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Multigroup actions on multiset

A.M. Ibrahim, P.A. Ejegwa

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Abstract. We propose the notion of actions of multigroup on multiset via count function of multiset with elaborate illustrations. The concepts of orbits and stabilizers in multigroup actions are introduced and some related results are obtained. The analogous of orbit-stabilizer theorem and class equation are established in multigroup actions on multiset.

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Corresponding Author: P.A. Ejegwa (ejegwa.augustine@uam.edu.ng)

1. Introduction

The notion of multisets emerged as a result of violating a basic underlying principle (i.e. principle of distinct element in a collection) in set theory. Multiset is an unordered collection of elements in which elements can occur more than once. The term multiset as Knuth [7] noted was first suggested by N.G. de Bruijn in a private correspondence to him. In literature, variety of terms such as list, heap, bunch, bag, sample, weighted set, occurrence set, etc are used in different contexts but conveying the same meaning as multiset. See [17, 18, 21] for details.

In [11], the concept of multigroups via multisets was introduced as a generalization of groups. Multigroup constitutes an application of multisets to the elementary theory of groups. The notion is consistent with other non-classical groups in [4, 13, 15, 16], etc.

Although other researchers in [3, 5, 10, 12, 14, 19, 20] earlier used the term multigroup as an extension of group theory (with each of them having a divergent view), the notion of multigroup in [11] is quite acceptable because it is in consonant with the aforementioned non-classical groups and defined over multiset. A complete survey on the concept of multigroups from various authors were reviewed in [9]. Further studies on the concept of multigroups via multisets and cuts of multigroups have been studied. See [1, 2, 6] for details.

In this paper, we introduce the notion of actions of multigroup on multiset via count function of multiset and obtain some related results. The analogous of orbit-stabilizer theorem and class equation are established in multigroup actions.

This paper is organized as follows: In Section 2, some preliminary definitions and results on multisets, multigroups and group actions are reviewed. Section 3 introduces the concept of actions in multigroup context and exemplify it. Meanwhile, Section 4 contains the main results.

2. Preliminaries

Definition 2.1 ([17]). Let X be a set. A multiset A is characterized by a count function

$$C_A : X \to \mathbb{N},$$

such that for $x \in Dom(A)$ implies $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$.

The set of all multisets of X is denoted by $MS(X)$.

Definition 2.2 ([18, 21]). Let $A, B \in MS(X)$. Then A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 2.3 ([11]). Let X be a group. A multiset A over X is called a multigroup of X if it satisfies the following conditions:

(i) $C_A(xy) \geq C_A(x) \wedge C_A(y) \; \forall x, y \in X$,
(ii) $C_A(x^{-1}) \geq C_A(x) \; \forall x \in X$.

It follows immediately from [11] that,

$$C_A(x^{-1}) = C_A(x) \; \forall x \in X$$

since

$$C_A(x) = C_A((x^{-1})^{-1}) \geq C_A(x^{-1}).$$

Also,

$$C_A(x) \leq C_A(e) \; \forall x \in X$$

because

$$C_A(e) = C_A(xx^{-1}) \geq C_A(x) \wedge C_A(x) = C_A(x),$$

where e is the identity element of X. We denote the set of all multigroups of X by $MG(X)$. Every multigroup is a multiset but the converse is not necessarily true.

Definition 2.4 ([6]). Let $A \in MG(X)$. A submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Definition 2.5 ([11]). Let $A \in MG(X)$. Then the sets A_* and A^* are defined as

$$A_* = \{x \in X \mid C_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid C_A(x) = C_A(e)\},$$

where e is an identity element of X.

A multigroup A is completely defined over X whenever $A_* = X$. From [11], A_* and A^* are subgroups of X.

Definition 2.6. Let A be a multigroup of a group X. Then the center of A is defined as

$$C(A) = \{x \in X \mid C_A([x, y]) = C_A(e) \forall y \in X\}.$$

Definition 2.7. Let $A \in MG(X)$ and $x, y \in X$. Then x and y are called conjugate elements in A if for some $z \in X$,

$$C_A(x) = C_A(zyz^{-1}).$$

Definition 2.8 ([11])**.** Let $A \in MG(X)$. Then A is said to be commutative if for all $x, y \in X$,

$$C_A(xy) = C_A(yx).$$

Definition 2.9. Let X be a group. For any submultigroup A of a multigroup G of X, the submultiset yA of G for $y \in X$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the left comultiset of A. Similarly, the submultiset Ay of G for $y \in X$ defined by

$$C_{Ay}(x) = C_A(xy^{-1}) \forall x \in A_*$$

is called the right comultiset of A.

Definition 2.10. Let X and Y be groups, $A \in MG(X)$ and $B \in MG(Y)$, respectively. The direct product of A and B depicted by $A \times B$ is a function

$$C_{A \times B} : X \times Y \to \mathbb{N}$$

defined by

$$C_{A \times B}(x, y) = C_A(x) \wedge C_B(y) \forall x \in X, \forall y \in Y.$$

Definition 2.11 ([8])**.** Let G be a group and T be a set. An action of a group G on a set T is a binary operation

$$\diamond : G \times T \to T,$$

satisfying $\forall\, e, g_1, g_2 \in G$ and $t \in T$

(i) $e \diamond t = t$,
(ii) $g_1 \diamond (g_2 \diamond t) = (g_1 g_2) \diamond t$.

Theorem 2.12 ([8])**.** *Let a group G act on a set T. Then $g \diamond t = t' \Leftrightarrow t = g^{-1} \diamond t$ $\forall\, g \in G$ and $t, t' \in T$.*

Theorem 2.13 ([8])**.** *Let a group G act on a set T. Suppose that $G(t)$ is the stabilizer of t. Then for any $g_1, g_2 \in G$, $g_1 \diamond t = g_2 \diamond t \Leftrightarrow g_2^{-1} g_1 \in G(t)$.*

3. Concept of multigroup actions

Let X be a group and Y be a set such that $X \subseteq Y$, $A \in MG(X)$ and $B \in MS(Y)$ respectively. We assume that $X = A_*$, $Y = B_*$ and the count of all elements in A is equal to the count of the corresponding elements in B with the exception of the identity of X in A.

Definition 3.1. Let X be a group and Y be a set. Suppose $A \in MG(X)$ and $B \in MS(Y)$ respectively. Then the action of A on B is an operation $\diamond$ that takes the count of $A \times B$ to B denoted by $C_{A\times B}(g, x) \mapsto C_B(g \diamond x)$ such that $\forall e, g, h \in X$ and $\forall x \in Y$, the following are satisfied;

(i) $C_B(e \diamond x) = C_B(x) \Leftrightarrow e \diamond x = x \in Y$,
(ii) $C_B(g \diamond (h \diamond x)) = C_B((gh) \diamond x) \Leftrightarrow g \diamond (h \diamond x) = (gh) \diamond x \in Y$.

This action of A on B is a left action; the right action is similar.

Remark 3.2. Let X be a group and Y be a set. Suppose $A \in MG(X)$ and $B \in MS(Y)$ respectively. If A acts on B, then A_* acts on B_*. In fact A acts on B if and only if A_* acts on B_*.

The notion of multigroup actions is indeed, the extension of group actions on a set. See [8] for details on group actions.

Remark 3.3. An action is trivial if $C_B(g \diamond x) = C_B(x) \forall g \in X$. An action is faithful or effective if for every two distinct $g, h \in X$ there exists an $x \in Y$ such that $C_B(g \diamond x) \neq C_B(h \diamond x)$; or equivalently, if for each $g \neq e$ in X there exists an $x \in Y$ such that $C_B(g \diamond x) \neq C_B(x)$.

Example 3.4. Let $X = \{0, 1, 2, 3\}$ be a group of modulo 4 with respect to addition and $Y = \{0, 1, 2, 3, 4\}$ be a set. Suppose

$$A = [0^4, 1^3, 2^2, 3^3]$$

and

$$B = [0^2, 1^3, 2^2, 3^3, 4^2]$$

be multigroup and multiset of X and Y respectively. It follows from Definition 3.1 that, A acts on B via the operation of X.

Example 3.5. Suppose B is a multiset over a set Y and consider A to be a multigroup of X. Then A acts on B as follows. For each $f \in X$ and each $x \in Y$, we define

$$C_B(f \diamond x) = C_B(f(x)).$$

(i) For $e \in X$, we have

$$C_B(e \diamond x) = C_B(e(x)) = C_B(x)$$

as desired.

(ii) Also, if $f, g \in X$, then

$$C_B(f \diamond (g \diamond x)) = C_B(f(g \diamond x)) = C_B(f(g(x))) = C_B(f \diamond g(x)) = C_B((fg) \diamond x)$$

as desired.

Example 3.6. Let A and B be multigroup and multiset of X respectively such that $B = A$. Then for $g, x \in X$, every multigroup A of a group X acting on itself by conjugation is defined by

$$C_B(g \diamond x) = C_B(gxg^{-1}) = C_A(gxg^{-1}).$$

Condition (i) holds since for $e \in X$

$$C_A(e \diamond x) = C_A(exe^{-1}) = C_A(x) \forall x \in X.$$

And condition (ii) holds as follows. Suppose $g, h, x \in X$. Then

$$\begin{aligned} C_A(g \diamond (h \diamond x)) &= C_A(g \diamond (hxh^{-1})) \\ &= C_A(g(hxh^{-1})g^{-1}) \\ &= C_A((gh)x(h^{-1}g^{-1}) \\ &= C_A((gh)x(gh)^{-1}) \\ &= C_A((gh) \diamond x) \end{aligned}$$

as desired.

Note that if A is commutative, then the multigroup action is trivial, in other words $C_A(g \diamond x) = C_A(x)\ \forall g, x \in X$.
From now henceforth, we simply write A and B as multigroup and multiset of X and Y respectively except otherwise stated.

Proposition 3.7. *Let A acts on B. If $x \in Y$, $g \in X$ and $C_B(y) = C_B(g \diamond x)$, then $C_B(x) = C_B(g^{-1} \diamond y)$. Again, $C_B(g \diamond x) \neq C_B(g \diamond x')$ if and only if $C_B(x) \neq C_B(x')$.*

Proof. Suppose $C_B(y) = C_B(g \diamond x)$. Then we have

$$C_B(g^{-1} \diamond y) = C_B(g^{-1} \diamond (g \diamond x)) = C_B((g^{-1}g) \diamond x) = C_B(x).$$

We show the proof of the second by contrapositive. Let $C_B(g \diamond x) = C_B(g \diamond x')$. Then $C_B(g^{-1} \diamond (g \diamond x)) = C_B(x')$, that is,

$$C_B((g^{-1}g) \diamond x) = C_B(x') \Rightarrow C_B(x) = C_B(x').$$

Conversely, suppose $C_B(x) = C_B(x')$. Then $C_B(e \diamond x) = C_B(x')$. Thus $C_B((g^{-1}g) \diamond x) = C_B(x')$. So $C_B(g^{-1} \diamond (g \diamond x)) = C_B(x')$. Hence $C_B(g \diamond x) = C_B(g \diamond x')$. □

4. Orbits and stabilizers in multigroup actions

Definition 4.1. Let A and B be multigroup and multiset of a group X and a set Y respectively. Suppose that A acts on B on the left and if we fix $x \in Y$. Then the set

$$\begin{aligned} Orb_A(x) &= \{y \in Y \mid C_B(y) = C_B(g \diamond x) \text{ for some } g \in X\} \\ &= \{g \diamond x \mid C_B(g \diamond x) \ exist\} \subset Y \end{aligned}$$

is called the orbit of x (under A).

Example 4.2. Using Example 3.4 and fixing $4 \in Y$, we have $Orb_A(4) = \{1, 3\}$ for $g = 3$.

Remark 4.3. Let A acts on B. Then for $x \in Y$, $Orb_A(x) = B_*$ for every element in A. Consequently, $|B_*| = \Sigma|Orb_A(x)|$. Again, two elements x and y in Y are equivalent if and only if their orbits are the same, that is, $Orb_A(x) = Orb_A(y)$.

Remark 4.4. The action of A on B is said to be transitive or A is said to act transitively on B, if there is only one orbit, that is, for any two elements x and y in Y, there exists a $g \in X$ such that $C_B(g \diamond x) = C_B(y)$. The action of A on B is doubly transitive if for any two pairs $(x_1, x_2), (y_1, y_2)$ of elements of B with $x_1 \neq x_2$ and $y_1 \neq y_2$, there exists $g \in X$ such that $C_B(g \diamond x_1) = C_B(y_1)$ and $C_B(g \diamond x_2) = C_B(y_2)$.

Lemma 4.5. *Let A acts on B. If $x \in Y$, then $x \in Orb_A(x)$.*

Proof. Let $x \in Y$. Since $C_B(x) = C_B(e \diamond x)$, $x \in Orb_A(x)$. □

Given an action, define a relation on B by $x \sim y$ if and only if $\exists\, g \in X$ such that $C_B(y) = C_B(g \diamond x)$. We prove that this defines an equivalence relation on B.

Proposition 4.6. *Let A acts on B. If a relation $x \sim y$ in B is defined by*

$$C_B(g \diamond x) = C_B(y),$$

for some $g \in X$ and $x, y \in Y$. Then $\sim$ is an equivalence relation. Furthermore, the equivalence class of any $x \in Y$ is an $Orb_A(x)$.

Proof. Since $A \neq \varnothing$, $e \in X$ and A acts on B such that

$$C_B(e \diamond x) = C_B(x),$$

$x \sim x$. Then $\sim$ is reflexive.

Suppose $x \sim y$. Then for $g, g^{-1} \in X$, we get

$$C_B(g \diamond x) = C_B(y) \Leftrightarrow C_B(g^{-1} \diamond y) = C_B(x).$$

Thus $y \sim x$, i.e., $\sim$ is symmetric.

Now, suppose $x \sim y$ and $y \sim z$. Then we have $g, g_1 \in X$ such that

$$C_B(g \diamond x) = C_B(y)$$

and

$$C_B(g_1 \diamond y) = C_B(z).$$

Thus we can find $g_2 = g_1 g \in X$ such that

$$C_B(z) = C_B(g_1 \diamond (g \diamond x)) = C_B((g_1 g) \diamond x) = C_B(g_2 \diamond x).$$

So $x \sim z$, i.e., $\sim$ is transitive. Hence $\sim$ is an equivalence relation.

For any $x \in Y$, the equivalence class of x is the set

$$\{y \in B \mid y \sim x\} = \{y \in B \mid C_B(y) = C_B(g \diamond x)\} = Orb_A(x).$$

□

Remark 4.7. We infer that B_* can be partitioned into disjoint equivalence classes called orbits. Since B_* can breaks up into a disjoint union of equivalence classes under an equivalence relation, it implies that if A acts on B, then B_* is a union of disjoint orbits.

Remark 4.8. Let A acts on itself by conjugation. Then the orbit of X, that is,

$$Orb_A(x) = \{gxg^{-1} \mid C_A(gxg^{-1}) \text{ exist for } g \in X\}$$

is the conjugacy class of x.

Theorem 4.9. *Let A acts on B (on the left). Then the (distinct) orbits of elements in B under A partition B_*.*

Proof. To prove this, we need to show that

(i) every element of B is in some orbit,
(ii) if $Orb_A(x) \cap Orb_A(y) \neq \varnothing$, then $Orb_A(x) = Orb_A(y)\ \forall x, y \in Y$.

These prove that B_* is covered by disjoint orbits. The proof of (i) holds since $x \in Orb_A(x)$ and so every element is in some orbit.

Now we prove (ii). Suppose $z \in Orb_A(x) \cap Orb_A(y)$, that is, $Orb_A(x)$ and $Orb_A(y)$ have a common element z. Then $\exists\, g_1, g_2 \in X$ such that

$$C_B(g_1 \diamond x) = C_B(z);\ C_B(g_2 \diamond y) = C_B(z),\ \text{i.e.,}$$

$$C_B(g_1 \diamond x) = C_B(z) = C_B(g_2 \diamond y).$$

We want to show that $Orb_A(x) = Orb_A(y)$. For any point $u \in Orb_A(x)$, we have $C_B(u) = C_B(g \diamond x)$ for some $g \in X$. Since $C_A(x) = C_B(g_1^{-1} \diamond z)$,

$$\begin{aligned} C_B(u) = C_B(g \diamond (g_1^{-1} \diamond z)) &= C_B((gg_1^{-1}) \diamond z) \\ &= C_B((gg_1^{-1}) \diamond (g_2 \diamond y)) \\ &= C_B((gg_1^{-1}g_2) \diamond y). \end{aligned}$$

Thus $u \in Orb_A(y)$. So $Orb_A(x) \subseteq Orb_A(y)$.

Similarly, we have $Orb_A(y) \subseteq Orb_A(x)$ by symmetry. Hence, the result follows. □

Definition 4.10. Let A acts on B. An element $g \in X$ stabilizes $x \in Y$ if $C_B(g \diamond x) = C_B(x)$. That is, the stabilizer of $x \in Y$, denoted by $Stab_A(x)$, is defined by

$$Stab_A(x) = \{g \in X \mid C_B(g \diamond x) = C_B(x)\} \subset X.$$

Remark 4.11. For every $x \in Y$ with $C_B(g \diamond x) = C_B(x)$, we say x is a fixed point of $g \in X$ and g fixes x. More generally, for any submultiset $T \subseteq B$, we can consider elements of A which fix T thus:

$$Fix_A(T) = \{g \in X \mid C_T(g \diamond x) = C_T(x) \forall x \in Y\}.$$

We note that $Stab_A(x)$ and $Fix_A(T)$ are subsets of X. An action is said to be free if

$$Stab_A(x) = \{e\} \forall x \in Y.$$

Example 4.12. Using Example 3.4 and fixing $4 \in Y$, we have $Stab_A(4) = \{0, 2\}$.

Proposition 4.13. *Let A acts on B. Then $Stab_A(g \diamond x) = gStab_A(x)g^{-1}$ for any $x \in Y$ and $g \in X$.*

Proof. For any $x \in Y$ and $g \in X$, it follows that

$$\begin{aligned} h \in Stab_A(g \diamond x) &\Leftrightarrow C_B(h \diamond (g \diamond x)) = C_B(g \diamond x) \\ &\Leftrightarrow C_B((hg) \diamond x) = C_B(g \diamond x) \\ &\Leftrightarrow C_B(g^{-1} \diamond (hg) \diamond x) = C_B(x) \\ &\Leftrightarrow C_B((g^{-1}hg) \diamond x) = C_B(x) \\ &\Leftrightarrow g^{-1}hg \in Stab_A(x) \\ &\Leftrightarrow h \in gStab_A(x)g^{-1}. \end{aligned}$$

This completes the proof. □

Theorem 4.14. *Let A acts on B and $T \subseteq B$. Then $Stab_A(x)$ and $Fix_A(T)$ are subgroups of a group X.*

Proof. If $g, h \in Stab_A(x)$, then we need to prove that $gh \in Stab_A(x)$. In other words, we need to show that

$$C_B((gh) \diamond x) = C_B(x).$$

Using the fact that there is a multigroup action and since g and h stabilize x, we have

$$C_B((gh) \diamond x) = C_B(g \diamond (h \diamond x)) = C_B(g \diamond x) = C_B(x).$$

This proves that fact $gh \in Stab_A(x)$. Also, $e \in Stab_A(x)$, since

$$C_B(e \diamond x) = C_B(x).$$

Finally, we need to prove the existence of inverse. We investigate whether if $g \in Stab_A(x)$, we can find $g^{-1} \in Stab_A(x)$. That is, whether $C_B(g \diamond x) = C_B(x)$ can yields $C_B(g^{-1} \diamond x) = C_B(x)$. Now,

$$C_B(g \diamond x) = C_B(x) \Rightarrow C_B(g^{-1} \diamond (g \diamond x)) = C_B(g^{-1} \diamond x).$$

However,

$$C_B(g^{-1} \diamond (g \diamond x)) = C_B((gg^{-1}) \diamond x) = C_B(x).$$

Then $C_B(g^{-1} \diamond x) = C_B(x)$. Thus $g^{-1} \in Stab_A(x)$. So $Stab_A(x)$ is a subgroup of X. Similarly, $Fix_A(T)$ is a subgroup of X. □

Remark 4.15. It follows that $\dfrac{\mid A_* \mid}{\mid Stab_A(x) \mid}$, that is, $[A_* : Stab_A(x)]$ and

$$\mid Stab_A(x) \mid = \mid gStab_A(x) \mid = \mid Stab_A(x)g \mid \forall g \in X.$$

While $Stab_A(x)$ is a subgroup of X, it is almost never normal (except A_* is abelian).

Theorem 4.16. *Let A and B be multigroup and multiset respectively. If A acts on B and $T \subseteq B$, then*

$$Fix_A(T) = \bigcap_{x \in Y} Stab_A(x).$$

Proof. Suppose $g \in Fix_A(T)$. Then $C_T(g \diamond x) = C_T(x), \forall x \in Y$. Thus $g \in Stab_A(x)$. So $g \in \bigcap_{x \in Y} Stab_A(x)$. Hence

$$Fix_A(T) \subseteq \bigcap_{x \in Y} Stab_A(x).$$

Also, suppose that $g \in \bigcap_{x \in Y} Stab_A(x)$. Then $g \in Stab_A(x) \forall x \in Y$. Thus

$$C_T(g \diamond x) = C_T(x) \forall x \in Y.$$

So $g \in Fix_A(T)$. Hence

$$\bigcap_{x \in Y} Stab_A(x) \subseteq Fix_A(T).$$

Therefore, the equality holds. □

Proposition 4.17. *Let X be a group. Suppose a multigroup A of X acts on itself by conjugation such that $C_A(g \diamond x) = C_A(gxg^{-1})$. Then the following statements are equivalent:*

(1) $x \in C(A)$,
(2) $Stab_A(x) = X$,
(3) $Orb_A(x) = \{x\}$,
(4) $\mid Orb_A(x) \mid = 1$.

Proof. Let $x \in C(A)$. Then $C_A(xg) = C_A(gx), \ \forall g \in X$. Thus

$$C_A(x) = C_A(gxg^{-1}) = C_A(g \diamond x), \ \forall g \in X.$$

This proves that (1) implies (2) and (3). Likewise since $x \in Orb_A(x)$ always, (4) and (3) are clearly equivalent.

Again, if (3) holds, then

$$C_A(g \diamond x) = C_A(x),$$

for all g. Thus

$$C_A(gxg^{-1}) = C_A(x),$$

for all g and clearly,

$$C_A(gx) = C_A(xg), \forall g \in X.$$

So (3) implies (1).

Finally, we need to prove that (2) implies (1). Suppose $Stab_A(x) = X$. Then

$$C_A(g \diamond x) = C_A(x), \ \forall g \in X$$

and the same logic holds as in (3) implies (1). This completes the proof. □

Theorem 4.18. *Let A and B be finite multigroup and finite multiset of X and Y respectively such that $C_B(e) \leq C_B(y) \ \forall y \in Y$. Suppose A acts on B (on the left). Then for at least one $x \in Y$ distinct from all $g \in X$,*

$$\mid A_* \mid = \mid Orb_A(x) \mid . \mid Stab_A(x) \mid .$$

Proof. Recall that $A_* = \{x \in X \mid C_A(x) > 0\}$. Clearly, $A_* = X$ meaning A_* is a group. Since $Stab_A(x)$ is a subgroup of A_*, by Theorem 4.14, $\mid Stab_A(x) \mid$ divides $\mid A_* \mid$. Thus,

$$\mid A_* \mid = (\text{number of cosets of } Stab_A(x)). \mid Stab_A(x) \mid,$$

by Lagrange's theorem. So, it is sufficient to prove that

$$\text{number of cosets of } Stab_A(x) = \mid Orb_A(x) \mid .$$

To show this, we prove there exists a bijection between these two sets. Define a function

$$f : \{\text{left cosets of } Stab_A(x)\} \to Orb_A(x)$$

by

$$f(pStab_A(x)) = p \diamond x,$$

for some $p \in X$. For the sake of simplicity, we take $H = Stab_A(x)$. Then $f(pH) = p \diamond x$. In order to establish this bijection, we check for well-definedness, injection and surjection.

Suppose that $pH = qH, \ \forall p, q \in X$. We show that

$$f(pH) = p \diamond x = q \diamond x = f(qH).$$

Since $pH = qH$, we know that $p \in qH$ so that $p = qh$ for some $h \in H = Stab_A(x)$. Note that

$$C_B(h \diamond x) = C_B(x), \ \forall x \in Y.$$

Then

$$C_B(p \diamond x) = C_B((qh) \diamond x) = C_B(q \diamond (h \diamond x)) = C_B(q \diamond x)$$

which proves that f is well-defined.

Next we show that f is injective. Suppose that

$$f(pH) = f(qH).$$

Then $C_B(p \diamond x) = C_B(q \diamond x)$. Thus

$$C_B(x) = C_B((p^{-1}p) \diamond x) = C_B(p^{-1} \diamond (p \diamond x)) = C_B(p^{-1} \diamond (q \diamond x)) = C_B((p^{-1}q) \diamond x).$$

So $p^{-1}q \in Stab_A(x) = H$, by definition. Hence $p^{-1}qH = H$, i.e., $qH = pH$. Therefore f is injective.

Finally, suppose that $C_B(y) = C_B(g \diamond x)$, that is, $y = g \diamond x \in Orb_A(x)$. Then $f(gH) = g \diamond x = y$ and it implies that f is surjective. Thus the equality holds. So

$$\mid A_* \mid = \mid Orb_A(x) \mid . \mid Stab_A(x) \mid .$$

□

Theorem 4.18 is analogous to orbit-stabilizer theorem. In order to establish the following analogous result (that is, class equation), we assume that A acts on $B = A$ by conjugation as in Example 3.6. That is, for $g, x \in X$, we define

$$C_B(g \diamond x) = C_B(gxg^{-1}).$$

Theorem 4.19. *Suppose that a multigroup A of a finite group X acts on itself by conjugation. Then*

$$\mid A_* \mid = \mid C(A) \mid + \Sigma_{x \text{ in disjoint nontrivial orbits}} \frac{|A_*|}{|Stab_A(x)|}.$$

Proof. By Remark 4.3, we infer that

$$A_* = \bigcup_{x \text{ in disjoint orbits}} Orb_A(x).$$

An element $x \in X$ has trivial orbit (that is, $Orb_A(x) = \{x\}$) if and only if $x \in C(A)$ by Proposition 4.17. Then we know that A_* is also the disjoint union, that is,

$$A_* = C(A) \cup \bigcup_{x \text{ in disjoint nontrivial orbits}} Orb_A(x).$$

Since this is a disjoint union, we have

$$| A_* |=| C(A) | +\Sigma_{x \text{ in disjoint nontrivial orbits}} | Orb_A(x) | \, .$$

Thus,

$$| A_* |=| C(A) | +\Sigma_{x \text{ in disjoint nontrivial orbits}} \frac{|A_*|}{|Stab_A(x)|},$$

by Theorem 4.18. □

5. Conclusions

We have introduced the idea of multigroup actions on multiset via count function of multiset. The notions of orbits and stabilizers in multigroup actions have been introduced and some related results were obtained. The analogous of orbit-stabilizer theorem and class equation were established in the context of multigroup actions.

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A.M. Ibrahim (amibrahim@abu.edu.ng; adekubash1@gmail.com)
Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria

P.A. Ejegwa (ejegwa.augustine@uam.edu.ng; ocholohi@gmail.com)
Department of Mathematics/Statistics/Computer Science, University of Agriculture, P.M.B. 2373, Makurdi, Nigeria