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# k-pseudo similar intuitionistic fuzzy matrices

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Abstract. In this paper, we shall define k-pseudo similarity (right k-pseudo similar or left k-pseudo similar) for intuitionistic fuzzy matrices and prove that, for a pair of intuitionistic fuzzy matrices $A, B \in (IF)_n$, if $A$ is said to be right (left) k-pseudo similar to $B$ then $A^s$ is said to be right (left) k-pseudo similar to $B^s$ for any integer $s \geq 1$, but the converse is not true which is illustrated by an example. Also prove that, $A$ is said to be right (left) k-pseudo similar to $B$ if and only if $B^T$ is said to be left (right) k-pseudo similar to $A^T$. We exhibit that the k-pseudo similarity on $A$ and $B$ preserve k-regularity of the intuitionistic fuzzy matrices $A$ and $B$. As a special case, for $k = 1$ it reduces to pseudo similar intuitionistic fuzzy matrices [3].

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## 1. Introduction

We deal with the fuzzy matrices that is the matrices over the fuzzy algebra with support $[0, 1]$ under the max-min operations $\{+, .\}$ defined as $a+b = max\{a, b\}$ and $a.b = min\{a, b\}$ for all $a, b \in \{F : F = [0, 1]\}$. Let $F_{mn}$ the set of all $m \times n$ fuzzy matrices over the fuzzy algebra $F$. A matrix $A \in F_{m\times n}$ is said to be regular if there exists $X$ such that $AXA = A$; $X$ is called a generalized $(g^-)$ inverse of A and is denoted by $A^-$. A development of theory of fuzzy matrices analogous to that of Boolean matrices is made by Kim and Roush [5]. Atanassov has introduced and developed the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [1]. A study on regularity and various g-inverse of intuitionistic fuzzy matrices over intuitionistic fuzzy algebra are discussed in [10]. Basic properties of intuitionistic fuzzy matrices as a generalization of the results on fuzzy matrices have been derived by Pal and Khan [4]. Meenakshi and Gandhimathi have studied on regularity of intuitionistic fuzzy matrices [8]. In [11], some properties on both idempotent intuitionistic fuzzy matrices and idempotent intuitionistic fuzzy matrices of T-type are discussed. In [12], a problem of reducing intuitionistic fuzzy matrices is examined and some useful properties are obtained with respect to nilpotent intuitionistic fuzzy matrices. In [6], some properties of a transitive fuzzy matrix are examined and the canonical form of the transitive fuzzy matrix is given using the properties also obtained a canonical form of the transitive intuitionistic fuzzy matrix. In [13], szpilrajn's theorem on ordering is generalized to intuitionistic fuzzy orderings. In [14], Riyaz Ahmad Padder and Murugadas have introduced the max-max operations on intuitionistic fuzzy matrices to study the conditions for convergence of intuitionistic fuzzy matrices. In [2], Cho has discussed the consistency of fuzzy matrix equations. Recently, Meenakshi and Jenita have introduced the concept of k-regular fuzzy matrix as a generalization of regular fuzzy matrix [9]. Further to learn about fuzzy matrix theory and applications one may refer [7]. In this paper, we have introduced the concept of k-pseudo similar intuitionistic fuzzy matrices(IFM) as a generalization of pseudo similar intuitionistic fuzzy matrices [3].

## 2. Preliminaries

In this paper, we are concerned with fuzzy matrices, that is matrices over a fuzzy algebra FM(FN) with support $[0, 1]$, under maxmin(minmax) operations and the usual ordering of real numbers. Let $(IF)_{m\times n}$ be the set of all intuitionistic fuzzy matrices of order $m \times n$, $F^M_{m\times n}$ be the set of all fuzzy matrices of order $m \times n$, under the maxmin composition and $F^N_{m\times n}$ be the set of all fuzzy matrices of order $m \times n$, under the minmax composition.

If $A = (a_{ij}) \in (IF)_{m\times n}$, then $A = (\langle a_{ij\mu}, a_{ij\vartheta}\rangle)$, where $a_{ij\mu}$ and $a_{ij\vartheta}$ are the membership values and non membership values of $a_{ij}$ in $A$ respectively with respect to the fuzzy sets $\mu$ and $\vartheta$, maintaining the condition $0 \leq a_{ij\mu} + a_{ij\vartheta} \leq 1$.

We shall follow the matrix operations on intuitionistic fuzzy matrices as defined in [8].

For $A, B \in (IF)_{m\times n}$,

$A + B = (\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\vartheta}, b_{ij\vartheta}\}\rangle)$,

$AB = \left(\left\langle \max_k \min\{a_{ik\mu}, b_{kj\mu}\}, \min_k \max\{a_{ik\vartheta}, b_{kj\vartheta}\}\right\rangle\right)$.

Let us define the order relation on $(IF)_{m\times n}$ as:

$A \leq B \Leftrightarrow a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\vartheta} \geq b_{ij\vartheta}$, for all $i$ and $j$.

In this work, we shall represent $A \in (IF)_{m\times n}$ as Cartesian product of fuzzy matrices.

For $A = (a_{ij}) \in (IF)_{m\times n}$. Let $A = (a_{ij}) = (\langle a_{ij\mu}, a_{ij\vartheta}\rangle) \in (IF)_{m\times n}$. We define $A_\mu = (a_{ij\mu}) \in F^M_{m\times n}$ as the membership part of $A$ and $A_\vartheta = (a_{ij\vartheta}) \in F^N_{m\times n}$ as the non-membership part of A. Thus $A$ is written as the Cartesian product $A_\mu$ and $A_\vartheta$, $A =< A_\mu, A_\vartheta >$ with $A_\mu \in F^M_{m\times n}, A_\vartheta \in F^N_{m\times n}$.

**Definition 2.1** ([8])**.** For $A, B \in (IF)_{m\times n}$, if $A =< A_\mu, A_\vartheta >$ and $B =< B_\mu, B_\vartheta >$, then $A + B =< A_\mu + B_\mu, A_\vartheta + B_\vartheta >$ .

**Definition 2.2** ([8])**.** For $A \in (IF)_{m\times p}, B \in (IF)_{p\times n}$ if $A =< A_\mu, A_\vartheta >$ and $B =< B_\mu, B_\vartheta >$, then

(i) $AB =< A_\mu B_\mu, A_\vartheta B_\vartheta >$, where $A_\mu B_\mu$ is the max min product in $F^M_{m\times n}$ and $A_\vartheta B_\vartheta$ is the min max product in $F^N_{m\times n}$,

(ii) $A^T =< A^T_\mu, A^T_\vartheta >$ .

**Definition 2.3** ([8]). A matrix $A \in (IF)_n$ is said to be invertible, if there exists $X \in (IF)_n$ such that $AX = XA = I_n =< I^M_n, I^N_n >$, where $I_n$ is the identity matrix in $(IF)_n$.

**Definition 2.4** ([8]). A square intuitionistic fuzzy matrix is called intuitionistic fuzzy permutation matrix, if every row and column contains exactly one $< 1, 0 >$ and all the other entries are $< 0, 1 >$. Let $P_n$ be the set of all $n \times n$ permutation matrices in $(IF)_n$.

**Definition 2.5** ([4]). An $A \in (IF)_{m\times n}$ is said to be regular, if there exists $X \in (IF)_{m\times n}$ satisfying $AXA = A$. In this case, $X$ is called a generalized inverses (g-inverse) of $A$ and is denoted by $\overline{A}$.

Let $A\{1\}$ be the set of all g-inverses of $A$.

**Definition 2.6** ([3]). $A \in (IF)_m$ and $B \in (IF)_n$ are said to be pseudo similar, denoted by $A \cong B$, if there exist $X \in (IF)_{mn}$ and $Y \in (IF)_{nm}$ such that

$$A = XBY, B = YAX \text{ and } XYX = X.$$

**Lemma 2.7** ([3]). *Let $A \in (IF)_m$ and $B \in (IF)_n$. Then the following are equivalent:*

(1) $A \cong B$,

(2) *there exist $X \in (IF)_{mn}, Y \in (IF)_{nm}$ such that $A = XBY, B = YAX, XYX = X$ and $YXY = Y$,*

(3) *There exist $X \in (IF)_{mn}, Y \in (IF)_{nm}$ such that $A = XBY, B = ZAX, XYX = X = XZX$.*

**Theorem 2.8** ([8]). *Let $A \in (IF)_{m\times n}$ be of the form $A = \langle A_\mu, A_\vartheta\rangle$. Then $A$ is regular $\Leftrightarrow$ $A_\mu$ is regular in $F^M_{m\times n}$ under max-min composition and $A_\vartheta$ is regular in $F^N_{m\times n}$ under min-max composition. $A_\mu = (a_{ij\mu}) \in F^M_{m\times n}$ as the membership part of $A$ and $A_\vartheta = (a_{ij\vartheta}) \in F^N_{m\times n}$ as the non-membership part of $A$.*

## 3. k-Pseudo Similar Intuitionistic Fuzzy Matrices

**Definition 3.1.** A matrix $A \in (IF)_n$, is said be right k-regular, if there exists a matrix $X \in (IF)_n$ such that $A^kXA = A^k$, for some positive integer $k$.

In this case, $X$ is called a right k-g-inverse of $A$.

Let $A_r\left\{1^k\right\} = \left\{X/A^kXA = A^k\right\}$.

**Definition 3.2.** A matrix $A \in (IF)_n$, is said be left k-regular, if there exists a matrix $Y \in (IF)_n$ such that $AYA^k = A^k$, for some positive integer $k$.

In this case, $Y$ is called a left k-g-inverse of $A$.

Let $A_\ell\left\{1^k\right\} = \left\{Y/AYA^k = A^k\right\}$.

In general, right k-regular is different from left k-regular. Then a right k-g-inverse need not be a left k-g-inverse (refer to Example 3.4). Thus forth we call a right k-regular (*or*) left k-regular IFM as a k-regular IFM.

**Example 3.3.** Let us consider $A = \begin{bmatrix} \langle 0.3, 0\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix} \in (IF)_2$, where $A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.2 \end{bmatrix} \in F_2^M$ and $A_\vartheta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in F_2^N$. Since each row of $A_\mu$ cannot be expressed as linear combination of the other row, by Definition 2.5 of (5) , the rows are linearly independent. Then by Definition 2.6 of (2) ,they form a standard basis for the row space of $A_\mu$.

For both permutation matrices $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_\mu P_1 A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.3 & 0.2 \end{bmatrix} \neq A_\mu$ and $A_\mu P_2 A_\mu = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.2 \end{bmatrix} \neq A_\mu$. Thus $A_\mu$ is not regular by step 3 in Algorithm 1 of (2) . Namely, $A_\mu$ is regular iff $A_\mu P A_\mu = A_\mu$, for some permutation matrix $P$. Since $A_\vartheta$ is idempotent, $A_\vartheta$ itself is a g-inverse of $A_\vartheta$, $A_\vartheta$ is regular under min max composition. So by Theorem 2.8, $A$ is not regular.

For this $A$, $A^2 = \begin{bmatrix} \langle 0.3, 0\rangle & \langle 0, 1\rangle \\ \langle 0.3, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$. For $X = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$, $A^2XA = A^2 = AXA^2$ holds. Hence $A$ is 2-regular.

**Example 3.4.** Let $A = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0.5\rangle & \langle 0, 0\rangle \\ \langle 0, 0\rangle & \langle 0, 1\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0, 0\rangle & \langle 0, 0\rangle \end{bmatrix}$. Then $A_\mu$ is not regular (See [9]). Thus by Theorem 2.8, $A$ is not regular.

For this $A$, $A^2 = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0\rangle & \langle 0.5, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0, 0.5\rangle & \langle 0, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0\rangle & \langle 0, 0\rangle \end{bmatrix}$, $A^3 = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0\rangle & \langle 0.5, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0\rangle & \langle 0, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0\rangle & \langle 0.5, 0\rangle \end{bmatrix}$.

For $X = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0, 1\rangle & \langle 0.5, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0.5\rangle & \langle 0.5, 0\rangle \end{bmatrix}$, $A^3XA = A^3 \neq AXA^3$ holds. So $X$ is a right 3-g inverse but $X$ is not a 3-g inverse of A.

**Theorem 3.5.** *Let $A = \langle A_\mu, A_\vartheta\rangle \in (IF)_n$. Then $A$ is right k-regular IFM $\Leftrightarrow$ $A_\mu, A_\vartheta \in F_n$ are right k-regular.*

*Proof.* Let $A = \langle A_\mu, A_\vartheta\rangle \in (IF)_n$. Since A is right k-regular IFM, there exists $X \in (IF)_n$, such that $A^kXA = A^k$.

Let$X = \langle X_\mu, X_\vartheta\rangle \in (IF)_n$ with $X_\mu, X_\vartheta \in F_n$. Then by Definition 2.2, $A^kXA = A^k$. Thus $\langle A_\mu, A_\vartheta\rangle^k \langle X_\mu, X_\vartheta\rangle \langle A_\mu, A_\vartheta\rangle = \langle A_\mu, A_\vartheta\rangle^k$,

$$\langle A_\mu^k, A_\vartheta^k\rangle \langle X_\mu, X_\vartheta\rangle \langle A_\mu, A_\vartheta\rangle = \langle A_\mu^k, A_\vartheta^k\rangle,$$
$$\langle A_\mu^k X_\mu A_\mu, A_\vartheta^k X_\vartheta A_\vartheta\rangle = \langle A_\mu^k, A_\vartheta^k\rangle,$$
$$A_\mu^k X_\mu A_\mu = A_\mu^k \text{ and } A_\vartheta^k X_\vartheta A_\vartheta = A_\vartheta^k.$$

So $A_\mu, A_\vartheta \in F_n$ are right k-regular.

Conversely, suppose $A_\mu, A_\vartheta \in F_n$ are right k-regular. Then $A_\mu^k X_\mu A_\mu = A_\mu^k$ and $A_\vartheta^k X_\vartheta A_\vartheta = A_\vartheta^k$, for some $X_\mu, X_\vartheta \in F_n$. Thus $X_\mu$ is a right k-g inverse of $A_\mu$ and $X_\vartheta$ is a right k-g inverse of $A_\vartheta$.

Now let us define the IFM $Z = \langle V, W\rangle$, where $V$ is a right k-g inverse of $A_\mu$ and $W$ is a right k-g inverse of $A_\vartheta$. We claim that Z is a right k-g inverse of A. Then by

Definition 2.2,

$$A^k ZA = \langle A_\mu, A_\vartheta\rangle^k \langle V, W\rangle \langle A_\mu, A_\vartheta\rangle = \langle A_\mu^k V A_\mu, A_\vartheta^k W A_\vartheta\rangle = \langle A_\mu^k, A_\vartheta^k\rangle = A^k.$$

Thus A is right k-regular IFM. So the proof is done. □

**Theorem 3.6.** *Let $A = \langle A_\mu, A_\vartheta\rangle \in (IF)_n$. Then A is left k-regular IFM $\Leftrightarrow A_\mu, A_\vartheta \in F_n$ are left k-regular.*

*Proof.* This can be proved along the same lines as that of Theorem 3.5. □

**Definition 3.7.** $A \in (IF)_n$ is said to be right k-pseudo similar to $B \in (IF)_n$, denoted by $A \cong_r^k B$, if there exist $X, Y \in (IF)_n$ such that $A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$.

**Definition 3.8.** $A \in (IF)_n$ is said to be left k-pseudo similar to $B \in (IF)_n$, denoted by $A \cong_\ell^k B$, if there exist $X, Y \in (IF)_n$ such that $A = X^kBY, B = YAX, XYX^k = X^k$ and $YXY = Y$.

**Remark 3.9.** In particular for k=1, Definitions 3.7 and 3.8 are identical. Then k-pseudo similar is reduced to Lemma 2.7. However, both right and left k-pseudo similarity of intuitionistic fuzzy matrices are not symmetric as in the case of pseudo similarity of intuitionistic fuzzy matrices.

**Lemma 3.10.** *Let $A, B \in (IF)_n$. If $A \cong_r^k B$, then we have the following:*
(1) $A^s = XB^sY$, *for any integer* $s \geq 1$,
(2) $BYX = YXB = B$,
(3) $AXY = XYA = A$,
(4) $B^s = YA^sX$, *for any integer* $s \geq 1$.

*Proof.* Since $A \cong_r^k B, A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$.
(1) Since $A = XBY$, $A^2 = (XBY)(XBY) = X(BYX)BY$. Thus

$$BYX = \left(YAX^k\right)YX = YA\left(X^kYX\right) = YAX^k = B.$$

So, $A^2 = (XBY)(XBY) = X(BYX)BY = XBBY = XB^2Y$. Hence in general, $A^s = XB^sY$, for any integer $s \geq 1$.
(2) $YXB = YX\left(YAX^k\right) = (YXY)AX^k = YAX^k = B$
and $BYX = \left(YAX^k\right)YX = YA\left(X^kYX\right) = YAX^k = B$.
(3) $AXY = (XBY)XY = XB(YXY) = XBY = A$
and $XYA = XY(XBY) = X(YXB)Y = XBY = A$.
(4) Clearly, $B = YXB$. Then $B^s = YXB^s$. Thus $B^s = YX(B^sYX) = Y(XB^sY)X = YA^sX$. □

**Lemma 3.11.** *Let $A, B \in (IF)_n$. If $A \cong_\ell^k B$. Then we have the following:*
(1) $B^s = YA^sX$, *for any integer* $s \geq 1$,
(2) $AXY = XYA = A$,
(3)) $BYX = YXB = B$,
(4) $A^s = XB^sY$, *for any integer* $s \geq 1$.

*Proof.* This can be proved as that of Lemma 3.10 and then omitted. □

**Theorem 3.12.** *Let $A, B \in (IF)_n$ such that $A \cong_r^k B$. $A$ is right(left) k-regular $\Leftrightarrow$ $B$ is right(left) k-regular.*

*Proof.* Since $A \cong_r^k B, A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. When the positive integers $k$ and $s$ are same in Lemma 3.10, we have

$$A^k = XB^kY, BYX = YXB = B, AXY = XYA = A$$

and

$$B^k = YA^kX, \text{ for any integer } k \geq 1.$$

Let A be right k-regular, i.e., $A^kGA = A^k$. Then $G$ is a right k-g-inverse of $A$. Choose $U = YGX$. We claim that $U$ is a right k-g-inverse of $B$. Then

$$\begin{aligned} B^kUB &= \left(YA^kX\right)(YGX)B = Y\left(A^kXY\right)G(XB) \\ &= YA^kG(XBYX) = YA^kG(AX) \\ &= Y\left(A^kGA\right)X = YA^kX = B^k. \end{aligned}$$

Conversely, assume that $B$ is right k-regular, i.e., $B^kUB = B^k$. Then $U$ is a right k-g-inverse of $B$. Choose $G = XUY$ We prove that , $G$ is a right k-g-inverse of $A$. Then

$$\begin{aligned} A^kGA &= \left(XB^kY\right)(XUY)(XBY) \\ &= X\left(B^kYX\right)U(YBX)Y \\ &= XB^kUBY = XB^kY \\ &= A^k. \end{aligned}$$

On the other hand, $A$ is left k-regular $\Leftrightarrow$ $B$ is left k-regular can be proved in the same manner. Thus the proof is done. □

**Theorem 3.13.** *Let $A, B \in (IF)_n$ such that $A \cong_\ell^k B$. Then $A$ is right(left) k-regular $\Leftrightarrow$ $B$ is right(left) k-regular.*

*Proof.* This can be proved as that of Theorem 3.12 and then omitted. □

**Remark 3.14.** For $k = 1$, Theorems 3.12 and 3.13 reduces to the following.

**Theorem 3.15** ([3])**.** *Let $A \in (IF)_m$ and $B \in (IF)_n$ such that $A \cong B$. Then $A$ is a regular matrix $\Leftrightarrow$ $B$ is a regular matrix.*

**Lemma 3.16.** *Let $A, B \in (IF)_n$ and suppose $A \cong_r^k B$. Then there exist $X, Y \in (IF)_n$ such that $A = XBY, B = YAX^k$ and $XY$ is $k$-potent.*

*Proof.* Since $A \cong_r^k B, A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. Then

$$\begin{aligned} (XY)^k &= (XY)^{k-1}XY \\ &= (XY)^{k-2}XYXY \\ &= (XY)^{k-2}X(YXY) \\ &= (XY)^{k-2}XY \\ &= \ldots\ldots. \\ &= XY. \end{aligned}$$

Thus the proof is done. □

**Remark 3.17.** The converse of the above Lemma need not be true. This is illustrated in the following.

**Example 3.18.** Let us consider $X = \begin{bmatrix} \langle 0.3, 0.3\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.2, 0.2\rangle \end{bmatrix}$ and $Y = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0\rangle \end{bmatrix}$. For $A = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.2, 0.3\rangle & \langle 0.2, 0.3\rangle \end{bmatrix}$ and $B = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.2, 0.3\rangle & \langle 0.2, 0.3\rangle \end{bmatrix}$, $A = XBY, B = YAX^2$ and $(XY)^2 = XY$. Then XY is 2-potent, but $X^2YX \neq X^2$ and $YXY \neq Y$. Here A is not right 2-pseudo similar to B.

**Lemma 3.19.** *Let $A, B \in (IF)_n$. If $A \cong_\ell^k B$, then there exist $X, Y \in (IF)_n$ such that $A = X^kBY, B = YAX$ and $YX$ is k-potent.*

*Proof.* Since $A \cong_\ell^k B$, $A = X^kBY, B = YAX, XYX^k = X^k$ and $YXY = Y$. Thus

$$\begin{aligned}(YX)^k &= (YX)^{k-1}YX \\ &= (YX)^{k-2}YXYX \\ &= (YX)^{k-2}(YXY)X \\ &= (YX)^{k-2}YX \\ &= \dots\dots \\ &= YX.\end{aligned}$$

So the proof is done. □

**Theorem 3.20.** *Let $A, B \in (IF)_n$. Then the following are equivalent:*
(1) $A \cong_r^k B$,
(2) $B^T \cong_\ell^k A^T$,
(3) $PAP^T \cong_r^k PBP^T$, *for some permutation matrix* $P \in (IF)_n$.

*Proof.* (1) ⇔ (2) : This is direct by taking transpose on both sides and by using $\left(A^T\right)^T = A$ and $(AX)^T = X^TA^T$.

(2) ⇔ (3) : Suppose $A \cong_r^k B$. Then $A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. Thus

Since $A = XBY$,

$$PAP^T = PXBYP^T = \left(PXP^T\right)\left(PBP^T\right)\left(PYP^T\right). \tag{3.1}$$

Since $B = YAX^k$,

$$\begin{aligned}PBP^T &= PYAX^kP^T = \left(PYP^T\right)\left(PAP^T\right)\left(PX^kP^T\right) \\ &= \left(PYP^T\right)\left(PAP^T\right)\left(PXP^T\right)^k.\end{aligned} \tag{3.2}$$

Since $X^kYX = X^k$, $PX^kP^T = PX^kYXP^T$. Thus

$$PX^kP^T = \left(PX^kP^T\right)\left(PYP^T\right)\left(PXP^T\right). \tag{3.3}$$

On the other hand, $\left(PXP^T\right)^k = \left(PXP^T\right)^k\left(PYP^T\right)\left(PXP^T\right)$. Since $Y = YXY$, $PYP^T = PYXYP^T$. Thus

$$PYP^T = \left(PYP^T\right)\left(PXP^T\right)\left(PYP^T\right). \tag{3.4}$$

So $PAP^T \cong_r^k PBP^T$.

Conversely, suppose $PAP^T \cong_r^k PBP^T$. Pre multiply by $P^T$ and post multiply by $P$ in Equations (3.1) to (3.4), we get $A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. Then $A \cong_r^k B$. Thus the proof is done. □

**Example 3.21.** The above Theorem 3.20 is illustrated in this example.
Let us consider $A = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}, B = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}, X = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.3, 0.5\rangle \end{bmatrix}$ and $Y = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$. Here $X \neq XYX$.
For this $X$, $X^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$.

Now $A = XBY = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$,

$$B = YAX^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix},$$

$$X^2 = X^2YX = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$$

and

$$Y = YXY = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}.$$

Then $A \cong_r^k B$.

For a given A and B,

$$A^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}, B^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix},$$

$$X^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.3, 0.5\rangle \end{bmatrix}, Y^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$$

and

$$\left(X^T\right)^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}. \text{ For this } X^T, Y^T \in (IF)_2\,,$$

$$A^T = Y^TB^TX^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix},$$

$$B^T = \left(X^T\right)^2 A^TY^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix},$$

$$\left(X^T\right)^2 = X^TY^T\left(X^T\right)^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix},$$

$$Y^T = Y^TX^TY^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}.$$

Thus $B^T \cong_\ell^k A^T$.

Consider a intuitionistic fuzzy permutation matrix $P = \langle P_\mu, P\vartheta\rangle = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 1, 0\rangle \end{bmatrix}$.
For this $P$, $P^T = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 1, 0\rangle \end{bmatrix}$. On the other hand,

$$PAP^T = X\left(PBP^T\right)Y = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$$

and

$$PBP^T = Y\left(PAP^T\right)X^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}.$$

So $PAP^T \cong_r^k PBP^T$ for some permutation matrix $P \in (IF)_n$.

**Theorem 3.22.** *Let $A, B \in (IF)_n$. Then the following are equivalent:*

(1) $A \cong_{\ell}^{k} B$,
(2) $B^T \cong_{r}^{k} A^T$,
(3)) $PAP^T \cong_{\ell}^{k} PBP^T$, *for some permutation matrix* $P \in (IF)_n$.

*Proof.* Proof of the theorem is similar to Theorem 3.20 and hence omitted. □

**Theorem 3.23.** *Let* $A, B \in (IF)_n$. *If* $A \cong_{r}^{k} B$, *then* $A^s \cong_{r}^{k} B^s$, *for any integer* $s \geq 1$.

*Proof.* Suppose $A \cong_{r}^{k} B$. Then $A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$.
Prove that, $A^s \cong_{r}^{k} B^s$. By Lemma 3.10(1), $A^s = XB^sY$, for any integer $s \geq 1$.
Next prove that, $B^s = YA^sX^k$.
By Lemma 3.10(2), $BYX = YXB = B$. Then

$$B^s = YXB^s = YXB^{s-1}B = YXB^{s-1}\left(YAX^k\right)$$
$$= Y\left(XB^{s-1}Y\right)AX^k = Y\left(A^{s-1}\right)AX^k$$
$$= YA^sX^k.$$

Thus $A^s \cong_{r}^{k} B^s$, for any integer $s \geq 1$. So the proof is done. □

**Remark 3.24.** The converse of the above theorem need not be true. This is illustrated in the following.

**Example 3.25.** Let us consider $X = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.3, 0.5\rangle \end{bmatrix}$
and $Y = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$. For $X^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$,
$XYX \neq X, X^2YX = X^2$ and $YXY = Y$.
For $A = \begin{bmatrix} \langle 0.3, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$ and $B = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$,
$A^2 = XB^2Y$ and $B^2 = YA^2X^2$.
Then $A^2$ is right 2-pseudo similar to $B^2$. But $A \neq XBY$ and $B = YAX^2$. Here A is not right 2-pseudo similar to B.

**Theorem 3.26.** *Let* $A, B \in (IF)_n$. *If* $A \cong_{\ell}^{k} B$, *then* $A^s \cong_{\ell}^{k} B^s$, *for any integer* $s \geq 1$.

*Proof.* This is similar to Theorem 3.23 and then omitted. □

**Theorem 3.27.** *Let* $A, B, C \in (IF)_n$. *If* $A \cong_{r}^{k} B$ *and* $B \cong_{r}^{k} C$, *then* $A \cong_{r}^{k} C$ *and if there exist matrices* $X, Y, Z$ *and* $L$ *with* $Y \in X\left\{1_{r}^{k}\right\}, Z \in L\left\{1_{r}^{k}\right\}, X \in Y\{1\}, L \in Z\{1\}$, *and* $XL = LX$ *satisfying any one of the following:*
(1) $LZY = Y$,
(2) $ZYX = Z$,
(3) $XLZ = X$,
(4) $YXL = L$.

*Proof.* Since $A \cong_{r}^{k} B, A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. Since $B \cong_{r}^{k} C, B = LCZ, C = ZBL^k, L^kZL = L^k$ and $ZLZ = Z$. Thern,

$$A = XBY = X\left(LCZ\right)Y = \left(XL\right)C\left(ZY\right).$$

and

$C = ZBL^k = Z\left(YAX^k\right)L^k = (ZY)\,A\left(X^kL^k\right) = (ZY)\,A\,(XL)^k\,.$

To prove, $A \cong_r^k C$, it is enough to prove that $ZY \in (XL)\left\{1_r^k\right\}$ and $XL \in (ZY)\{1\}$.

Suppose (1) holds. Then

$$\begin{aligned}(XL)^k(ZY)(XL) &= X^kL^k(ZY)(XL) = X^kL^{k-1}(LZY)(XL)\\ &= X^kL^{k-1}(Y)(XL) = L^{k-1}X^k(Y)(XL)\\ &= L^{k-1}\left(X^kYX\right)L = L^{k-1}X^kL\\ &= (XL)^k\end{aligned}$$

and

$$(ZY)(XL)(ZY) = ZYX(LZY) = ZYXY = Z(YXY) = ZY.$$

Suppose (2) holds. Then

$$\begin{aligned}(XL)^k(ZY)(XL) &= (XL)^k(ZYX)L\\ &= (XL)^kZY = X^kL^kZL\\ &= X^kL^k = (XL)^k\end{aligned}$$

and

$$(ZY)(XL)(ZY) = (ZYX)LZY = ZLZY = (ZLZ)Y = ZY.$$

Suppose (3) holds. Then

$$\begin{aligned}(XL)^k(ZY)(XL) &= (XL)^{k-1}(XL)(ZY)(XL)\\ &= (XL)^{k-1}(XLZ)(YXL)\\ &= (XL)^{k-1}XYXL = L^{k-1}X^{k-1}XYXL\\ &= L^{k-1}X^kYXL = L^{k-1}X^kL\\ &= (XL)^k\end{aligned}$$

and

$$(ZY)(XL)(ZY) = ZY(XLZ)Y = ZYXY = Z(YXY) = ZY.$$

Suppose (4) holds. Then

$$\begin{aligned}(XL)^k(ZY)(XL) &= (XL)^kZ(YXL)\\ &= (XL)^kZL = X^kL^kZL = X^kL^k\\ &= (XL)^k\end{aligned}$$

and $\quad (ZY)(XL)(ZY) = Z(YXL)ZY = ZLZY = (ZLZ)Y = ZY.$

Thus the proof is done. □

**Theorem 3.28.** *Let $A, B, C \in (IF)_n$. If $A \cong_\ell^k B$ and $B \cong_\ell^k C$, then $A \cong_\ell^k C$ and if there exist matrices $X, Y, Z$ and $L$ with $Y \in X\left\{1_\ell^k\right\}, Z \in L\left\{1_\ell^k\right\}, X \in Y\{1\}, L \in Z\{1\}$, and $XL = LX$ satisfying any one of the following:*

(1) $LZY = Y$,
(2) $ZYX = Z$,
(3) $XLZ = X$,
(4) $YXL = L$.

*Proof.* This is similar to that of Theorem 3.27 and then omitted. □

## 4. Conclusion

In this paper, the concept of k-regular intuitionistic fuzzy matrix as a generalization of regular intuitionistic fuzzy matrix is introduced. k-pseudo similar intuitionistic fuzzy matrix is defined and the properties are discussed.

## References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and System 20 (1986) 87–96.
[2] H. H. Cho, Regular fuzzy matrices and fuzzy equations, Fuzzy Sets and Systems 105 (1999) 445–451.
[3] T. Gandhimathi and AR. Meenakshi, Pseudo similar intuitionistic fuzzy matrices, Applied and Computational Mathematics, 4 (1-2) (2015) 15–19.
[4] S. Khan and Anita paul, The Genaralised inverse of intuitionistic fuzzy matrices, Journal of Physical Science II (2007) 62–67.
[5] K. H. Kim and F. W. Roush, On generalized fuzzy matrices, Fuzzy Sets and Systems 4 (1980) 293–375.
[6] H. Y. Lee and N. G. Jeong, Canonical form of a transitive intuitionistic fuzzy matrices, Honam Mathem. J. 27 (4) (2005) 543–550
[7] AR. Meenakshi, Fuzzy Matrix Theory and Applications, MJP Publishers, Chennai 2008.
[8] AR. Meenakshi and T. Gandhimathi, On regular intuitionistic fuzzy matrices, International Journal of Fuzzy Mathematics 19 (2) (2011) 599–605.
[9] AR. Meenakshi and P. Jenita, Generalized regular fuzzy matrices, Iranian Journal of Fuzzy Systems 8 (2) (2011) 133–141.
[10] Riyaz Ahmad Padder and P. Murugadas, On idempotent intuitionistic fuzzy matrices, of T-type, International Journal of Fuzzy Logic and Intelligent Systems 16 (3) (2016) 181–187.
[11] Riyaz Ahmad Padder and P. Murugadas, Reduction of a nilpotent intuitionistic fuzzy matrix using implication operator, Application of Applied Mathematics 11 (2) (2016) 614–631.
[12] Riyaz Ahmad Padder and P. Murugadas, Generalization of Szpilrajn's theorem on intuitionistic fuzzy matrix, Journal of Mathematics and Informatics 6 (2016) 7–14.
[13] Riyaz Ahmad Padder and P. Murugadas, Max-max operations on intuitionistic fuzzy matrix, Ann. Fuzzy Math.Inform. 12 (6) (2016) 757–766.
[14] M. Pal, S. K. Khan and A. K. Shyamal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets 8 (2) (2002) 51–62.

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