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Soft differentiation in vector soft topology

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Abstract. In the present paper a notion of soft differentiation in vector soft topologies has been introduced and some basic properties of soft differentiable functions have been studied.

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1. Introduction

The theory of differential calculus in linear topological spaces has important applications to general differential geometry, general dynamics and general continuous group theory. The first definition of derivative of a function whose arguments and values lie in linear topological spaces was proposed by Michal and Paxon (1936) [12]. After that various definitions were proposed by several authors [10, 4, 5, 6] etc. The notion of differentiation was extended in fuzzy topological vector spaces by Ferraro and Foster [2]. Molodtsov [13] initiated a novel concept of soft set theory and then this concept is discussed and studied its applications by various authors [7, 8, 17]. In recent years, some soft separation axioms in soft topological spaces are introduced and studied [11, 3]. In 2015, the notion of vector soft topology is introduced and separation properties of vector soft topology are studied [1]. As a continuation of [1], in this paper we attempt to introduce the concept of soft differential in vector soft topologies using soft continuous function and one of the soft separation axioms that is soft T_1. Here, we shall consider the soft topology of the range space of a soft differentiable function is soft T_1 and contains a balanced soft neighbourhood base at the soft point corresponding to the null vector.

2. Preliminaries

Definition 2.1 ([13])**.** Let X be a universal set, A be a set of parameters, $P(X)$ denote the power set of X and $B \subseteq A$. A pair (F, B) is called a soft set over X, where F is a mapping given by $F : B \to P(X)$.
In [9] the soft sets are redefined as follows: Let B be the set of parameters and $B \subseteq A$. Then for each soft set (F, B) over X a soft set (H, A) is constructed over X, where $\forall \alpha \in A$, $H(\alpha) = \begin{cases} F(\alpha) & if\ \alpha \in B \\ \phi & if\ \alpha \in A \setminus B. \end{cases}$
Thus the soft sets (F, B) and (H, A) are equivalent to each other and the usual set operations of the soft sets $(F_i, B_i), i \in \Delta$ is the same as those of the soft sets $(H_i, A), i \in \Delta$. For this reason, in this paper, we have considered our soft sets over the same parameter set A.

Set theoretic operations are considered as in [7, 8, 13] considering the same parameter set A.

Unless otherwise stated, X will be assumed to be an initial universal set, A will be taken to be a set of parameters and $S(X, A)$ denote the set of all soft sets over X.

Definition 2.2 ([15])**.** A soft point E^x_α (soft element [15]) is a soft set (E, A) such that $E(\alpha)$ is a singleton, say, $\{x\}$ and $E(\beta) = \phi$, $\forall \beta \in A \setminus \{\alpha\}$. E^x_α is said to be in (F, A), denoted by $E^x_\alpha \tilde{\in} (F, A)$, if $x \in F(\alpha)$. $\Im$ denotes the set of all soft points of X.

Definition 2.3 ([15])**.** Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Then for $(F, A) \in S(X, A)$ and $(G, A) \in, S(Y, A)$

(i) $f[(F, A)] = (f(F), A)$, where $[f(F)](\alpha) = f[F(\alpha)], \forall \alpha \in A$,
(ii) $f^{-1}[(G, A)] = (f^{-1}(G), A)$, where $\left[f^{-1}(G)\right](\alpha) = f^{-1}[G(\alpha)], \forall \alpha \in A$.

Definition 2.4 ([16])**.** Let τ be a collection of soft sets over X. Then τ is said to be a soft topology on X, if

(i) $(\tilde{\Phi}, A)$, $(\tilde{X}, A) \in \tau$, where $\tilde{\Phi}(\alpha) = \phi$ and $\tilde{X}(\alpha) = X$, $\forall \alpha \in A$,
(ii) the intersection of any two soft sets in τ belongs to τ,
(iii) the union of any number of soft sets in τ belongs to τ.

The triplet (X, A, τ) is called a soft topological space over X.

Definition 2.5 ([15])**.** A soft topology τ on X is said to be an enriched soft topology, if the condition (i) of Definition 2.4 is replaced by (i)$'$:

(i)$'$ $(F, A) \in \tau$, for all pseudo constant soft set (F,A)(i. e. $F(\alpha) = X$ or ϕ, $\forall \alpha \in A$). The triplet (X, A, τ) is called an enriched soft topological space.

Proposition 2.6 ([1])**.** *Let for each $\alpha \in A$, τ^α is a crisp topology on X. Then $\tau^* = \{(G, A) \in S(X, A) : G(\alpha) \in \tau^\alpha, \forall \alpha \in A\}$ is an enriched soft topology on X.*

Definition 2.7 ([15])**.** Let (X, A, τ) be a soft topological space. $\mathscr{B} \subseteq \tau$ is said to be an open base of τ, if each $(F, A) \in \tau$ can be expressed as the union of some members of $\mathscr{B}$.

Definition 2.8 ([14])**.** $f : (X, A, \tau) \to (Y, A, \nu)$ is said to be soft continuous, if for any $(G, A) \in \nu$, there is $(F, A) \in \tau$ such that $f(F, A) \tilde{\subseteq} (G, A)$.

Proposition 2.9 ([14]). *$f : (X, A, \tau) \to (Y, A, \nu)$ is soft continuous if and only if $\forall x \in X$, $\alpha \in A$ and $\forall (V, A) \in \nu$ such that $E_\alpha^{f(x)} \tilde{\in} (V, A)$, $\exists (U, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (U, A)$ and $f[(U, A)] \tilde{\subseteq} (V, A)$.*

Definition 2.10 ([14]). The soft topology on $X_1 \times X_2$ induced by the open base $\mathcal{F} = \{(F, A) \tilde{\times} (G, A) : (F, A) \in \tau_1, (G, A) \in \tau_2\}$ is said to be the product soft topology of τ_1 and τ_2. It is denoted by $\tau_1 \tilde{\times} \tau_2$ and $(X_1 \times X_2, A, \tau_1 \tilde{\times} \tau_2)$ is said to be the soft topological product of the soft topological spaces (X_1, A, τ_1) and (X_2, A, τ_2).

Proposition 2.11 ([14]). *The projection mappings $\pi_i : (X_1 \times X_2, A, \tau_1 \tilde{\times} \tau_2) \to (X_i, A, \tau_i)$, $i = 1, 2$ are soft continuous and soft open. Further for any soft topological space (Y, A, ν), $f : (Y, A, \nu) \to (X_1 \times X_2, A, \tau_1 \tilde{\times} \tau_2)$ is soft continuous if and only if $\pi_i \circ f : (Y, A, \nu) \to (X_i, A, \tau_i), i = 1, 2$ are soft continuous.*

Definition 2.12 ([15]). (X, A, τ) is said to be soft T_1, if for $E_\alpha^x, E_\beta^y \in \Im$ with $E_\alpha^x \neq E_\beta^y$, $\exists (F, A), (G, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (F, A), E_\beta^y \tilde{\notin} (F, A)$ and $E_\beta^y \tilde{\in} (G, A), E_\alpha^x \tilde{\notin} (G, A)$.

Throughout the rest of the paper we use the notation V for the vector space $(V, +, \cdot)$ over the scalar field K, where K is the field of real or complex numbers, A is the parameter set. Also, we use the notation xy instead of $x \cdot y$.

Definition 2.13 ([1]). For $(F, A), (G, A) \in S(V, A)$, $k \in K$, $x \in V$, $(H, A) \in S(K, A)$,

$$(F, A) + (G, A) = (F + G, A),$$
$$k(F, A) = (kF, A),$$
$$x + (F, A) = (x + F, A)$$

and

$$(H, A) \cdot (F, A) = (H \cdot F, A)$$

are defined parameterwise as in [1].

Definition 2.14 ([1]). Let ν^α be the usual topology on K, $\forall \alpha \in A$. Then the soft topology ν defined as in Proposition 2.5 is called the soft usual topology on K.

Definition 2.15. [1] Let V be a vector space over the scalar field K endowed with the soft usual topology ν, A be the parameter set and τ be a soft topology on V. Then τ is said to be a vector soft topology on V, if the mappings:

(i) $f : (V \times V, A, \tau \tilde{\times} \tau) \to (V, A, \tau)$, defined by $f(x, y) = x + y$ and
(ii) $g : (K \times V, A, \nu \tilde{\times} \tau) \to (V, A, \tau)$, defined by $g(k, x) = kx$

are soft continuous, $\forall x, y \in V$ and $\forall k \in K$.

Definition 2.16 ([1]). Let (V, A, τ) be a vector soft topology. A balanced soft set (F, A) (i.e. $k(F, A) \tilde{\subseteq} (F, A)$ for all $k \in K$ with $| k | \leq 1$) is said to be a balanced neighbourhood of a soft point E_α^x if there exists $(G, A) \in \tau$ such that $E_\alpha^x \tilde{\in} (G, A) \tilde{\subseteq} (F, A)$.

Proposition 2.17 ([1]). *Let (V, A, τ) be a soft topological space over V and the field K is equipped with the soft usual topology ν. Then τ is a vector soft topology if and only if*

(1) $\forall x, y \in V$, $\forall \alpha \in A$ *and* $\forall (W, A) \in \tau$ *with* $E_\alpha^{x+y} \tilde{\in} (W, A)$, $\exists (F, A), (G, A) \in \tau$ *such that* $E_\alpha^x \tilde{\in} (F, A), E_\alpha^y \tilde{\in} (G, A)$ *and* $(F + G, A) \tilde{\subseteq} (W, A)$,

(2) $\forall x \in V$, $\forall k \in K$, $\forall \alpha \in A$ *and* $\forall (W, A) \in \tau$ *with* $E_\alpha^{kx} \tilde{\in} (W, A)$, $\exists (G, A) \in \nu, (F, A) \in \tau$ *such that* $E_\alpha^x \tilde{\in} (F, A), E_\alpha^k \tilde{\in} (G, A)$ *and* $(G \cdot F, A) \tilde{\subseteq} (W, A)$.

Definition 2.18 ([1]). A collection $\mathscr{B}$ of soft neighbourhoods of E_α^x is said to be a soft neighbourhood base of E_α^x, if for any soft neighbourhood (F, A) of E_α^x, $\exists (H, A) \in \mathscr{B}$ such that $(H, A) \tilde{\subseteq} (F, A)$.

Proposition 2.19 ([1]). *Let (V, A, τ) be an enriched vector soft topology. Then $\exists$ a balanced soft neighbourhood base of the soft point E_α^θ in (V, A, τ).*

Proposition 2.20. *Let (V_1, A, τ_1) and (V_2, A, τ_2) be two vector soft topologies. Then their product $(V_1 \times V_2, A, \tau_1 \tilde{\times} \tau_2)$ is also a vector soft topology.*

Proof. (i) Let $\alpha \in A$, $E_\alpha^{(x_1+y_1, x_2+y_2)} \tilde{\in} (V_1, A) \tilde{\times} (V_2, A)$ and let (F, A) be any soft neighbourhood of $E_\alpha^{(x_1+y_1, x_2+y_2)}$. Then $(F, A) \tilde{\supseteq} (F_1, A) \tilde{\times} (F_2, A)$, where $(F_1, A), (F_2, A)$ are soft neighbourhoods of $E_\alpha^{(x_1+y_1)}$ and $E_\alpha^{(x_2+y_2)}$. Thus by Proposition 2.17, there exist soft neighbourhoods (G_i, A) and (H_i, A) of $E_\alpha^{x_i}$ and $E_\alpha^{y_i}$, respectively such that $(G_i, A) + (H_i, A) \tilde{\subseteq} (F_i, A)$, $i = 1, 2$. So, $(G_1, A) \tilde{\times} (G_2, A)$ and $(H_1, A) \tilde{\times} (H_2, A)$ are soft neighbourhoods of $E_\alpha^{(x_1, x_2)}$ and $E_\alpha^{(y_1, y_2)}$, respectively in $(V_1 \times V_2, A, \tau_1 \tilde{\times} \tau_2)$. Again,

$$\begin{aligned} &[(G_1, A) + (H_1, A)] \tilde{\times} [(G_2, A) + (H_2, A)] \\ &= \left[(G_1, A) \tilde{\times} (G_2, A)\right] + [(H_1, A) \tilde{\times} (H_2, A)] \\ &\tilde{\subseteq} (F_1, A) \tilde{\times} (F_2, A) \tilde{\subseteq} (F, A). \end{aligned}$$

Clearly $\alpha \in A$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$ are arbitrary. Hence this is true for all $\alpha \in A$, $\forall x_1, y_1 \in V_1$, $\forall x_2, y_2 \in V_2$.

(ii) Let $\alpha \in A$, $k \in K$, $(x_1, x_2) \in V_1 \times V_2$, $E_\alpha^{(kx_1, kx_2)} \tilde{\in} (V_1, A) \tilde{\times} (V_2, A)$ and let (F, A) be any soft neighbourhood of $E_\alpha^{(kx_1, kx_2)}$. Then $(F, A) \tilde{\supseteq} (F_1, A) \tilde{\times} (F_2, A)$, where (F_i, A) is a soft neighbourhood of $E_\alpha^{kx_i}$, $i = 1, 2$. Thus by Proposition 2.17, there exist soft neighbourhoods (G_1, A) and (H_1, A) of E_α^k and $E_\alpha^{x_1}$, respectively such that $(G_1 \cdot H_1, A) \tilde{\subseteq} (F_1, A)$. Similarly, there exist soft neighbourhoods (G_2, A) and (H_2, A) of E_α^k and of $E_\alpha^{x_2}$ such that $(G_2 \cdot H_2, A) \tilde{\subseteq} (F_2, A)$. Set $(G, A) = (G_1, A) \tilde{\cap} (G_2, A)$. Then $(G \cdot H_1, A) \tilde{\subseteq} (F_1, A)$ and $(G \cdot H_2, A) \tilde{\subseteq} (F_2, A)$. Thus

$$(G \cdot (H_1 \times H_2), A) \tilde{\subseteq} (G \cdot H_1, A) \tilde{\times} (G \cdot H_2, A) \tilde{\subseteq} (F_1, A) \tilde{\times} (F_2, A) \tilde{\subseteq} (F, A).$$

Since $\alpha \in A$, $k \in K$ $x_1, y_1 \in V_1$ are arbitrary, this is true for all $\alpha \in A, k \in K$ and $\forall x_1, y_1 \in V_1$. So, by Proposition 2.17, $(V_1 \times V_2, A, \tau_1 \tilde{\times} \tau_2)$ is a vector soft topology. $\square$

Definition 2.21 ([6]). A real valued function of a real variable t defined on some neighbourhood of 0 is said to be $o(t)$, if $\lim\limits_{t \to 0} \frac{o(t)}{t} = 0$.

3. Soft tangent

Definition 3.1. Let $(V_i, A, \tau_i), i = 1, 2$ be vector soft topologies and $\theta \in V_1, \theta' \in V_2$ are null vectors. A function $\phi : (V_1, A, \tau_1) \to (V_2, A, \tau_2)$ with $\phi(\theta) = \theta'$ is said to be a soft tangent to E_α^θ, if for any soft neighbourhood (G, A) of $E_\alpha^{\theta'}$ in (V_2, A, τ_2), there exists a soft neighbourhood (F, A) of E_α^θ in (V_1, A, τ_1) such that $\phi(t(F, A)) \tilde{\subseteq} o(t)(G, A)$, for some function $o(t)$.

Lemma 3.2. *Let E_α^θ be a soft point in a vector soft topology (V, A, τ) and (F, A) be any soft set containing E_α^θ. If there is a point $a \in V$ such that $E_\alpha^{ka} \tilde{\notin} (F, A)$, for all non-zero scalar $k \in K$, then (F, A) is not a soft neighbourhood of E_α^θ.*

Proof. Suppose that (F, A) be a soft open neighbourhood of E_α^θ. Consider the function $g : (k, a) \to ka$ and let E_α^a be any soft point. For $k = 0$, the point $E_\alpha^{ka} \tilde{\in} (F, A)$. Since g is soft continuous, there exist soft neighbourhood (G, A) and (H, A) of E_α^0 and E_α^a, respectively such that $(G, A) \cdot (H, A) = (G \cdot H, A) \subseteq (F, A)$. Now $G(\alpha) = (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$. Then, $E_\alpha^{\delta a} \tilde{\in} (F, A)$, for some $\delta(\neq 0) \in (-\varepsilon, \varepsilon)$. Thus the result holds. □

Lemma 3.3. *In a vector soft topology (V_1, A, τ_1), where τ_1 is enriched, if (F, A) is a soft neighbourhood of E_α^θ, then there is a soft neighbourhood (G, A) of E_α^θ such that $k(G, A) \tilde{\subseteq} (F, A)$, for each $k \in K, \mid k \mid \leq 1$.*

Proof. Let (F, A) be a soft neighbourhood of E_α^θ. Since the scalar product is soft continuous, there exists an $\varepsilon > 0$ and a soft neighbourhood (H, A) of E_α^θ such that for $\xi \in K$, $\mid \xi \mid < \varepsilon$, $\xi H(\alpha) \subseteq F(\alpha)$. Let (J, A) be the soft set with $J(\alpha) = H(\alpha)$ and $J(\beta) = \phi$, $\beta(\neq \alpha) \in A$. Then $\xi(J, A) \tilde{\subseteq} (F, A)$. By hypothesis, $\mid k \mid \leq 1$. Thus $\mid k\xi \mid < \varepsilon$ and $k\xi(J, A) \tilde{\subseteq} (F, A)$. Set $\xi(J, A) = (G, A)$. So the result follows. □

Proposition 3.4. *If the function $\phi : (V_1, A, \tau_1) \to (V_2, A, \tau_2)$ is soft tangent to E_α^θ, where τ_1, τ_2 are enriched, then ϕ is soft continuous at E_α^θ.*

Proof. By Lemma 3.3, for every soft neighbourhood (F, A) of $E_\alpha^{\theta'}$, there exists a soft neighbourhood (F', A) of $E_\alpha^{\theta'}$ such that $o(t)(F', A) \tilde{\subseteq} (F, A)$, for $\mid o(t) \mid \leq 1$. By Definition 3.1, for each (F', A) there exists a soft neighbourhood (G, A) of E_α^θ in (V_1, A, τ_1) such that $\phi(t(G, A)) \tilde{\subseteq} o(t)\ (F', A) \tilde{\subseteq} (F, A)$. Since $t(G, A)$ is also a soft neighbourhood of E_α^θ, ϕ is soft continuous at E_α^θ. □

Proposition 3.5. *If the functions $\phi, \psi : (V_1, A, \tau_1) \to (V_2, A, \tau_2)$ are soft tangents to E_α^θ, then $\phi + \psi$ is soft tangent to E_α^θ.*

Proof. For every soft neighbourhood of (G, A) of $E_\alpha^{\theta'}$, there exists a soft neighbourhood (G', A) of $E_\alpha^{\theta'}$ in (V_2, A, τ_2) such that $(G', A) + (G', A) \tilde{\subseteq} (G, A)$.
Then, $o(t)(G', A) + o(t)(G', A) \tilde{\subseteq} o(t)\ (G, A)$. Since $\phi, \psi : (V_1, A, \tau_1) \to (V_2, A, \tau_2)$ is soft tangent to E_α^θ, for any soft neighbourhood (G', A) of $E_\alpha^{\theta'}$ in (V_2, A, τ_2), there exist soft neighbourhoods $(F', A), (F'', A)$ of E_α^θ in (V_1, A, τ_1) such that

$$\phi(t(F', A)) \tilde{\subseteq} o(t)(G', A) \text{ and } \psi(t(F'', A)) \tilde{\subseteq} o(t)(G', A).$$

Let $(F, A) = ((F', A) \tilde{\cap} (F'', A)$. Then $\phi(t(F, A)) \tilde{\subseteq}\ o(t)(G', A)$ and $\psi(t(F, A))$ $\tilde{\subseteq}\ o(t)\ (G', A)$. Thus,

$$\begin{aligned}(\phi + \psi)\,(t(F, A)) &= \phi(t(F, A)) + \psi(t(F, A)) \\ &\tilde{\subseteq}\ o(t)(G', A) + o(t)(G', A) \\ &\tilde{\subseteq}\ o(t)\ (G, A).\end{aligned}$$

□

Proposition 3.6. *Let (V_1, A, τ_1), (V_2, A, τ_2) and (V_3, A, τ_3) be three vector soft topologies. If $\phi : (V_1, A, \tau_1) \to (V_2, A, \tau_2)$ is soft tangent to E_α^θ and $f : (V_2, A, \tau_2) \to (V_3, A, \tau_3)$ is linear soft continuous, then $f \circ \phi$ is soft tangent to E_α^θ.*

On the other hand, if $f : (V_1, A, \tau_1) \to (V_2, A, \tau_2)$ is linear soft continuous and $\phi : (V_2, A, \tau_2) \to (V_3, A, \tau_3)$ is soft tangent to E_α^θ, then $\phi \circ f$ is soft tangent to E_α^θ .

Proof. By the soft continuity of f, for every soft neighbourhood (F, A) of $E_\alpha^{\theta''}$ in (V_3, A, τ_3), there exists a soft neighbourhood (G, A) of $E_\alpha^{\theta'}$ in (V_2, A, τ_2) such that

$f(G,A)\tilde{\subseteq}(F,A)$. Since ϕ is a soft tangent to E_α^θ, for every such (G,A), there is a soft neighbourhood (H,A) of E_α^θ such that $\phi(t(H,A))\tilde{\subseteq}o(t)(G,A)$. Then

$f(\phi(t(H,A)))\tilde{\subseteq}\ f(o(t)(G,A)) = o(t)f(G,A)\ \tilde{\subseteq}o(t)(F,A)$.

Thus $f\circ\phi$ is soft tangent to E_α^θ.

The other part of the Proposition 3.6 proceeds in a similar way. □

4. Soft differentiation

Definition 4.1. Let (V_1,A,τ_1) and (V_2,A,τ_2) be two vector soft topologies of which (V_2,A,τ_2) is soft T_1 and contains a balanced neighbourhood base at $E_\alpha^{\theta'}$, θ' is the null vector of V_2. Then a soft continuous function $f:(V_1,A,\tau_2)\to(V_2,A,\tau_2)$ is said to be soft differentiable at a soft point $E_\alpha^x\tilde{\in}(\tilde{V_1},A)$, if there exists a linear soft continuous function $u:(V_1,A,\tau_1)\to(V_2,A,\tau_2)$ such that we can write

$f(E_\alpha^{x+y}) = f(E_\alpha^x)+u(E_\alpha^y)+\phi(E_\alpha^y)$, $y\in V_1$, where ϕ is a soft tangent to E_α^θ.

The mapping u is called the soft derivative of f at E_α^x and denoted by $f'(E_\alpha^x)$. Here the function u depends on x and α both.

Henceforth, we shall consider the soft topology of the range space of a soft differentiable function is soft T_1 and contains a balanced soft neighbourhood base at the soft point corresponding to the null vector.

Example 4.2. Consider the vector space $\mathbb{R}$ over the field $\mathbb{R}$ and A be the parameter set. Let τ^α be the usual topology on $\mathbb{R}$ for all $\alpha\in A$ and τ be the soft topology on $\mathbb{R}$ as of Proposition 2.6. Then $(\mathbb{R},A,\tau)$ is an enriched vector soft topology. Now for any $r\in\mathbb{R}$, define the mapping $U_r:\mathbb{R}\to\mathbb{R}$ by $U_r(x)=rx$. Then obviously, $U_r:(\mathbb{R},A,\tau)\to(\mathbb{R},A,\tau)$ is linear soft continuous mapping. Also, for any $E_\alpha^x\tilde{\in}\left(\tilde{\mathbb{R}},A\right)$, $U_r\left(E_\alpha^{x+y}\right)=E_\alpha^{r(x+y)}=E_\alpha^{rx}+E_\alpha^{ry}+O\left(E_\alpha^y\right)$, $y\in\mathbb{R}$, where O, the zero function (i.e. $O(x)=0\in\mathbb{R},\forall x\in\mathbb{R}$), is a soft tangent to E_α^0. So, U_r is soft differentiable at every soft point $E_\alpha^x\tilde{\in}\left(\tilde{\mathbb{R}},A\right)$.

Proposition 4.3. *The soft derivative of a function $f:(V_1,A,\tau_1)\to(V_2,A,\tau_2)$ at a soft point E_α^x is unique.*

Proof. Suppose if possible that the derivative of f at soft point E_α^x is not unique. Then there exist two linear soft continuous functions $u_1,\ u_2$ such that

$$u_1(E_\alpha^y)+\phi(E_\alpha^y)=u_2(E_\alpha^y)+\psi(E_\alpha^y), y\in V_1,$$

where ϕ,ψ are each tangent to E_α^θ.

Let $\eta:V_1\to V_2$ such that $\eta(y)=u_1(y)-u_2(y)$, $y\in V_1$. Then clearly, η is a linear function such that $\eta(E_\alpha^y)=u_1(E_\alpha^y)-u_2(E_\alpha^y)=\psi(E_\alpha^y)-\phi(E_\alpha^y)$, $y\in V_1$. Thus by Proposition 3.5, η is soft tangent to E_α^θ. By assumption, η is not zero. Let $a\in V_1$ such that $\eta(a)=r\neq\theta'$. Since (V_2,A,τ_2) is soft T_1, for $E_\alpha^r\tilde{\in}(\widetilde{V_2},A)$, there exists a soft open set (G,A) such that $E_\alpha^r\tilde{\notin}(G,A)$, $E_\alpha^{\theta'}\tilde{\in}(G,A)$.

If $\mathscr{B}$ is a balanced soft neighbourhood base of $E_\alpha^{\theta'}$ in (V_2,A,τ_2), then there is a $(H,A)\in\mathscr{B}$, $E_\alpha^{\theta'}\tilde{\in}(H,A)\tilde{\subseteq}(G,A)$ with $(\varepsilon H,A)\tilde{\subseteq}(H,A)$, for all $|\ \varepsilon\ |\leq 1$.

If $\xi=\frac{1}{\varepsilon}$, for $\varepsilon\neq 0$, then $E_\alpha^{\xi r}\tilde{\notin}(H,A)$. Since η is soft tangent to E_α^θ, there must be a soft neighbourhood (J,A) of E_α^θ such that $\eta(t(J,A))\tilde{\subseteq}o(t)(H,A)$. Thus

$\eta(J,A)\tilde{\subseteq}\frac{o(t)}{t}(H,A)$, as η is linear. Put $\frac{t}{o(t)}=\xi$. Then

$$\eta(J,A)\tilde{\subseteq}\frac{1}{\xi}(H,A)=(\varepsilon H,A)\tilde{\subseteq}(H,A).$$

Thus $E_\alpha^{\xi r}\tilde{\notin}\eta(J,A)$,i.e., $E_\alpha^{\xi a}\tilde{\notin}(J,A)$. For $|\xi|\leq 1,\ \xi\neq 0$, as $E_\alpha^{r}\tilde{\notin}(H,A)$, $E_\alpha^{\xi r}\tilde{\notin}\xi(H,A)$. Since $\eta(J,A)\tilde{\subseteq}\frac{o(t)}{t}(H,A)$, $\eta(J,A)\tilde{\subseteq}\xi(H,A)$. So $E_\alpha^{\xi r}\tilde{\notin}\eta(J,A)$ and thus $E_\alpha^{\xi a}\tilde{\notin}(J,A)$. Setting $\xi=k$, we get that there is a point $a\in V_1$, such that for any k, $k\neq 0$, $E_\alpha^{ak}\tilde{\notin}(J,A)$. Hence by Lemma 3.2, (J,A) is not a soft neighbourhood of E_α^{θ}, a contradiction. Therefore the soft derivative of f at soft point E_α^x is unique. □

Proposition 4.4. *Let (V_1,A,τ_1), (V_2,A,τ_2) be vector soft topologies. Then any soft continuous constant function $f:(V_1,A,\tau_1)\to(V_2,A,\tau_2)$ is soft differentiable at every soft point of $(\tilde{V}_1,A)$.*

Proposition 4.5. *The soft derivative of a linear soft continuous mapping $u:(V_1,A,\tau_1)\to(V_2,A,\tau_2)$ exists at every soft point $E_\alpha^x\tilde{\in}(\tilde{V}_1,A)$.*

Proposition 4.6. *Let $(W,A,\nu)=\prod_{j=1}^{n}(W_j,A,\nu_j)$ be the product vector soft topology of a finite family of vector soft topologies (W_j,A,ν_j), $j=1,2,...,n$, and (V,A,τ) be any vector soft topology. Then a soft continuous mapping $f:(V,A,\tau)\to(W,A,\nu)$ is soft differentiable at $E_\alpha^x\tilde{\in}(\tilde{V},A)$ iff $\pi_j\circ f$ is soft differentiable at E_α^x.*

Proof. Let f be soft differentiable at E_α^x. Then

$$f(E_\alpha^{x+y})=f(E_\alpha^x)+u(E_\alpha^y)+\phi(E_\alpha^y),y\in V,$$

where ϕ is a soft tangent to E_α^θ.

By linearity of projection mapping π_j, we can write for every j,

$$\pi_j(f(E_\alpha^{x+y}))-\pi_j(f(E_\alpha^x))=\pi_j(f'(E_\alpha^x)(E_\alpha^y))+\pi_j(\phi(E_\alpha^y)),\ y\in V.$$

Since π_j and f' both are linear and soft continuous, $\pi_j\circ f'$ is linear and soft continuous and by Proposition 3.6, $\pi_j\circ\phi$ is soft tangent to E_α^θ, $j=1,2,..,n$.

Conversely, let $\pi_j\circ f$ be soft differentiable at E_α^x, for every $j\in\{1,2,..,n\}$. Then, for every j, we can write,

$$\pi_j(f(E_\alpha^{x+y}))-\pi_j(f(E_\alpha^x))=u_j(E_\alpha^y)+\phi_j(E_\alpha^y),$$

where u_j is a linear soft continuous mapping and ϕ_j is soft tangent to E_α^θ. Let (G,A) be a soft neighbourhood of $E_\alpha^{\theta'}$ in (W,A,ν). By definition of soft product topology, $(G,A)\tilde{\supseteq}\prod_{j=1}^{n}(G_j,A)$, where (G_j,A) are soft neighbourhoods of $E_\alpha^{\theta'_j}$ in (W_j,A,ν_j).

Now, for every (G_j,A), there exists a soft neighbourhood (F_j,A) of E_α^θ in (V,A,τ) such that $\phi_j(t(F_j,A))\tilde{\subseteq}o(t).(G_j,A)$.
Setting $(F,A)=\tilde{\cap}(F_j,A)$, we have $\phi_j(t(F,A))\ \tilde{\subseteq}o(t).(G_j,A),\ \forall j=1,2,..,n$.
Again, $o(t)(G,A)\tilde{\supseteq}o(t)\prod_{j=1}^{n}(G_j,A)=\prod_{j=1}^{n}o(t)(G_j,A)$.

Let $\phi=\prod_{j=1}^{n}\phi_j$. Then

$$\phi(t(F,A))=\prod_{j=1}^{n}\phi_j(t(F,A))\tilde{\subseteq}\prod_{j=1}^{n}o(t).(G_j,A)\tilde{\subseteq}o(t)(G,A).$$

Thus ϕ is soft tangent to E_α^θ.

Define $u = \prod_{j=1}^{n} u_j$. This mapping is linear and soft continuous by linearity and soft continuity of the functions u_j. The uniqueness of $f'(E_\alpha^x)$ follows by the uniqueness of u_j. □

Proposition 4.7. *Let $(V_1, A, \tau_1), (V_2, A, \tau_2), (V_3, A, \tau_3)$ be three vector soft topologies, $f : (V_1, A, \tau_1) \to (V_2, A, \tau_2)$ and $g : (V_2, A, \tau_2) \to (V_3, A, \tau_3)$ be two soft continuous mapping. Let $x \in V_1$ and $y = f(x)$. If f is soft differentiable at E_α^x and g is soft differentiable at E_α^y, then the composition $h = g \circ f$ is soft differentiable at E_α^x.*

Proof. By hypothesis, f and g are soft differentiable. Then,

$$f(E_\alpha^{x+r}) = f(E_\alpha^x) + f'(E_\alpha^x)(E_\alpha^r) + \phi(E_\alpha^r), r \in V_1$$

and

$$g(E_\alpha^{y+s}) = g(E_\alpha^y) + g'(E_\alpha^y)(E_\alpha^s) + \psi(E_\alpha^s), s \in V_2,$$

where ϕ, ψ are soft tangents to E_α^θ and $E_\alpha^{\theta'}$, respectively.

Defining $h = g \circ f$, we obtain, after substitution,

$$h(E_\alpha^{x+r}) - h(E_\alpha^x)$$
$$= g'(E_\alpha^y)(f'(E_\alpha^x)(E_\alpha^r)) + g'(E_\alpha^y)(\phi(E_\alpha^r)) + \psi(f'(E_\alpha^x)(E_\alpha^r) + \phi(E_\alpha^r)),\ r \in V_1.$$

By Proposition 3.6, $g'(E_\alpha^y) \circ \phi$ is soft tangent to E_α^θ. Consider the mapping $\psi \circ (f'(E_\alpha^x) + \phi)$. For every soft neighbourhood (G, A) of $E_\alpha^{\theta''}$ in (V_3, A, τ_3), there is a soft neighbourhood (F, A) in $E_\alpha^{\theta'}$ in (V_2, A, τ_2) such that $\psi(t(F, A)) \tilde{\subseteq} o(t)(G, A)$. Given (F, A) in (V_2, A, τ_2), there exists a soft neighbourhood (F', A) of $E_\alpha^{\theta'}$ such that $(F', A) + (F', A) \tilde{\subseteq} (F, A)$. Without loss of generality, suppose that both (F, A) and (F', A) are balanced. By soft continuity of $f'(E_\alpha^x)$, there is a soft neighbourhood (H, A) of E_α^θ in (V_1, A, τ_1) such that $f'(E_\alpha^x)((H, A)) \tilde{\subseteq} (F', A)$, which implies that $tf'(E_\alpha^x)((H, A)) \tilde{\subseteq} t(F', A)$, i.e., $f'(E_\alpha^x)(t(H, A)) \tilde{\subseteq} t(F', A)$. For every (F', A), there exists a soft neighbourhood (J, A) of E_α^θ in (V_1, A, τ_1), such that $\phi(t(J, A)) \tilde{\subseteq} o(t)(F', A)$ and for $\mid \frac{o(t)}{t} \mid \leq 1$, $o(t)(F', A) \tilde{\subseteq} t(F', A)$.

Setting $(N, A) = (H, A) \tilde{\cap} (J, A)$, we get $f'(E_\alpha^x)(t(N, A)) + \phi(t(N, A)) \tilde{\subseteq} t(F, A)$ and which implies that

$$\psi[f'(E_\alpha^x)(t(N, A)) + \phi(t(N, A))] \tilde{\subseteq} \psi(t(F, A)) \tilde{\subseteq} o(t)(G, A).$$

Then the mapping $\psi \circ (f'(E_\alpha^x) + \phi)$ is soft tangent to E_α^θ. Thus we can write

$$h(E_\alpha^{x+r}) - h(E_\alpha^x) = g'(E_\alpha^y) \circ f'(E_\alpha^x)(E_\alpha^r) + \chi(E_\alpha^r), r \in V_1,$$

where $g'(E_\alpha^y) \circ f'(E_\alpha^x)$ is linear soft continuous and χ, the sum of two mappings which are soft tangent to E_α^θ, is soft tangent to E_α^θ. So the result holds. □

Proposition 4.8. *Let $(V_1, A, \tau_1), (V_2, A, \tau_2)$ be two vector soft topologies and $f, g : (V_1, A, \tau_1) \to (V_2, A, \tau_2)$ be two soft continuous mappings. If f and g are soft differentiable at E_α^x, so are $f + g$ and kf, $k \in K$.*

Proof. The mapping $f + g$ is composition of $x \to (f(x), g(x))$ from (V_1, A, τ_1) into $(V_2 \tilde{\times} V_2, A, \tau_2 \tilde{\times} \tau_2)$ and of $(u, v) \to u + v$ from $(V_2 \tilde{\times} V_2, A, \tau_2 \tilde{\times} \tau_2)$ into (V_2, A, τ_2). The first is soft differentiable, by Proposition 4.6 and the second by the definition of sum; the result follows from Proposition 4.7. For kf it is sufficient to note that the mapping $u \to ku$ of (V_2, A, τ_2) into itself is soft differentiable, by Proposition 4.5. □

Remark 4.9. In Definition 4.1, if we replace soft T_1 by soft Tychonoff or soft T_2, then all the results in section 4 also hold.

5. Conclusion

There is a future scope of studying higher order soft differentiation in vector soft topologies and other properties of soft differentiable functions.

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