omega_{\uparrow X}$ is defined by:

$$\omega_{\uparrow X} = \{A \subseteq X : A \cap Y \in \omega\}.$$

Theorem 3.1. *Let X, Y be given two non-empty ordinary sets, where $Y \subset X$. Then*

(1) *every generalized filter $\mathbb{F}$ on the set X restricted the generalized filter $\mathbb{F}_{\downarrow Y} = \{Y, A \subseteq Y : A \in \mathbb{F}\}$ on the set Y,*

(2) *every generalized filter $\mathbb{G}$ on the set Y extended the generalized filter $\mathbb{G}_{\uparrow X} =< \mathbb{G} >$ on the set X.*

Proof. (1) It is clear that $\phi \notin \mathbb{F}_{\downarrow Y}$. If $A \subseteq B \subseteq Y$ and $A \in \mathbb{F}_{\downarrow Y}$, then $A \in \mathbb{F}$. and $B \in \mathbb{F}$. Thus $B \in \mathbb{F}_{\downarrow Y}$.

(2) The proof is clear, since $\mathbb{G}$ is a generalized filter. □

Theorem 3.2. *Let (X, τ) and (Y, ω) be two generalized topological spaces, and let $Y \subseteq X$. Then*

(1) *every neighbourhood generalized filter $\mathbb{N}_x$ in the space (X, τ), restricted a neighbourhood generalized filter $(\mathbb{N}_x)_{\downarrow Y} = \{Y, A \subseteq Y : A \in \mathbb{N}_x\}$ in the space $(Y, \tau_{\downarrow Y})$, whenever $x \in Y$,*

(2) *every generalized neighbourhood filter $\mathbb{N}_y$ in the space (Y, ω), extended a neighbourhood generalized filter $(\mathbb{N}_y)_{\uparrow X} = \{B \subseteq X : B \cap Y \in \mathbb{N}_y\}$ in the space $(X, \omega_{\uparrow X})$.*

Proof. (1) Let $B \subseteq Y, B \in (\mathbb{N}_x)_{\downarrow Y}$. Then there exists $C \in \tau_{\downarrow Y}$ such that $x \in C \subseteq B$. It is follows that $C \in \mathbb{N}_x$ and $B \in \mathbb{N}_x$. Thus, $B \in \{Y, A \subseteq Y : A \in \mathbb{N}_x\}$, and $(\mathbb{N}_x)_{\downarrow Y} \subseteq \{Y, A \subseteq Y : A \in \mathbb{N}_x\}$.

Now, let $B \subseteq Y, B \in \{Y, A \subseteq Y : A \in \mathbb{N}_x\}$. Then $B \in \mathbb{N}_x$ and there exists $O \in \tau such that x \in O \subseteq B$. Since $O \subseteq Y$, $O \in \tau_{\downarrow Y}$ and $B \in (\mathbb{N}_x)_{\downarrow Y}$. Thus, $\{Y, A \subseteq Y : A \in \mathbb{N}_x\} \subseteq (\mathbb{N}_x)_{\downarrow Y}$. So the proof of (1) is obtained.

(2) Let $A \subseteq X, A \in (\mathbb{N}_y)_{\uparrow X}$. Then there exists $H \in \omega_{\uparrow X}$ such that $y \in H \subseteq A$. It is follows that $y \in H \cap Y \subseteq A \cap Y$. Thus $A \cap Y \in N_y$. So

$$(\mathbb{N}_y)_{\uparrow X} \subseteq \{B \subseteq X : B \cap Y \in \mathbb{N}_y\}.$$

Now, let $A \subsetneq X$ and $A \in \{B \subseteq X : B \cap Y \in \mathbb{N}_y\}$. Then $A \cap Y \in N_y$. Thus there exists $K \in \omega, y \in K \subseteq A \cap Y$. Since $K \subseteq Y$, $K \in \omega_{\uparrow X}$. So $K \in (\mathbb{N}_y)_{\uparrow X}$. Hence, $A \in (\mathbb{N}_y)_{\uparrow X}$ and $\{B \subseteq X : B \cap Y \in \mathbb{N}_y\} \subseteq (\mathbb{N}_y)_{\uparrow X}$. Therefore the proof of (2) is obtained. □

Theorem 3.3. *Let $\mathbb{F}$ and $\mathbb{G}$ be the two ordinary generalized filters on two the generalized topological spaces (X, τ) and (Y, ω), respectively, and let $\phi \neq Y \subseteq X$. Then*

(1) *$\mathbb{F} \to x$ in $(X, \tau) \Rightarrow \mathbb{F}_{\downarrow Y} \to x$ in $(Y, \tau_{\downarrow Y})$, where $x \in Y$,*

(2) *$\mathbb{G} \to y$ in $(Y, \omega) \Rightarrow \mathbb{G}_{\uparrow X} \to y$ in $(X, \omega_{\uparrow X})$.*

Proof. (1) Let $\mathbb{F} \to x$ in (X, τ) and let $N \in (\mathbb{N}_x)_{\downarrow Y}$. Then there exists $O \in \tau_{\downarrow Y}$ and $x \in O \subseteq N$. Thus $O \in \tau$ and $O \in \mathbb{N}_x \subseteq \mathbb{F}$. So $O \in \mathbb{F}_{\downarrow Y}$ and thus $N \in \mathbb{F}_{\downarrow Y}$, which implies that $(\mathbb{N}_x)_{\downarrow Y} \subseteq \mathbb{F}_{\downarrow Y}$. Hence $\mathbb{F}_{\downarrow Y} \to x$.

(2) Let $\mathbb{G} \to y$ in (Y, ω) and let $M \in (\mathbb{N}_y)_{\uparrow X}$. Then there exists $H \in \omega_{\uparrow X}$ and $y \in H \subseteq M$. Thus $y \in H \cap Y \subseteq M \cap Y$. So $M \cap Y \in \mathbb{N}_y$ and thus $M \cap Y \in \mathbb{G}$, which implies that $M \in \mathbb{G}_{\uparrow X}$ and $(\mathbb{N}_y)_{\uparrow X} \subseteq \mathbb{G}_{\uparrow X}$. Hence $\mathbb{G}_{\uparrow X} \to y$ in $(X, \omega_{\uparrow X})$. □

4. $L-$generalized filters

Definition 4.1. Let X be a non-empt universal set. Then we call a non-zero function $\mathfrak{F} : 2^X \longrightarrow L$ an $L-$generalized filter on X, if

(i) $\mathfrak{F}(\phi) = 0_L$,

(ii) $\forall A, B \subseteq X, A \subseteq B \Rightarrow \mathfrak{F}(B) \geq \mathfrak{F}(A)$.

An $L-$generalized filter $\mathfrak{F}$ is called $L-$filter, if $\forall A, B \subseteq X, \mathfrak{F}(A \cap B) \geq \mathfrak{F}(A) \wedge \mathfrak{F}(B)$.

In general, an $L-$generalized filter may not be an $L-$filter as the following example.

Example 4.2. Let the set $X = \{a, b, c\}$ and define a function $\mathfrak{F} : 2^X \longrightarrow P^*([0,1])$ by:

$$\mathfrak{F}(A) = \begin{cases} [0,1]; & A \in \{X, \{b\}, \{a,c\}, \{b,c\}, \{a,b\}\}, \\ \{0\}; & A \in \{\phi, \{c\}, \{a\}\}. \end{cases}$$

Then $\mathfrak{F}$ is a $P^*([0,1])-$generalised filter, but not a $P^*([0,1])-$filter, since

$$\mathfrak{F}(\{a,c\} \cap \{b,c\}) = \mathfrak{F}(\{c\} = \{0\} \text{ and } \mathfrak{F}(\{a,c\}) \cap \mathfrak{F}(\{b,c\}) = [0,1].$$

Definition 4.3. . Let X be a non-empty universal set. Then we call a non-zero function $\sigma : 2^X \longrightarrow L$ an $L-$generalized topology on X, if

(i) $\sigma(\phi) = 1_L$,

(ii) $\forall A_i \subseteq X, \sigma(\bigcup_{i\in\Delta} A_i) \geq \bigwedge_{i\in\Delta} \sigma(A_i)$.

An $L-$generalized topology σ is called an $L-$topology or an $L-$fuzzifying topology on X, if

$$\forall A, B \subseteq X, \sigma(A \cap B) \geq \sigma(A) \wedge \sigma(B).$$

Definition 4.4. Let (X, σ) be an $L-$generalized topological space and let $x \in X$. Then the neighbourhood $L-$generalized filter $\mathrm{N}_x : 2^X \longrightarrow L$ is defined by: for all $A \subseteq X$,

$$\mathrm{N}_x(A) = \begin{cases} \bigvee_B \sigma(B); & x \in B \subset A, \\ 0_L; & x \notin A. \end{cases}$$

Definition 4.5. The $L-$generalized filter $\mathfrak{F}$ in the $L-$generalized topological space (X, σ) converges to a point $x \in X$, if $\mathrm{N}_x(A) \leq \mathfrak{F}(A)$, for all $A \subseteq X$ and we write $\mathbb{F} \to x$. If $\mathfrak{F}$ does not converge to a point $x \in X$, then we write $\mathfrak{F} \nrightarrow x$.

Example 4.6. Let the set $X = \{a, b, c\}$ and define the $P^*([0,1])-$generalized topology $\sigma : 2^X \longrightarrow P^*([0,1])$ as follows: for each $A \in 2^X$,

$$\sigma(A) = \begin{cases} [0,1]; & A \in \{X, \phi, \{a\}, \{a,b\}\}, \\ \{0\}; & otherwise. \end{cases}$$

Also, define the $P^*([0,1])-$generalized filter $\mathfrak{F} : 2^X \longrightarrow P^*([0,1])$: for each $A \in 2^X$,

$$\mathfrak{F}(A) = \begin{cases} [0,1]; & A \in \{X, \{a,b\}, \{a,c\}\}, \\ \{0\}; & otherwise. \end{cases}$$

Then,

$$\mathrm{N}_a(A) = \begin{cases} [0,1]; & A \in \{X, \{a\}, \{a,b\}, \{a,c\}\}, \\ \{0\}; & otherwise, \end{cases}$$

$$\mathrm{N}_b(A) = \begin{cases} [0,1]; & A \in \{X, \{a,b\}\}, \\ \{0\}; & otherwise, \end{cases}$$

$$\mathrm{N}_c(A) = \begin{cases} [0,1]; & A = X, \\ \{0\}; & otherwise. \end{cases}$$

Thus, $\mathfrak{F} \to b$, $\mathfrak{F} \to c$ and $\mathfrak{F} \nrightarrow a$.

Remark 4.7. The above example shows that in general the limit of an $L-$generalized filter need not be unique.

The following example shows that the limit of an $L-$generalized filter may be unique.

Example 4.8. Let the set $X = \{a, b\}$, and define the $P^*([0,1])-$generalized topology

$$\sigma : 2^X \longrightarrow P^*([0,1]); \sigma(A) = [0,1], \text{ for all } A \subseteq X.$$

Also, define the $P^*([0,1])-$generalized filter $\mathfrak{F} : 2^X \longrightarrow P^*([0,1])$:

$$\mathfrak{F}(A) = \begin{cases} [0,1]; & A \in \{X, \{a\}\}, \\ \{0\}; & A \in \{\phi, \{b\}\}, \end{cases}$$

Then,

$$\mathrm{N}_a(A) = \begin{cases} [0,1]; & A \in \{X, \{a\}\}, \\ \{0\}; & A \in \{\phi, \{b\}\}. \end{cases}$$

$$\mathrm{N}_b(A) = \begin{cases} [0,1]; & A \in \{X, \{b\}\}, \\ \{0\}; & A \in \{\phi, \{a\}\}. \end{cases}$$

Thus, $\mathfrak{F} \to a$ and $\mathfrak{F} \nrightarrow b$.

Theorem 4.9. *Let X, Y be given non-empty ordinary sets, where $Y \subseteq X$. Then*

(1) *every an $L-$generalized topology $\sigma : 2^X \longrightarrow L$ on the set X restricted an $L-$generalized topology $\sigma_{\downarrow Y}$ on the set Y, defined by: for every $A \subseteq Y$,*

$$\sigma_{\downarrow Y}(A) = \sigma(A),$$

(2) *every an $L-$generalized topology $\ell : 2^Y \longrightarrow L$ on the set Y extended an $L-$generalized topology $\ell_{\uparrow X}$ on the set X, defined by: for every $V \subseteq X$,*

$$\ell_{\uparrow X}(V) = \ell(V \cap Y).$$

Proof. (1) $\sigma_{\downarrow Y}(\phi) = \sigma(\phi) = 1_L$, and for all $A_i \subseteq Y, i \in \Delta$, then

$$\sigma_{\downarrow Y}(\bigcup_{i\in\Delta} A_i) = \sigma(\bigcup_{i\in\Delta} A_i) \geq \bigwedge_{i\in\Delta} \sigma(A_i) = \bigwedge_{i\in\Delta} \sigma_{\downarrow Y}(A_i).$$

(2) $\ell_{\uparrow X}(\phi) = \ell(\phi) = 1_L$, and for all $B_i \subseteq X, i \in \Delta$, then

$$\ell_{\uparrow X}(\bigcup_{i\in\Delta} B_i) = \ell((\bigcup_{i\in\Delta} B_i) \cap Y) = \ell(\bigcup_{i\in\Delta}(B_i \cap Y) \geq \bigwedge_{i\in\Delta} \ell(B_i \cap Y) = \bigwedge_{i\in\Delta} \ell_{\uparrow X}(B_i).$$

□

The proof of the following theorem is obtained easily.

Theorem 4.10. *Let X, Y be given ordinary sets, where $Y \subset X$. Then*

(1) *every $L-$generalized filter $\mathfrak{F}$ on the set X restricted an $L-$generalized filter $\mathfrak{F}_{\downarrow Y}$ on the set Y, defined by: for all $A \subseteq Y$,*

$$\mathfrak{F}_{\downarrow Y}(A) = \mathfrak{F}(A),$$

(2) *every $L-$generalized filter $\mathcal{H}$ on the set Y extended an $L-$generalized filter $\mathcal{H}_{\uparrow X}$ on the set X, defined by: for all $B \subseteq X$,*

$$\mathcal{H}_{\uparrow X}(B) = \mathcal{H}(B \cap Y).$$

Theorem 4.11. *Let (X, σ) and (Y, ℓ) be two $L-$generalized topological spaces, where $Y \subseteq X$. Then*

(1) *every neighbourhood $L-$generalized filter N_x in (X, σ) restricted the neighbourhood $L-$generalized filter $(\mathrm{N}_x)_{\downarrow Y}$ in $(Y, \sigma_{\downarrow Y})$, where $x \in Y$, defined by: for all $A \subseteq Y$,*

$$(\mathrm{N}_x)_{\downarrow Y}(A) = \mathrm{N}_{\mathrm{x}}(A),$$

(2) *every neighbourhood $L-$generalized filter N_y in (Y, ℓ) extended the neighbourhood $L-$generalized filter $(\mathrm{N}_y)_{\uparrow X}$ in $(X, \ell_{\uparrow X})$, where $y \in X$, defined by: for all $B \subseteq X$,*

$$(\mathrm{N}_y)_{\uparrow X}(B) \leq \mathrm{N}_y(B \cap Y).$$

Proof. (1) Let $A \subseteq Y, x \in A$. Then

$$(\mathrm{N}_x)_{\downarrow Y}(A) = \bigvee_{x \in V \subseteq A} \sigma_{\downarrow Y}(V) = \bigvee_{x \in V \subseteq A} \sigma(V) = \mathrm{N}_x(A).$$

Moreover, if $A \subset Y, x \notin A$, then $(\mathrm{N}_x)_{\downarrow Y}(A) = \mathrm{N}_x(A) = 0_L$.
Thus, $(\mathrm{N}_x)_{\downarrow Y}(A) = \mathrm{N}_{\mathrm{x}}(A)$, for all $A \subseteq Y$.

(2) Let $B \subseteq X, y \in B$. Then

$$\begin{aligned}(\mathrm{N}_y)_{\uparrow X}(B) &= \textstyle\bigvee_{y \in U \subseteq B} \ell_{\uparrow X}(U) \\ &= \textstyle\bigvee_{y \in U \subseteq B} \ell(U \cap Y) \\ &\leq \textstyle\bigvee_{y \in (U \cap Y) \subseteq (B \cap Y)} \ell(U \cap Y) \\ &\leq \textstyle\bigvee_{y \in K \subseteq (B \cap Y)} \ell(K) = \mathrm{N}_y(B \cap Y).\end{aligned}$$

Moreover, if $B \subset X, y \notin B$, then $(\mathrm{N}_y)_{\uparrow X}(B) = \mathrm{N}_y(B \cap Y) = 0_L$.
Thus, $(\mathrm{N}_y)_{\uparrow X}(B) \leq \mathrm{N}_y(B \cap Y)$, for all $B \subseteq X$. □

Theorem 4.12. *Let $\mathfrak{F}, \mathcal{H}$ be two $L-$generalized filters on two $L-$generalized topological spaces (X, σ) and (Y, ℓ), respectively and let $Y \subseteq X$. Then*

(1) *$\mathfrak{F} \to x$ in $(X, \sigma) \Rightarrow \mathfrak{F}_{\downarrow Y} \to x$ in $(Y, \sigma_{\downarrow Y})$, where $x \in Y$,*

(2) *$\mathcal{H} \to y$ in $(Y, \ell) \Rightarrow \mathcal{H}_{\uparrow X} \to y$ in $(X, \ell_{\uparrow X})$.*

Proof. (1) Let $\mathfrak{F} \to x$ in (X, σ). Then $\mathrm{N}_x(B) \leq \mathfrak{F}(B)$, for all $B \subseteq X$. Let $A \subseteq Y$. Then

$$(\mathrm{N}_x)_{\downarrow Y}(A) = \mathrm{N}_x(A) \leq \mathfrak{F}(A) = \mathfrak{F}_{\downarrow Y}(A).$$

Thus, $\mathfrak{F}_{\downarrow Y} \to x$ in $(Y, \Im_{\downarrow Y})$.

(2) Let $\mathcal{H} \to y$ in (Y, ℓ). Then $\mathrm{N}_y(A) \leq \mathcal{H}(A)$, for all $A \subseteq Y$. Let $B \subseteq X$. Then

$$(\mathrm{N}_y)_{\uparrow X}(B) \leq \mathrm{N}_y(B \cap Y) \leq \mathcal{H}(B \cap Y) = \mathcal{H}_{\uparrow X}(B).$$

Thus, $\mathcal{H}_{\uparrow X} \to y$ in $(X, \ell_{\uparrow X})$. □

Definition 4.13. The mapping $S : 2^X \longrightarrow L$ is called an $L-$generalized filter base for an $L-$generalized filter $\mathfrak{F}$ on the set X, if it satisfies the following conditions:

(i) $S(\phi) = 0_L$,

(ii) $\mathfrak{F}(A) = \bigvee_{B \subseteq A} S(B)$, for all $A \subseteq X$.

The above definition shows that any mapping $S : 2^X \longrightarrow L$, where $S(\phi) = 0_L$ is an $L-$generalized filter base for the $L-$generalized filter $\mathfrak{F}_S$ on the set X, which is defined by:

$$\mathfrak{F}_S(A) = \bigvee_{B \subseteq A} S(B).$$

Definition 4.14. An $L-$generalized filter $\mathfrak{F}$ on an $L-$generalized topological space (X,σ) converges to a point $x \in X$ at level $\alpha \in L - \{0_L\}$, if $\alpha \leq \mathrm{N}_x(A) \leq \mathfrak{F}(A)$, for all $A \subseteq X$ such that $\mathrm{N}_x(A) > 0_L$.

It is clear that an $L-$generalized filter $\mathfrak{F}$ converges to x at level γ, whenever, $\mathfrak{F}$ converges to x at level α, for all $\alpha \geq \gamma, \gamma \in L - \{0_L\}$.

Example 4.15. Let the set $X = \{a, b, c\}$, and define the $P^*([0,1])-$generalized topology $\sigma : 2^X \longrightarrow P^*([0,1])$ as follows: for each $A \in 2^X$,

$$\sigma(A) = \begin{cases} [0,1]\,; & A \in \{X, \phi\}, \\ [0,0.3]\,; & A \in \{\{a\},\{b\},\{a,b\}\}, \\ \{0\}; & otherwise. \end{cases}$$

Also, define the $P^*([0,1])-$generalized filter $\mathfrak{F} : 2^X \longrightarrow P^*([0,1])$ as follows: for each $A \in 2^X$,

$$\mathfrak{F}(A) = \begin{cases} [0,1]\,; & A = X, \\ [0,0.6]\,; & A \in \{\{a\},\{a,b\},\{a,c\}\}, \\ [0,0.4]\,; & A \in \{\{b\},\{c\},\{b,c\}\}, \\ \{0\}; & A = \phi. \end{cases}$$

Then,

$$\mathrm{N}_a(A) = \begin{cases} [0,1]\,; & A = X, \\ [0,0.3]\,; & A \in \{\{a\},\{a,b\},\{a,c\}\}, \\ \{0\}; & otherwise, \end{cases}$$

$$\mathrm{N}_b(A) = \begin{cases} [0,1]\,; & A = X, \\ [0,0.3]\,; & A \in \{\{b\},\{a,b\},\{a,c\}\}, \\ \{0\}; & otherwise, \end{cases}$$

$$\mathrm{N}_c(A) = \begin{cases} [0,1]\,; & A = X, \\ \{0\}; & otherwise. \end{cases}$$

Thus, $\mathfrak{F}$ converges to a, b at level $[0,0.3]$ and converges to c at level $[0,1]$.

Theorem 4.16. *Let $\sigma, \mathfrak{F}$ be $L-$generalized topology and $L-$generalized filter on the non-empty set X, respectively and let $\alpha \in L - \{0_L\}$. Then*
(1) $\sigma_\alpha = \{A \in 2^X : \sigma(A) \geq \alpha\}$ *is the generalized topology on* X,
(2) $\mathfrak{F}_\alpha = \{A \in 2^X : \mathfrak{F}(A) \geq \alpha\}$ *is the generalized filter on* X.

Corollary 4.17. *Let N_x be a neighbourhood $L-$generalized filter in an $L-$generalized topological space (X,σ). Then $(\mathrm{N}_x)_\alpha = \{A \in 2^X : \mathrm{N}_x(A) \geq \alpha\}$ is the neighbourhood generalized filter in the generalized topological space (X, σ_α).*

Theorem 4.18. *Let $\mathfrak{F}$ be an $L-$generalized filter in an $L-$generalized topological space (X,σ). Then $\mathfrak{F} \to x$ at level α in (X,σ) if and only if $\mathfrak{F}_\alpha \to x$ in (X,σ_α).*

Proof. Let $\mathfrak{F} \to x$ in (X,σ). Then $\alpha \leq \mathrm{N}_x(A) \leq \mathfrak{F}(A)$, for all $A \subseteq X$.

Now, Let $A \in (\mathrm{N}_x)_\alpha$. Then $\mathrm{N}_x(A) \geq \alpha$ implies that $\alpha \leq \mathfrak{F}(A)$. Thus, $A \in \mathfrak{F}_\alpha$ and $(\mathrm{N}_x)_\alpha \subseteq \mathfrak{F}_\alpha$, which implies that $\mathfrak{F}_\alpha \to x$ in (X,σ_α).

Conversely, let $\mathfrak{F}_\alpha \to x$ in (X,σ_α). Then $(\mathrm{N}_x)_\alpha \subseteq \mathfrak{F}_\alpha$. Thus, $\alpha \leq \mathrm{N}_x(B) \leq \mathfrak{F}(B)$, for all $B \subseteq X$. So, $\mathfrak{F} \to x$ at level α in (X,σ). □

Definition 4.19. Let (X,σ) be an $L-$generalized topological space and let $\phi \neq A \subseteq X$. Then the closure $\overline{A}$ of the set A is defined as follows:

$$x \in \overline{A} \text{ if and only if } B \cap A \neq \phi, \text{ for all } B \subseteq X \text{ with } \mathrm{N}_x(B) = 1_L.$$

It is clear that $x \notin \overline{A}$, if there exists $B \subset X$ such that $B \cap A = \phi$ with $\mathrm{N}_x(B) = 1_L$. Moreover, for all non-empty two sets $A, B \subseteq X$, one can show that $A \subseteq \overline{A}$ and $\overline{A} \subseteq \overline{B}$ if $A \subseteq B$.

Definition 4.20. $A \subseteq X$ in the $L-$generalized topological space (X,σ) is called closed set, if $A = \overline{A}$.

Definition 4.21. $A \subseteq X$ in the $L-$generalized topological space (X,σ) is called open set, if A^C is closed.

Theorem 4.22. *Let (X,σ) be an $L-$generalized topological space and let $\phi \neq A \subseteq X$. then $x \in \overline{A}$ if and only if there exists an $L-$generalized filter $\mathfrak{F}$ on the set X such that $\mathfrak{F} \to x$ with $\mathfrak{F}(A \cap B) = 1_L$, for all $B \subseteq X$ and $\mathrm{N}_x(B) = 1_L$.*

Proof. Let $x \in \overline{A}$. then $B \cap A \neq \phi$, for all $B \subseteq X$ with $\mathrm{N}_x(B) = 1_L$. Define the mapping $S : 2^X \longrightarrow L$ as follows: for each $H \in 2^X$,

$$S(H) = \begin{cases} 1_L; & if\, H = A \cap G, \mathrm{N}_x(G) = 1_L, \\ \mathrm{N}_x(H); & otherwise. \end{cases}$$

Since $S(\phi) = \mathrm{N}_x(\phi) = 0_L$, S generates an $L-$generalized filter $\mathfrak{F}$ on the set X, which is defined by:

$$\mathfrak{F}(K) = \bigvee_{H \subseteq K} S(H).$$

If $\mathrm{N}_x(G) = 1_L$, for some $G \subseteq X$, then $G \cap A \neq \phi$. Thus $S(G \cap A) = 1_L$, implies that $\mathfrak{F}(G \cap A) = 1_L$. So $\mathfrak{F}(G) = 1_L$.

If $\mathrm{N}_x(G) \neq 1_L$, for some $G \subset X$, and $G \neq \phi$, then the definition of S implies that $\mathfrak{F}(G) \geq S(G) = \mathrm{N}_x(G)$. Thus, $\mathfrak{F}(G) \geq \mathrm{N}_x(G)$, for all $G \subset X$. So, $\mathfrak{F} \to x$.

Conversely, let $\mathfrak{F}$ be an $L-$generalized filter on the set X such that $\mathfrak{F} \to x$ with $\mathfrak{F}(A \cap B) = 1_L$, for all $B \subseteq X$ and $\mathrm{N}_x(B) = 1_L$. Let $G \subseteq X$ such that $\mathrm{N}_x(G) = 1_L$. Then $\mathfrak{F}(A \cap G) = 1_L$. Thus $G \cap A \neq \phi$. So $x \in \overline{A}$. □

Corollary 4.23. *The nonempty set A is open in the $L-$generalized topological space (X,σ), if $\mathrm{N}_x(A) = 1_L$, for all $x \in A$.*

Proof. Since $\mathrm{N}_x(A) = 1_L$, for all $x \in A$ and $A \cap A^C = \phi$, from the above theorem, $x \notin \overline{A^C}$, for all $x \notin A^C$ which implies that $\overline{A^C} \subseteq A^C$. Then $\overline{A^C} = A^C$. Thus, A^C is a closed set in (X,σ). So, A is an open set in (X,σ). □

Corollary 4.24. *The nonempty set A is closed in the $L-$generalized topological space (X,σ) if and only if for all $x \in A$, if there exists an $L-$generalized filter $\mathfrak{F}$ on the set X, $\mathfrak{F} \to x$ with $\mathfrak{F}(G \cap A) = 1_L$ and $\mathrm{N}_x(G) = 1_L$, for all $G \subseteq X$, then $x \in A$.*

Theorem 4.25. *The nonempty set A is open in the $L-$generalized topological space (X,σ) if and only if $\mathfrak{F}(A) = 1_L$, for each an $L-$generalized filter $\mathfrak{F}$ on the set X with $\mathfrak{F} \to x, x \in A$.*

Proof. Let A be an open in the $L-$generalized topological space (X, σ) and let $\mathfrak{F}$ be an $L-$generalized filter on the set X with $\mathfrak{F} \to x_0, x_0 \in A$. Then, $x_0 \notin \overline{A^C} = A^C$, which implies that $B \cap A^C = \phi$ with $\mathrm{N}_{x_0}(B) = 1_L$, for some $B \subset X$. Thus $\mathfrak{F}(B) = 1_L,\ B \subseteq A$, which implies that $\mathfrak{F}(A) = 1_L$.

Conversely, let $\mathfrak{F}(A) = 1_L$, for each an $L-$generalized filter $\mathfrak{F}$ on the set X with $\mathfrak{F} \to x$. Then $x \in A$. Since N_x is an $L-$generalized filter and $\mathrm{N}_x \to x, x \in X$, $\mathrm{N}_x(A) = 1_L$, for each $x \in A$. Thus, Corollary 4.23 shows that A is open set in (X, σ). □

Corollary 4.26. *Let A be the non-empty set in the $L-$generalized topological space (X, σ). Then $\mathrm{N}_x(A) = 1_L$, for all $x \in A$, if A is an open set.*

5. $L-$GINERALIZED NETS

Definition 5.1. Let D be a partially ordered set under the relation $\leq$ and let X be a non-empty set. Then any non zero mapping $\mathcal{B} : D \times X \longrightarrow L$ is called an $L-$generalized net on the set X.

If D is a directed set under the relation $\leq$, then $\mathcal{B} : D \times X \longrightarrow L$ is called an $L-$net on the set X.

Theorem 5.2. *Any crisp generalized net on the set X associates a unique $L-$generalized net on X.*

Proof. Let $\mathcal{A} : D \longrightarrow X$ be a crisp generalized net on the set X. Define the $L-$generalized net $\mathcal{B} : D \times X \longrightarrow L$ to be associated with $\mathcal{A}$ and is defined by:

$$\mathcal{B}(\lambda, x) = \begin{cases} 1_L; & \mathcal{A}(\lambda) = x, \\ 0_L; & \mathcal{A}(\lambda) \neq x. \end{cases}$$

Now, we show that $\mathcal{B}$ is the unique $L-$generalized net on X associated with $\mathcal{A}$.

Let $\mathcal{A}_1, \mathcal{A}_2$ be two crisp generalized nets on X such that $\mathcal{A}_1 \neq \mathcal{A}_2$. Then there exists $\lambda_0 \in D$ such that $\mathcal{A}_1(\lambda_0) \neq \mathcal{A}_2(\lambda_0)$, which implies that there exist two associated $L-$generalized nets $\mathcal{B}_1, \mathcal{B}_2$ with $\mathcal{A}_1, \mathcal{A}_2$, respectively and $\mathcal{B}_1(\lambda_0, x) \neq \mathcal{B}_2(\lambda_0, x)$. Thus, $\mathcal{B}_1 \neq \mathcal{B}_2$, which implies that $\mathcal{B}$ is the unique associated $L-$generalized net with $\mathcal{A}$. □

In general, an $L-$generalized net on the set X need not associate the crisp generalized net on X (See the following example).

Example 5.3. Let $D = 3N, X = R$, where N, R are two sets of all natural and real numbers respectively. Define the $L-$generalized net $\mathcal{B} : 3N \times R \longrightarrow [0, 1]$, which is defined by:

$$\mathcal{B}(3n, x) = \begin{cases} 1; & x = 3n, \\ \frac{1}{3}; & x = n, \\ 0; & otherise. \end{cases}$$

Then, it is impossible to fined a crisp generalized net on R, which is associated with $\mathcal{B}$.

The following theorem shows the condition to find the crisp generalized net which is uniquely associated with any given an $L-$generalized net on the set X.

Theorem 5.4. *Any $L-$generalized net $\mathcal{B} : D \times X \longrightarrow L$ on the set X associates the unique crisp net $\mathcal{A} : D \longrightarrow X$ on X, if the following condition is true: for each $\lambda \in D$, there exists a unique element $x \in X$ such that*

$$\mathcal{B}(\lambda, y) = \begin{cases} 1_L; & y = x, \\ 0_L; & y \neq x. \end{cases}$$

Proof. Define the crisp generalized net $\mathcal{A} : D \longrightarrow X$ on X as follows:
$\mathcal{A}(\lambda) = x$, whenever $\mathcal{B}(\lambda, x) = 1_L$. Then, $\mathcal{A}$ is uniquely associated with $\mathcal{B}$. □

Definition 5.5. Let (X, σ) be an $L-$generalized topological space. Then the $L-$generalized net $\mathcal{B}$ on the set X converges to the element $x_0 \in X$ at level $\alpha \in L - \{0_L\}$, if for each $U \subseteq X, \mathrm{N}_x(U) \geq \alpha$, there exists $\lambda_0 \in D$ such that if $\mathcal{B}(\lambda, x) \geq \alpha$, for all $\lambda \in D, \lambda \geq \lambda_0$, then $x \in U$.

Example 5.6. Let Z, R be two sets of all integer and real numbers and let $\mathcal{U}$ be the usual topology on R. Define the $[0,1]-$generalized topology σ on R as follows:

$$\sigma(A) = \begin{cases} 1; & A \in \mathcal{U}, \\ 0; & otherise. \end{cases}$$

Then the neighbourhood $[0,1]-$generalized filter at $0 \in R$ is defined by:

$$\mathrm{N}_x(A) = \begin{cases} 1; & A \in]\epsilon, -\epsilon[, \ \forall \epsilon > 0, \\ 0; & otherise. \end{cases}$$

Moreover, define the $[0,1]-$generalized net $\mathcal{B} : Z \times R \longrightarrow [0,1]$ on R as follows:

$$\mathcal{B}(n, x) = \begin{cases} \frac{2}{3}; & n \leq o, \\ 1; & x = \frac{1}{n}, n > 0, \\ 0; & x \neq \frac{1}{n}, n > 0. \end{cases}$$

Let $A \subseteq X$, $\mathrm{N}_x(A) \geq \frac{1}{2}$ and let $n_0 \in Z$, $n_0 > \frac{1}{\varepsilon} > 0, n \geq n_0$ and $\mathcal{B}(n, x) \geq \frac{1}{2}$. Then $A =]\epsilon, -\epsilon[, n > \frac{1}{\varepsilon} > 0$ and $\mathcal{B}(n, x) = 1$, which implies that $\epsilon > \frac{1}{n}, x = \frac{1}{n}$. Thus, $x \in]\epsilon, -\epsilon[= A$, and so $\mathcal{B}$ converges to 0 at level $\frac{1}{2}$.

For any topological space (X, τ), there exists the $L-$generalized topological space (X, τ_g), where

$$\tau_g(A) = \begin{cases} 1_L; & A \in \tau, \\ 0_L; & otherwise. \end{cases}$$

Moreover, for any neighbourhood system $\mathbb{N}_x$ in (X, τ),

$$(\mathbb{N}_x)_g(A) = \begin{cases} 1_L; & A \in \mathbb{N}_x, \\ 0_L; & otherwise \end{cases}$$

is the neighbourhood $L-$generalized system in (X, τ_g).

Theorem 5.7. *Let $\mathcal{A} : D \longrightarrow X$ be a crisp generalized net on X in a generalized topological space (X, τ) and let $\mathcal{B} : D \times X \longrightarrow L$ be an $L-$generalized net associated with $\mathcal{A}$ on X. Then $\mathcal{A}$ converges to $x_0 \in X$ in (X, τ) if and only if $\mathcal{B}$ converges to x_0 at level $\alpha \in L - \{0_L\}$ in (X, τ_g).*

Proof. Let $\mathcal{A}$ converge to $x_0 \in X$ in (X, τ) and let $G \subseteq X, (\mathbb{N}_{x_0})_g(G) \geq \alpha,\ \alpha \in L - \{0_L\}$. Then $(\mathbb{N}_x)_g(G) = 1_L$ and $G \in \mathbb{N}_x$. Since $\mathcal{A}$ converges to x_0, there exists $\lambda_0 \in D$ such that $\mathcal{A}(\lambda) \in G$, for each $\lambda \geq \lambda_0$. Let $\mathcal{B}(\lambda, x) \geq \alpha$, for each $\lambda \geq \lambda_0$. Then the definition of $\mathcal{B}$ implies that $\mathcal{B}(\lambda, x) = 1_L,\ x = \mathcal{A}(\lambda) \in G$. Thus, $\mathcal{B}$ converges to x_0 at level $\alpha \in L - \{0_L\}$ in (X, τ_g).

Conversely, let $\mathcal{B}$ converges to x_0 at level $\alpha \in L - \{0_L\}$ in (X, τ_g) and let $H \subseteq X, H \in \mathbb{N}_x$. Then $(\mathbb{N}_x)_g(H) = 1_L \geq \alpha$. Thus there exists $\lambda_0 \in D$ such that if $\mathcal{B}(\lambda, x) \geq \alpha$, for each $\lambda \geq \lambda_0$, then $x \in H$. So, $\mathcal{B}(\lambda, x) = 1_L,\ x = \mathcal{A}(\lambda) \in H$, for each $\lambda \geq \lambda_0$. Hence $\mathcal{A}$ converges to x_0 in (X, τ). □

Let X, Y be two non-empty ordinary sets, where $Y \subset X$. Then

(1) every an $L-$generalized net $\mathcal{B} : D \times X \longrightarrow L$ on the set X restricted an $L-$generalized net $\mathcal{B}_{\downarrow Y} : D \times Y \longrightarrow L$ on the set Y, which is defined by:

$$\mathcal{B}_{\downarrow Y}(\lambda, y) = \mathcal{B}(\lambda, y), \text{ for all } \lambda \in D, y \in Y,$$

(2) every an $L-$generalized net $\mathcal{C} : D \times Y \longrightarrow L$ on the set Y extended an $L-$generalized net $\mathcal{C}_{\uparrow X} : D \times X \longrightarrow L$ on the set X, which is defined by:

$$\mathcal{C}_{\uparrow X}(\lambda, x) = \begin{cases} \mathcal{C}(\lambda, x); & x \in Y, \\ o_L; & x \notin Y. \end{cases}$$

.

Theorem 5.8. *Let $\mathcal{B}, \mathcal{C}$ be two $L-$generalized nets on the two $L-$generalized topological spaces (X, σ) and (Y, ℓ), respectively and let $\phi \neq Y \subseteq X$. Then*

(1) $\mathcal{B}$ converges to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ), implies $\mathcal{B}_{\downarrow Y}$ converges to x_0 at level $\alpha \in L - \{0_L\}$ in $(Y, \sigma_{\downarrow Y})$, where $x_0 \in Y$,

(1) $\mathcal{C}$ converges to $y_0 \in Y$ at level $\alpha \in L - \{0_L\}$ in (Y, ℓ), implies $\mathcal{C}_{\uparrow X}$ converges to y_0 at level $\alpha \in L - \{0_L\}$ in $(X, \ell_{\uparrow X})$.

Proof. (1) Let $\mathcal{B}$ converges to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ) and let $H \subseteq Y, (\mathrm{N}_{x_0})_{\downarrow Y}(H) \geq \alpha$. Since $(\mathrm{N}_{x_0})_{\downarrow Y}(H) = \mathrm{N}_{x_0}(H)$, $\mathrm{N}_{x_0}(H) \geq \alpha$, where N_{x_0} is the negibourhood $L-$generalized filter system in (X, σ). Then there exists $\lambda_0 \in D$ such that if $\mathcal{B}(\lambda, x) \geq \alpha$, for all $\lambda \in D, \lambda \geq \lambda_0$, then $x \in H$. Thus $\mathcal{B}_{\downarrow Y}$ converges to x_0 at level $\alpha \in L - \{0_L\}$ in $(Y, \sigma_{\downarrow Y})$.

(2) Let $\mathcal{C}$ converges to $y_0 \in Y$ at level $\alpha \in L - \{0_L\}$ in (Y, ℓ) and let $G \subseteq X, (\mathrm{N}_{y_0})_{\uparrow X}(G) \geq \alpha$. Since $(\mathrm{N}_{y_0})_{\uparrow X}(G) \leq \mathrm{N}_{y_0}(G \cap Y)$, $\mathrm{N}_{y_0}(G \cap Y) \geq \alpha$, where N_{y_0} is the negibourhood $L-$generalized filter system in (Y, ℓ). Then there exists $\lambda_0 \in D$ such that if: $\mathcal{C}(\lambda, y) \geq \alpha$, for all $\lambda \in D, \lambda \geq \lambda_0$, then $y \in G \cap Y \subseteq G$. Thus $\mathcal{C}_{\uparrow X}$ converges to y_0 at level $\alpha \in L - \{0_L\}$ in $(X, \ell_{\uparrow X})$. □

6. Relation between $L-$gineralized filters and $L-$gineralized nets

Definition 6.1. Let $\mathcal{B} : D \times X \longrightarrow L$ be an $L-$generalized net on the set X and let $\lambda_0 \in D$, $\alpha \in L - \{0_L\}$. Then the subfamily $T(\lambda_0, \alpha)$ of X, which is defined as:

$$T(\lambda_0, \alpha) = \{x \in X : \mathcal{B}(\lambda, x) \geq \alpha,\ for\ all\ \lambda \in D, \lambda \geq \lambda_0\}$$

is called a tail of $\mathcal{B}$.

Remark 6.2. Since any given an $L-$generalized net $\mathcal{B}$ is a non-zero mapping, there exists at least one tail, which is non-empty set. Moreover, in general, it may happen that there exists a tail of $\mathcal{B}$, which is empty set as the following example.

Example 6.3. Let Z, R be two sets of all integer and real numbers. Define the $L-$generalized net $\mathcal{B} : Z \times R \longrightarrow [0,1]$, which is defined by:

$$\mathcal{B}(n,x) = \begin{cases} 0; & x < 0, n < 0, \\ \\ \frac{1}{2}; & x = 0, n = 0 \\ \\ \frac{2}{3}; & x > 0, n > 0. \end{cases}$$

Then $\mathcal{B}$ is the $L-$generalized net on R, and $T(\lambda, 1) = \phi$, for all $n \in Z$.

Theorem 6.4. *Every an $L-$generalized net $\mathcal{B}$ on the set X, having the family of all non empty tails induces an $L-$generalized filter $\mathfrak{F}_{\mathcal{B}}$ on X.*

Proof. Let $\Gamma = \{T(\lambda, \alpha) : \lambda \in D, \alpha \in L\}$ be the family of all non-empty tails of the given generalized net $\mathcal{B}$ on X. Define the non zero mapping $S : 2^X \longrightarrow L$ by:

$$S(A) = \begin{cases} 1_L; & A \in \Gamma or A = X, \\ 0_L; & A \notin \Gamma. \end{cases}$$

It is clear that $S(\phi) = 0_L$. Then it generates an $L-$generalized filter $\mathfrak{F}_{\mathcal{B}}$ on X and it is defined by:

$$\mathfrak{F}_{\mathcal{B}}(B) = \bigvee_{A \subseteq B} S(A).$$

□

Example 6.5. Let $X = \{r, s, t\}, D \subset 2^X$, where $D = \{\{r\}, \{s\}, \{t\}, \{r, s\}, \{s, t\}\}$. Then D is a partially ordered set with respect to the inclusion relation $\subseteq$. Let $L = \{0, 1\}$ and define the $\{0, 1\}-$generalized net $\mathcal{B} : D \times X \longrightarrow \{0, 1\}$ on X, as follows:

$$\mathcal{B}(\lambda, x) = \begin{cases} 1; & (\lambda, x) \in \{(\{r\}, r), (\{s\}, s), (\{t\}, t), (\{r, s\}, s), (\{s, t\}, s)\}, \\ 0; & otherwise. \end{cases}$$

Thus,

$$T(\{r\}, 1) = \{r, s\}, T(\{s\}, 1) = \{s\}, T(\{t\}, 1) = \{t, s\}, T(\{r, s\}, 1) = \{s\}, T(\{s, t\}, 1) = \{s\}.$$

So,

$$S(A) = \begin{cases} 1; & A \in \{X, \{s\}, \{s, t\}, \{r, s\}\}, \\ 0; & otherwise. \end{cases}$$

It is clear that S is the $\{0, 1\}-$generalized filter. Hence the $\{0, 1\}-$generalized filter induced by the $\{0, 1\}-$generalized net $\mathcal{B}$ is $\mathfrak{F}_{\mathcal{B}} = S$.

Theorem 6.6. *Let (X, σ) be an $L-$generalized topological space and let $\mathfrak{F}_{\mathcal{B}}$ be an induced $L-$generalized filter of an $L-$generalized net $\mathcal{B}$ on X. Then $\mathcal{B}$ converges to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ) if and only if $\mathfrak{F}_{\mathcal{B}}$ converges to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ).*

Proof. Let $\mathcal{B}$ converges to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ). Then for each $H \subseteq X, \mathrm{N}_{x_0}(H) \geq \alpha$ and there exists $\lambda_0 \in D$ such that if $\mathcal{B}(\lambda, x) \geq \alpha$, for each $\lambda \in D,\ \lambda \geq \lambda_0$, then $x \in H$. Thus the definition of the tail of the net $\mathcal{B}$ shows that $T(\lambda_0, \alpha) \subset H$, which implies that $\mathfrak{F}_{\mathcal{B}}(H) = 1_L$. So, $\alpha \leq \mathrm{N}_{x_0}(H) \leq \mathfrak{F}_{\mathcal{B}}(H)$, for each

$H \subset X$ such that $N_{x_0}(H) > 0_L$. Hence $\mathfrak{F}_{\mathcal{B}}$ converges to $x_0 \in X$ at level $\alpha \in L-\{0_L\}$ in (X, σ).

Conversely, let $\mathfrak{F}_{\mathcal{B}}$ converges to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ). Then $\alpha \leq N_{x_0}(H) \leq \mathfrak{F}_{\mathcal{B}}(H)$, for each $H \subseteq X$ such that $N_{x_0}(H) > 0_L$. Let $G \subseteq X, N_{x_0}(G) \geq \alpha$. Then $\mathfrak{F}_{\mathcal{B}}(G) \geq \alpha$, which implies that $\mathfrak{F}_{\mathcal{B}}(G) = 1_L$. Thus the definition of the induced $L-$generalized filter $\mathfrak{F}_{\mathcal{B}}$ shows that $\phi \neq T(\lambda_0, \alpha) \subseteq G$, for some $\lambda_0 \in D$ and $\alpha \in L - \{0_L\}$. So, for each $\lambda \geq \lambda_0, \mathcal{B}(\lambda, x) \geq \alpha$, it is follows that $x \in T(\lambda_0, \alpha) \subseteq G$. Hence, $\mathcal{B}$ converges to $x_0 \in X$ at the level $\alpha \in L - \{0_L\}$. □

Theorem 6.7. *Every an $L-$generalized filter $\mathfrak{F}$ on the set X, induces an $L-$generalized net $\mathcal{B}_{\mathfrak{F}}$ on X.*

Proof. Let $\mathfrak{F}$ be an $L-$generalized filter on the set X. Define the set

$$D = \{(x, B) \in X \times 2^X : x \in B, \mathfrak{F}(B) = 1_L\}$$

and define the relation $\geq$ on the set D as: $(x_1, B_1) \geq (x_2, B_2)$, if $B_1 \subseteq B_2$. Then, the set D is a partially ordered set under the relation $\geq$. Define the mapping $\mathcal{B}_{\mathfrak{F}} : D \times 2^X \longrightarrow L$, to be an induced $L-$generalized net of an $L-$generalized filter $\mathfrak{F}$ and is defined by:

$$\mathcal{B}_{\mathfrak{F}}((x, B), y) = \begin{cases} 1_L; & x = y, \\ 0_L; & x \neq y. \end{cases}$$

□

Example 6.8. Let $X = \{r, s, t\}, L = P^*([0, 1])$. Define the $L-$generalized filter $\mathfrak{F} : 2^X \longrightarrow P^*([0, 1]$ on X, as follows:

$$\mathfrak{F}(A) = \begin{cases} [0, 1]; & A \in \{\{r, s\}, \{r, t\}, X\}, \\ \{0\}; & otherwise. \end{cases}$$

Moreover, the partially ordered set $D \subseteq X \times 2^X$ is defined as follows:

$$D = \{(r, \{r, s\}), (r, \{r, t\}), (r, X), (s, \{r, s\}), (s, X), (t, \{r, t\}), (t, X)\}.$$

Define the family $\Upsilon \subset D \times X$, where

$$\Upsilon = \{((r, \{r, s\}), r), ((r, \{r, t\}), r), ((r, X), r), ((s, \{r, s\}), s), ((s, X), s), ((t, \{r, t\}), t), ((t, X), t)\}.$$

Then, the induced $L-$generalized net $\mathcal{B}_{\mathfrak{F}} : D \times X \longrightarrow P^*([0, 1])$ of the $L-$generalized filter $\mathfrak{F}$ is defined by:

$$\mathcal{B}_{\mathfrak{F}}(H) = \begin{cases} [0, 1]; & H \in \Upsilon, \\ \{0\}; & otherwise. \end{cases}$$

Theorem 6.9. *Let (X, σ) be an $L-$generalized topological space and let $\mathcal{B}_{\mathfrak{F}} : D \times X \longrightarrow L$ be an induced $L-$generalized net of an $L-$generalized filter $\mathfrak{F} : 2^X \longrightarrow L$ on X. then $\mathfrak{F}$ converges to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ) if and only if $\mathcal{B}_{\mathfrak{F}}$ converges to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ).*

Proof. Let $\mathfrak{F}$ converge to $x_0 \in X$ at level $\alpha \in L - \{0_L\}$ in (X, σ). Then for each $G \subseteq X, N_{x_0}(G) \geq \alpha$. Thus $x_0 \in G, \alpha \leq N_{x_0}(G) \leq \mathfrak{F}(G)$.

Now, let $(y, H) \in D, (y, H) \geq (x_0, G)$. Then $H \subseteq G, y \in H$ and $\mathfrak{F}(H) = 1_L$, which implies that $\mathfrak{F}(G) = 1_L$ and $(x_0, G) \in D$. Let $\mathcal{B}_{\mathfrak{F}}((y, H), x) \geq \alpha$. Then

$y \in H \subseteq G, \mathcal{B}_{\mathfrak{F}}((y,H),x) = 1_L$. Thus, $x = y \in G,$. So $\mathcal{B}_{\mathfrak{F}}$ converges to $x_0 \in X$ at the level $\alpha \in L - \{0_L\}$ in (X, σ).

Conversely, let $\mathcal{B}_{\mathfrak{F}}$ converges to $x_0 \in X$ at the level $\alpha \in L - \{0_L\}$ in (X, σ) and let $U \subseteq X, \mathrm{N}_{x_0}(U) \geq \alpha$. Then there exists $(z, K) \in D$ such that if $\mathcal{B}_{\mathfrak{F}}((t,U),x) \geq \alpha$, for each $(t, M) \in D, (t, M) \geq (z, K)$, then $x \in U$. Thus, from the definition of the induced $L-$generalized net $\mathcal{B}_{\mathfrak{F}}$ of the $L-$generalized filter $\mathfrak{F}$, it is follows that $\mathcal{B}_{\mathfrak{F}}((t,U),x) = 1_L$ and $x = t \in U$, which implies that $t \in U$, for each $t \in M$. So $M \subseteq U$. Since $(t, M) \in D$, $\mathfrak{F}(M) = 1_L$. Hence, $\mathfrak{F}(U) = 1_L$. Therefore, $\alpha \leq \mathrm{N}_{x_0}(U) \leq \mathfrak{F}(U)$, for all $U \subseteq X$ such that $\mathrm{N}_{x_0}(U) > 0_L$, and so $\mathfrak{F}$ converges to $x_0 \in X$ at the level $\alpha \in L - \{0_L\}$ in (X, σ). □

7. Conclusion

From the above discussions, we can advocate that the $L-$generalized filters and the $L-$generalized nets on the non empty universal set X are the natural generalization of the crisp generalized filters and the crisp generalized nets respectively. Moreover, the convergence of the $L-$generalized filters and the $L-$generalized nets are the natural generalization of the convergence of the crisp generalized filters and the crisp generalized net respectively.

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