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Pairwise fuzzy functions and pairwise fuzzy Volterra Spaces

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Abstract. In this paper, by using pairwise fuzzy functions and somewhat pairwise fuzzy functions, a study on the conditions under which a fuzzy bitopological space becomes pairwise fuzzy Volterra and pairwise fuzzy weakly Volterra spaces, has been carried out. Results concerning pairwise fuzzy functions that preserve pairwise fuzzy Volterra and pairwise fuzzy weakly Volterra spaces in the context of images and pre-images are also studied.

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1. Introduction

Many mathematical concepts can be represented by the notion of set theory, which dichotomize the situation into two conditions: either "yes" or "no". Till 1965, Mathematicians were concerned only about "well-defined" things and smartly avoided any other possibility which are more realistic in nature. In 1965, Zadeh [19] introduced fuzzy set to accommodate real life situations by giving partial membership to each element of a situation under consideration. The usual notion of set topology was generalized with introduction of fuzzy topology by Chang [2] in 1968, based on the concept of fuzzy sets invented by Zadeh.

In 1989, Kandil [9] introduced the concept of fuzzy bitopological spaces and since then various notions in classical topology have extended to fuzzy bitopological spaces. The concepts of Volterra spaces have been studied extensively in classical topology in [4, 5, 6, 7, 8]. The concept of pairwise Volterra spaces in fuzzy setting was introduced and studied by the authors in [15]. Somewhat fuzzy continuous functions, somewhat fuzzy open functions on fuzzy topological spaces were introduced and studied by G.Thangaraj and G.Balasubramanian in [12]. The results concerning pairwise fuzzy functions that preserve pairwise fuzzy Volterra and pairwise fuzzy weakly Volterra spaces in the context of images and pre-images in fuzzy bitopological spaces are studied in this paper.

2. Preliminaries

Now we give some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). By a fuzzy bitopological space (Kandil, 1989), we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are two fuzzy topologies on a non-empty set X. Throughout this paper, the indices i and j take values in $\{1, 2\}$ and $i \neq j$.

Definition 2.1 ([2])**.** Let λ and μ be fuzzy sets in X. Then for all $x \in X$,
(i) $\lambda = \mu \Leftrightarrow \lambda(x) = \mu(x)$,
(ii) $\lambda \leq \mu \Leftrightarrow \lambda(x) \leq \mu(x)$,
(iii) $\psi = \lambda \vee \mu \Leftrightarrow \psi(x) = max\{\lambda(x), \mu(x)\}$,
(iv) $\delta = \lambda \wedge \mu \Leftrightarrow \delta(x) = min\{\lambda(x), \mu(x)\}$,
(v) $\eta = \lambda^c \Leftrightarrow \eta(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i \mid i \in I\}$ of fuzzy sets in X, the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined, respectively by:

$$\psi(x) = sup_i\{\lambda_i(x) \mid x \in X\} \text{ and } \delta(x) = inf_i\{\lambda_i(x) \mid x \in X\}.$$

The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.2 ([2])**.** Let (X, T) be any fuzzy topological space and λ be any fuzzy set in (X, T). The closure and interior of a fuzzy set λ in a fuzzy topological space (X, T) are respectively denoted as $cl(\lambda)$ and $int(\lambda)$ and defined, respectively as:
(i) $cl(\lambda) = \wedge\{\mu \mid \lambda \leq \mu, 1 - \mu \in T\}$,
(ii) $int(\lambda) = \vee\{\mu \mid \mu \leq \lambda, \mu \in T\}$.

Lemma 2.3 ([1])**.** *For a fuzzy set λ of a fuzzy space X,*
(1) $1 - cl(\lambda) = int(1 - \lambda)$,
(2) $1 - int(\lambda) = cl(1 - \lambda)$.

Definition 2.4 ([15])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy open set, if $\lambda \in T_i,\ (i = 1, 2)$.

Definition 2.5 ([15])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy closed set, if $1 - \lambda \in T_i,\ (i = 1, 2)$.

Definition 2.6 ([15])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy G_δ-set, if $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2).

Definition 2.7 ([11])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy dense set, if $cl_{T_1} cl_{T_2}(\lambda) = 1$ and $cl_{T_2} cl_{T_1}(\lambda) = 1$.

Definition 2.8 ([13])**.** A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy strongly irresolvable space, if $cl_{T_1} int_{T_2}(\lambda) = 1 = cl_{T_2} int_{T_1}(\lambda)$ for each pairwise fuzzy dense set λ in (X, T_1, T_2).

Definition 2.9 ([2])**.** Let (X, T) and (Y, S) be any two fuzzy topological spaces and $f : (X, T) \to (Y, S)$ be a function. If λ is a fuzzy set in X, we define $f(\lambda)$ as

$$f(\lambda)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \lambda(x), & \text{if } f^{-1}(y) \text{ is non-empty} \\ 0, & \text{if otherwise,} \end{cases}$$

for each $y \in Y$ and if μ is a fuzzy set of Y, we define $f^{-1}(\mu)$ as $f^{-1}(\mu)(x) = \mu(f(x))$, for each $x \in X$.

Definition 2.10 ([10])**.** A function $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is said to be pairwise fuzzy continuous (resp. pairwise fuzzy open), if $f : (X, T_1) \to (Y, S_1)$ and $f : (X, T_2) \to (Y, S_2)$ are fuzzy continuous (resp. fuzzy open).

Definition 2.11 ([16])**.** Let (X, T_1, T_2) and (Y, S_1, S_2) be any two fuzzy bitopological spaces. A function $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is called somewhat pairwise fuzzy continuous, if $\lambda \in S_1$ or $\lambda \in S_2$ and $f^{-1}(\lambda) \neq 0$, implies there exists $\mu \in T_1$ or $\mu \in T_2$ such that $\mu \neq 0$ and $\mu \leq f^{-1}(\lambda)$.

Definition 2.12. [16] Let (X, T_1, T_2) and (Y, S_1, S_2) be any two fuzzy bitopological spaces. A function $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is called somewhat pairwise fuzzy open if $\lambda \in T_1$ or $\lambda \in T_2$ and $\lambda \neq 0$ implies there exists $\mu \in S_1$ or $\mu \in S_2$ such that $\mu \neq 0$ and $\mu \leq f^{-1}(\lambda)$. That is, $int_{S_1} f(\lambda) \neq 0$ or $int_{S_2} f(\lambda) \neq 0$.

Theorem 2.13 ([17])**.** *Let X be a fuzzy topological space and λ, μ be fuzzy sets in X. Then we have*

(1) *λ is fuzzy closed (resp., fuzzy open) $\Leftrightarrow cl(\lambda) = \lambda$ (resp., $int(\lambda) = \lambda$),*
(2) *$\lambda \leq \mu \Rightarrow cl(\lambda) \leq cl(\mu) \quad (int(\lambda) \leq int(\mu))$,*
(3) *$cl\ cl(\lambda) = cl(\lambda) \quad (int\ int(\lambda) = int(\lambda))$,*
(4) *$cl(\lambda) \vee cl(\mu) = cl(\lambda \vee \mu)$,*
(5) *$cl(\lambda) \wedge cl(\mu) \geq cl(\lambda \wedge \mu)$,*
(6) *$int(\lambda) \vee int(\mu) \leq int(\lambda \vee \mu)$,*
(7) *$int(\lambda) \wedge int(\mu) = int(\lambda \wedge \mu)$.*

Lemma 2.14 ([1])**.** *Let (X, T) and (Y, S) be any two fuzzy topological spaces and $f : X \to Y$ be a function and $\{\lambda_\alpha\}$ be a family of fuzzy sets of Y, then*

(1) *$f^{-1}(\vee \lambda_\alpha) = \vee f^{-1}(\lambda_\alpha)$,*
(2) *$f^{-1}(\wedge \lambda_\alpha) = \wedge f^{-1}(\lambda_\alpha)$.*

Lemma 2.15 ([3])**.** *Let (X, T) and (Y, S) be any two fuzzy topological spaces and f be a function from a set X to a set Y and let $\{A_j\}, j \in J$, be a family of fuzzy sets in X. Then*

(1) *$f(\vee_{j \in J}[A_j]) = \vee_{j \in J}[f(A_j)]$,*
(2) *$f(\wedge_{j \in J}[A_j]) \leq \wedge_{j \in J}[f(A_j)]$.*

Theorem 2.16 ([18])**.** *Let (X, T) and (Y, S) be any two fuzzy topological spaces and $f : (X, T) \to (Y, S)$ be a fuzzy open function. Then, for every fuzzy set β in (Y, S), $f^{-1}[cl(\beta)] \leq cl\left[f^{-1}(\beta)\right]$.*

Theorem 2.17 ([16])**.** *Let (X, T_1, T_2) and (Y, S_1, S_2) be any two fuzzy bitopological spaces. Let $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ be any function. Then the following are equivalent:*

(1) *f is somewhat pairwise fuzzy continuous,*

(2) *if λ is S_1-fuzzy closed or S_2-fuzzy closed such that $f^{-1}(\lambda) \neq 1$, then there exists a proper T_1-fuzzy closed or T_2-fuzzy closed set μ such that $\mu > f^{-1}(\lambda)$,*

(3) *if λ is a pairwise fuzzy dense set in (X, T_1, T_2), then $f(\lambda)$ is a pairwise fuzzy dense set in (Y, S_1, S_2).*

Theorem 2.18 ([16])**.** *Suppose (X, T_1, T_2) and (Y, S_1, S_2) are any two fuzzy bitopological spaces. Let $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ be a 1-1 and onto function. Then the following conditions are equivalent:*

(1) *f is somewhat pairwise fuzzy open,*

(1) *if λ is a pairwise fuzzy dense set in (Y, S_1, S_2), then $f^{-1}(\lambda)$ is a pairwise fuzzy dense set in (X, T_1, T_2).*

Theorem 2.19 ([14])**.** *If $cl_{T_1}cl_{T_2}(\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) = 1$, for a fuzzy set λ in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then $cl_{T_1}(\lambda) = 1$ and $cl_{T_2}(\lambda) = 1$ in (X, T_1, T_2).*

Theorem 2.20 ([2])**.** *Let (X, T) and (Y, S) be any two fuzzy topological spaces and $f : (X, T) \rightarrow (Y, S)$ be a function. For fuzzy sets λ and μ of (X, T) and (Y, S) respectively, the following statements hold:*

(1) $ff^{-1}(\mu) \leq \mu$,
(2) $f^{-1}f(\lambda) \geq \lambda$,
(3) $f(1 - \lambda) \geq 1 - f(\lambda)$,
(4) $f^{-1}(1 - \mu) = 1 - f^{-1}(\mu)$,
(5) *if f is 1-1, then $f^{-1}f(\lambda) = \lambda$,*
(6) *if f is onto, then $ff^{-1}(\mu) = \mu$,*
(7) *if f is both 1-1 and onto, then $f(1 - \lambda) = 1 - f(\lambda)$.*

3. Pairwise fuzzy functions, somewhat pairwise fuzzy functions and pairwise fuzzy (weakly) Volterra spaces

Definition 3.1 ([15])**.** A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy Volterra space, if $cl_{T_i}\big(\wedge_{k=1}^{N} (\lambda_k)\big) = 1$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Definition 3.2 ([15])**.** A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy weakly Volterra space, if $\wedge_{k=1}^{N}(\lambda_k) \neq 0$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Proposition 3.3. *If $f : (X, T_1, T_2) \rightarrow (Y, S_1, S_2)$ is a pairwise fuzzy continuous function from a fuzzy bitopological space (X, T_1, T_2) into another fuzzy bitopological space (Y, S_1, S_2) and if λ is a pairwise fuzzy open set in (Y, S_1, S_2), then $f^{-1}(\lambda)$ is a pairwise fuzzy open set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy open set in (Y, S_1, S_2). Since f is a pairwise fuzzy continuous function, $f : (X, T_1) \rightarrow (Y, S_1)$ and $f : (X, T_2) \rightarrow (Y, S_2)$ are fuzzy continuous functions. Since λ is a pairwise fuzzy open set in (Y, S_1, S_2), $\lambda \in S_1$

and $\lambda \in S_2$. Since $f : (X, T_1) \to (Y, S_1)$ is fuzzy continuous, $f^{-1}(\lambda) \in T_1$ and $f : (X, T_2) \to (Y, S_2)$ is fuzzy continuous, $f^{-1}(\lambda) \in T_2$. Then $f^{-1}(\lambda)$ is a pairwise fuzzy open set in (X, T_1, T_2). □

Proposition 3.4. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a pairwise fuzzy continuous function from a fuzzy bitopological space (X, T_1, T_2) into another fuzzy bitopological space (Y, S_1, S_2) and if λ is a pairwise fuzzy G_δ-set in (Y, S_1, S_2), then $f^{-1}(\lambda)$ is a pairwise fuzzy G_δ-set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy G_δ-set in (Y, S_1, S_2). Then $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy open sets in (Y, S_1, S_2). Thus $f^{-1}(\lambda) = f^{-1}\left(\wedge_{k=1}^{\infty}(\lambda_k)\right) = \wedge_{k=1}^{\infty}\left(f^{-1}(\lambda_k)\right)$ [by Lemma 2.14]. By Proposition 3.3, $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy open sets in (X, T_1, T_2), for the pairwise fuzzy open sets (λ_k)'s in (Y, S_1, S_2). So $f^{-1}(\lambda) = \wedge_{k=1}^{\infty}\left(f^{-1}(\lambda_k)\right)$, where $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy open sets in (X, T_1, T_2), implies that $f^{-1}(\lambda)$ is a pairwise fuzzy G_δ-set in (X, T_1, T_2). □

Proposition 3.5. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a pairwise fuzzy open, 1-1 function from a fuzzy bitopological space (X, T_1, T_2) onto another fuzzy bitopological space (Y, S_1, S_2) and if μ is a pairwise fuzzy G_δ-set in (X, T_1, T_2), then $f(\mu)$ is a pairwise fuzzy G_δ-set in (Y, S_1, S_2).*

Proof. Let μ be a pairwise fuzzy G_δ-set in (X, T_1, T_2). Then $\mu = \wedge_{k=1}^{\infty}(\mu_k)$, where (μ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2). Since f is a pairwise fuzzy open function, (μ_k)'s are pairwise fuzzy open sets in (X, T_1, T_2), implies that $(f(\mu_k))$'s are pairwise fuzzy open sets in (Y, S_1, S_2). Thus $\wedge_{k=1}^{\infty}(f(\mu_k))$ is a pairwise fuzzy G_δ-set in (Y, S_1, S_2). Now $f^{-1}\left(\wedge_{k=1}^{\infty}(f(\mu_k))\right) = \wedge_{k=1}^{\infty}\left(f^{-1}(f(\mu_k))\right) = \wedge_{k=1}^{\infty}(\mu_k) = \mu$ [since f is 1-1, $f^{-1}(f(\mu_k)) = \mu_k$]. That is, $\mu = f^{-1}\left(\wedge_{k=1}^{\infty}(f(\mu_k))\right)$. So $f(\mu) = f\left(f^{-1}\left(\wedge_{k=1}^{\infty}(f(\mu_k))\right)\right) = \wedge_{k=1}^{\infty}(f(\mu_k))$ [since f is onto]. Hence $f(\mu)$ is a pairwise fuzzy G_δ-set in (Y, S_1, S_2). □

Proposition 3.6. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a pairwise fuzzy open function from a fuzzy bitopological space (X, T_1, T_2) into another fuzzy bitopological space (Y, S_1, S_2) and if λ is a pairwise fuzzy dense set in (Y, S_1, S_2), then $f^{-1}(\lambda)$ is a pairwise fuzzy dense set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy dense set in the fuzzy bitopological space (Y, S_1, S_2). Then $cl_{T_1}cl_{T_2}(\lambda) = 1$ and $cl_{T_2}cl_{T_1}(\lambda) = 1$. But $f^{-1}[cl_{T_1}cl_{T_2}(\lambda)] \leq cl_{T_1}f^{-1}[cl_{T_2}(\lambda)] \leq cl_{T_1}cl_{T_2}[f^{-1}(\lambda)]$ [by Theorem 2.16]. That is, $f^{-1}[cl_{T_1}cl_{T_2}(\lambda)] \leq cl_{T_1}cl_{T_2}[f^{-1}(\lambda)]$. Thus $f^{-1}[1] \leq cl_{T_1}cl_{T_2}[f^{-1}(\lambda)]$, implies that $1 \leq cl_{T_1}cl_{T_2}[f^{-1}(\lambda)]$. That is, $cl_{T_1}cl_{T_2}$ $[f^{-1}(\lambda)] = 1$. Also, $f^{-1}[cl_{T_2}cl_{T_1}(\lambda)] \leq cl_{T_2}f^{-1}[cl_{T_1}(\lambda)] \leq cl_{T_2}cl_{T_1}[f^{-1}(\lambda)]$. So $f^{-1}[1] \leq cl_{T_2}cl_{T_1}[f^{-1}(\lambda)]$, implies that $1 \leq cl_{T_2}cl_{T_1}[f^{-1}(\lambda)]$. That is, $cl_{T_2}cl_{T_1}$ $[f^{-1}(\lambda)] = 1$. Hence $f^{-1}(\lambda)$ is a pairwise fuzzy dense set in (X, T_1, T_2). □

Proposition 3.7. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a pairwise fuzzy continuous, 1-1 function from a fuzzy bitopological space (X, T_1, T_2) into another fuzzy bitopological space (Y, S_1, S_2) and if μ is a pairwise fuzzy dense set in (X, T_1, T_2), then $f(\mu)$ is a pairwise fuzzy dense set in (Y, S_1, S_2).*

Proof. Let μ be a pairwise fuzzy dense set in (X, T_1, T_2). It is to be proved that $f(\mu)$ is a pairwise fuzzy dense set in (Y, S_1, S_2). Assume the contrary. Suppose that $f(\mu)$

is not a pairwise fuzzy dense set in (Y, S_1, S_2). Then there exists a pairwise fuzzy closed set ν in (Y, S_1, S_2) such that $f(\mu) < \nu < 1$. But $f^{-1}f(\mu) < f^{-1}(\nu) < f^{-1}(1)$. Since f is 1-1, $\mu < f^{-1}(\nu) < f^{-1}(1)$. Since f is a pairwise fuzzy continuous function and ν is a pairwise fuzzy closed set in (Y, S_1, S_2), $f^{-1}(\nu)$ is a pairwise fuzzy closed set in (X, T_1, T_2). Now $\mu \leq f^{-1}(\nu)$, implies that $cl_{T_1}cl_{T_2}(\mu) \leq cl_{T_1}cl_{T_2}f^{-1}(\nu) = cl_{T_1}f^{-1}(\nu) = f^{-1}(\nu)$. That is, $cl_{T_1}cl_{T_2}(\mu) \leq f^{-1}(\nu) < 1$. Also, $cl_{T_2}cl_{T_1}(\mu) \leq cl_{T_2}cl_{T_1}f^{-1}(\nu) = cl_{T_2}f^{-1}(\nu) = f^{-1}(\nu)$. That is, $cl_{T_2}cl_{T_1}(\mu) \leq f^{-1}(\nu) < 1$. Thus $cl_{T_1}cl_{T_2}(\mu) \neq 1 \neq cl_{T_2}cl_{T_1}(\mu)$, which is a contradiction to μ being a pairwise fuzzy dense set in (X, T_1, T_2). So $f(\mu)$ is a pairwise fuzzy dense set in (Y, S_1, S_2). □

The following proposition gives conditions on f underwhich the image of a pairwise fuzzy Volterra space to be a pairwise fuzzy Volterra space.

Proposition 3.8. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a pairwise fuzzy continuous, 1-1, pairwise fuzzy open function from a pairwise fuzzy Volterra space (X, T_1, T_2) onto a pairwise fuzzy strongly irresolvable space (Y, S_1, S_2), then (Y, S_1, S_2) is a pairwise fuzzy Volterra space.*

Proof. Let (λ_k)'s $(k = 1 \text{ to } N)$ be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2), by Proposition 3.4, $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since f is a pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy dense sets in (Y, S_1, S_2), by Proposition 3.6, $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Then $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Volterra space, $cl_{T_i}\left(\wedge_{k=1}^{N}\left(f^{-1}(\lambda_k)\right)\right) = 1$, $(i = 1, 2)$ where $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). By Lemma 2.14, $\wedge_{k=1}^{N}\left(f^{-1}(\lambda_k)\right) = f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)$ and thus $cl_{T_i}\left(f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)\right) = 1$.

Now $cl_{T_1}cl_{T_2}\left(f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)\right) = cl_{T_1}(1) = 1$. Also, $cl_{T_2}cl_{T_1}\left(f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)\right) = cl_{T_2}(1) = 1$. That is, $cl_{T_1}cl_{T_2}\left(f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)\right) = 1 = cl_{T_2}cl_{T_1}\left(f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)\right)$. Thus $f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)$ is a pairwise fuzzy dense set in (X, T_1, T_2). Since f is a pairwise fuzzy continuous and 1-1 function, by Proposition 3.7, $f\left(f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)\right)$ is a pairwise fuzzy dense set in (Y, S_1, S_2). Since f is onto, $f\left(f^{-1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)\right) = \wedge_{k=1}^{N}(\lambda_k)$. So $\wedge_{k=1}^{N}(\lambda_k)$ is a pairwise fuzzy dense set in (Y, S_1, S_2). Hence $cl_{S_1}cl_{S_2}\left(\wedge_{k=1}^{N}(\lambda_k)\right) = 1 = cl_{S_2}cl_{S_1}\left(\wedge_{k=1}^{N}(\lambda_k)\right)$. Since (Y, S_1, S_2) is a pairwise fuzzy strongly irresolvable space, by Theorem 2.19, for the pairwise fuzzy dense set $\wedge_{k=1}^{N}(\lambda_k)$, $cl_{S_i}\left(\wedge_{k=1}^{N}(\lambda_k)\right) = 1$, $(i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Therefore (Y, S_1, S_2) is a pairwise fuzzy Volterra space. □

The following proposition gives conditions on f underwhich the image of a pairwise fuzzy weakly Volterra space to be a pairwise fuzzy weakly Volterra space.

Proposition 3.9. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a pairwise fuzzy continuous, 1-1, pairwise fuzzy open function from a pairwise fuzzy weakly Volterra space (X, T_1, T_2) onto a fuzzy bitopological space (Y, S_1, S_2), then (Y, S_1, S_2) is a pairwise fuzzy weakly Volterra space.*

Proof. Let (λ_k)'s $(k = 1 \text{ to } N)$ be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2), by Proposition 3.4, $(f^{-1}(\lambda_k))$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since f is a pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy dense sets in (Y, S_1, S_2), by Proposition 3.6, $(f^{-1}(\lambda_k))$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Then $(f^{-1}(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space, $\wedge_{k=1}^{N} (f^{-1}(\lambda_k)) \neq 0$, where $(f^{-1}(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). By Lemma 2.14, $\wedge_{k=1}^{N} (f^{-1}(\lambda_k)) = f^{-1}(\wedge_{k=1}^{N}(\lambda_k))$. Thus $f^{-1}(\wedge_{k=1}^{N}(\lambda_k)) \neq 0$. So $f(f^{-1}(\wedge_{k=1}^{N}(\lambda_k))) \neq f(0) = 0$. That is, $f(f^{-1}(\wedge_{k=1}^{N}(\lambda_k))) \neq 0$........(1). Since f is onto, $f(f^{-1}(\wedge_{k=1}^{N}(\lambda_k))) = \wedge_{k=1}^{N}(\lambda_k)$. From (1), $\wedge_{k=1}^{N}(\lambda_k) \neq 0$. Hence $\wedge_{k=1}^{N}(\lambda_k) \neq 0$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2), implies that (Y, S_1, S_2) is a pairwise fuzzy weakly Volterra space. □

Proposition 3.10. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a pairwise fuzzy continuous, 1-1 and somewhat pairwise fuzzy open function from a fuzzy bitopological space (X, T_1, T_2) onto a pairwise fuzzy strongly irresolvable space (Y, S_1, S_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space if and only if (Y, S_1, S_2) is a pairwise fuzzy Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy Volterra space and (λ_k)'s ($k = 1$ to N) be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2), by Proposition 3.4, $(f^{-1}(\lambda_k))$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since f is a somewhat pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy dense sets in (Y, S_1, S_2), by Theorem 2.18, $(f^{-1}(\lambda_k))$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Then $(f^{-1}(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Volterra space, $cl_{T_i}(\wedge_{k=1}^{N}(f^{-1}(\lambda_k))) = 1$, $(i = 1, 2)$ where $(f^{-1}(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Thus, by Lemma 2.14, $cl_{T_i}(f^{-1}(\wedge_{k=1}^{N}(\lambda_k))) = 1$.

Now $cl_{T_1}cl_{T_2}(f^{-1}(\wedge_{k=1}^{N}(\lambda_k))) = cl_{T_1}(1) = 1$. Also, $cl_{T_2}cl_{T_1}(f^{-1}(\wedge_{k=1}^{N}(\lambda_k))) = cl_{T_2}(1) = 1$. That is, $cl_{T_1}cl_{T_2}(f^{-1}(\wedge_{k=1}^{N}(\lambda_k))) = 1 = cl_{T_2}cl_{T_1}(f^{-1}(\wedge_{k=1}^{N}(\lambda_k)))$. So $f^{-1}(\wedge_{k=1}^{N}(\lambda_k))$ is a pairwise fuzzy dense set in (X, T_1, T_2). Since f is a pairwise fuzzy continuous and 1-1 function, by Proposition 3.7, $f(f^{-1}(\wedge_{k=1}^{N}(\lambda_k)))$ is a pairwise fuzzy dense set in (Y, S_1, S_2). Since f is onto, $f(f^{-1}(\wedge_{k=1}^{N}(\lambda_k))) = \wedge_{k=1}^{N}(\lambda_k)$ and hence $\wedge_{k=1}^{N}(\lambda_k)$ is a pairwise fuzzy dense set in (Y, S_1, S_2). Hence $cl_{S_1}cl_{S_2}(\wedge_{k=1}^{N}(\lambda_k)) = 1 = cl_{S_2}cl_{S_1}(\wedge_{k=1}^{N}(\lambda_k))$. Since (Y, S_1, S_2) is a pairwise fuzzy strongly irresolvable space, by Theorem 2.19, for the pairwise fuzzy dense set $\wedge_{k=1}^{N}(\lambda_k)$, $cl_{S_i}(\wedge_{k=1}^{N}(\lambda_k)) = 1$, $(i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Therefore (Y, S_1, S_2) is a pairwise fuzzy Volterra space.

Conversely, let (Y, S_1, S_2) be a pairwise fuzzy Volterra space and (λ_k)'s ($k = 1$ to N) be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2), by Proposition 3.4, $(f^{-1}(\lambda_k))$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since f is a somewhat pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy dense sets in (Y, S_1, S_2), by Theorem 2.18, $(f^{-1}(\lambda_k))$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Then $(f^{-1}(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy

G_δ-sets in (X, T_1, T_2). It has to be proved that $cl_{T_i}\left(\wedge_{k=1}^N\left(f^{-1}(\lambda_k)\right)\right) = 1$, $(i = 1, 2)$. Assume the contrary. Suppose that $cl_{T_i}\left(\wedge_{k=1}^N\left(f^{-1}(\lambda_k)\right)\right) \neq 1$. Then, $1 - cl_{T_i}\left(\wedge_{k=1}^N\left(f^{-1}(\lambda_k)\right)\right) \neq 0$. This implies that $int_{T_i}\left(\vee_{k=1}^N\left(1 - f^{-1}(\lambda_k)\right)\right) \neq 0$. That is, $int_{T_i}\left(\vee_{k=1}^N\left(f^{-1}(1-\lambda_k)\right)\right) \neq 0$. Then there will be a non-zero pairwise fuzzy open set μ_k in (X, T_1, T_2) such that $\mu_k \leq \vee_{k=1}^N\left(f^{-1}(1-\lambda_k)\right)$. This implies that $f(\mu_k) \leq f\left(\vee_{k=1}^N\left(f^{-1}(1-\lambda_k)\right)\right) = \vee_{k=1}^N\left(ff^{-1}(1-\lambda_k)\right)$ [by Lemma 2.15]. Thus $f(\mu_k) \leq \vee_{k=1}^N\left(ff^{-1}(1-\lambda_k)\right)$. Since f is onto, $ff^{-1}(1-\lambda_k) = 1-\lambda_k$. So $f(\mu_k) \leq \vee_{k=1}^N(1-\lambda_k) = 1 - \wedge_{k=1}^N(\lambda_k)$. That is, $f(\mu_k) \leq 1 - \wedge_{k=1}^N(\lambda_k)$.

Now $int_{S_i}(f(\mu_k)) \leq int_{S_i}\left(1 - \wedge_{k=1}^N(\lambda_k)\right) = 1 - cl_{S_i}\left(\wedge_{k=1}^N(\lambda_k)\right) = 1 - 1 = 0$ [since (Y, S_1, S_2) is a pairwise fuzzy Volterra space, $cl_{S_i}\left(\wedge_{k=1}^N(\lambda_k)\right) = 1$, $(i = 1, 2)$ and by Theorem 2.13 and Lemma 2.3]. Hence, $int_{S_i}(f(\mu_k)) \leq 0$. That is, $int_{S_i}(f(\mu_k)) = 0$, $(i = 1, 2)$, a contradiction to f being a somewhat pairwise fuzzy open function in which $int_{S_1}(f(\mu_k)) \neq 0$ or $int_{S_2}(f(\mu_k)) \neq 0$. Therefore, it must be $cl_{T_i}\left(\wedge_{k=1}^N\left(f^{-1}(\lambda_k)\right)\right) = 1$, $(i = 1, 2)$. Therefore $cl_{T_i}\left(\wedge_{k=1}^N\left(f^{-1}(\lambda_k)\right)\right) = 1$, $(i = 1, 2)$, where $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2), implies that (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

Proposition 3.11. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a somewhat pairwise fuzzy continuous, 1-1 and pairwise fuzzy open function from a fuzzy bitopological space (X, T_1, T_2) onto another fuzzy bitopological space (Y, S_1, S_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space if and only if (Y, S_1, S_2) is a pairwise fuzzy Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy Volterra space and (λ_k)'s ($k = 1$ to N) be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since f is a pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2), by Proposition 3.5, $(f(\lambda_k))$'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a somewhat pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2), by Theorem 2.17, $(f(\lambda_k))$'s are pairwise fuzzy dense sets in (Y, S_1, S_2). Then $(f(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2).

Now it has to be proved that $cl_{S_i}\left(\wedge_{k=1}^N(f(\lambda_k))\right) = 1$, $(i = 1, 2)$. Assume the contrary. Suppose that $cl_{S_i}\left(\wedge_{k=1}^N(f(\lambda_k))\right) \neq 1$. Then $1 - cl_{S_i}\left(\wedge_{k=1}^N(f(\lambda_k))\right) \neq 0$, implies that $int_{S_i}\left(1 - \wedge_{k=1}^N(f(\lambda_k))\right) = int_{S_i}\left(\vee_{k=1}^N(1 - f(\lambda_k))\right) \neq 0$ [by Lemma 2.3]. Since f is a 1-1 and onto function, $1 - f(\lambda_k) = f(1-\lambda_k)$ [by Theorem 2.20]. Thus $int_{S_i}\left(\vee_{k=1}^N(f(1-\lambda_k))\right) \neq 0$. So there will be a non-zero pairwise fuzzy open set μ_k in (Y, S_1, S_2) such that $\mu_k \leq \vee_{k=1}^N(f(1-\lambda_k))$. This implies that $f^{-1}(\mu_k) \leq f^{-1}\left(\vee_{k=1}^N(f(1-\lambda_k))\right) = \vee_{k=1}^N\left(f^{-1}(f(1-\lambda_k))\right)$ [by Lemma 2.14]. Then $f^{-1}(\mu_k) \leq \vee_{k=1}^N\left(f^{-1}(f(1-\lambda_k))\right)$. Since f is 1-1, $f^{-1}(f(1-\lambda_k)) = (1-\lambda_k)$ and thus $f^{-1}(\mu_k) \leq \vee_{k=1}^N(1-\lambda_k)$. So $int_{T_i}\left(f^{-1}(\mu_k)\right) \leq int_{T_i}\left(\vee_{k=1}^N(1-\lambda_k)\right)$ [by Theorem 2.13]. Since f is a somewhat pairwise fuzzy continuous function and $\mu_k \in S_i$, $(i = 1, 2)$, $int_{T_i}\left(f^{-1}(\mu_k)\right) \neq 0$. Hence $int_{T_i}\left(\vee_{k=1}^N(1-\lambda_k)\right) \neq 0$. This implies that $int_{T_i}\left(1 - \wedge_{k=1}^N(\lambda_k)\right) = 1 - cl_{T_i}\left(\wedge_{k=1}^N(\lambda_k)\right) \neq 0$ [by Lemma 2.3]. Therefore $cl_{T_i}\left(\wedge_{k=1}^N(\lambda_k)\right) \neq 1$, a contradiction [since (X, T_1, T_2) being a pairwise fuzzy Volterra space, $cl_{T_i}\left(\wedge_{k=1}^N(\lambda_k)\right) = 1$]. Hence, it must be $cl_{S_i}\left(\wedge_{k=1}^N(f(\lambda_k))\right) = 1$. Therefore $cl_{S_i}\left(\wedge_{k=1}^N(f(\lambda_k))\right) = 1$, $(i = 1, 2)$, where $(f(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2), implies that (Y, S_1, S_2) is a pairwise fuzzy Volterra space.

Conversely, let (Y, S_1, S_2) be a pairwise fuzzy Volterra space and (λ_k)'s ($k = 1$ to N) be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since f is a pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2), by Proposition 3.5, $(f(\lambda_k))$'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a somewhat pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2), by Theorem 2.17, $(f(\lambda_k))$'s are pairwise fuzzy dense sets in (Y, S_1, S_2). Then $(f(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2).

It has to be proved that $cl_{T_i}\left(\wedge_{k=1}^N(\lambda_k)\right) = 1$, $(i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Assume the contrary. Suppose that $cl_{T_i}\left(\wedge_{k=1}^N(\lambda_k)\right) \neq 1$. Then $1 - cl_{T_i}\left(\wedge_{k=1}^N(\lambda_k)\right) \neq 0$, implies that $int_{T_i}\left(1 - \wedge_{k=1}^N(\lambda_k)\right) = int_{T_i}\left(\vee_{k=1}^N(1 - \lambda_k)\right) \neq 0$ [by Lemma 2.3]. Thus there will be a non-zero pairwise fuzzy open set μ_k in (X, T_1, T_2) such that $\mu_k \leq \vee_{k=1}^N(1-\lambda_k)$. This implies that $f(\mu_k) \leq f\left(\vee_{k=1}^N(1 - \lambda_k)\right) = \vee_{k=1}^N(f(1 - \lambda_k))$ [by Lemma 2.15]. Then $f(\mu_k) \leq \vee_{k=1}^N(f(1 - \lambda_k))$. Since f is a 1-1 and onto function, $f(1 - \lambda_k) = 1 - f(\lambda_k)$ [by Theorem 2.20]. Thus $f(\mu_k) \leq \vee_{k=1}^N(1 - f(\lambda_k)) = 1 - \wedge_{k=1}^N(f(\lambda_k))$. That is, $f(\mu_k) \leq 1 - \wedge_{k=1}^N(f(\lambda_k))$. Now $int_{S_i}(f(\mu_k)) \leq int_{S_i}\left(1 - \wedge_{k=1}^N((\lambda_k))\right) = 1 - cl_{S_i}\left(\wedge_{k=1}^N(\lambda_k)\right) = 1 - 1 = 0$, [since (Y, S_1, S_2) is a pairwise fuzzy Volterra space, $cl_{S_i}\left(\wedge_{k=1}^N(\lambda_k)\right) = 1$, $(i = 1, 2)$ and by Theorem 2.13 and Lemma 2.3]. This implies that $int_{S_i}(f(\mu_k)) \leq 0$. So $int_{S_i}(f(\mu_k)) = 0$, $(i = 1, 2)$, a contradiction [since $\mu_k \in T_i$, $(i = 1, 2)$ and f is a pairwise fuzzy open function, implies that $f(\mu_k) \in S_i$ and thus $int_{S_i}(f(\mu_k)) = f(\mu_k)$, $(i = 1, 2)$]. Hence, it must be $cl_{T_i}\left(\wedge_{k=1}^N(\lambda_k)\right) = 1$. Therefore $cl_{T_i}\left(\wedge_{k=1}^N(\lambda_k)\right) = 1$, $(i = 1, 2)$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2), implies that (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

Proposition 3.12. *Let (X, T_1, T_2) and (Y, S_1, S_2) be any two fuzzy bitopological spaces and $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ be a pairwise fuzzy continuous, 1-1 and somewhat pairwise fuzzy open function. If (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space, then (Y, S_1, S_2) is a pairwise fuzzy weakly Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy weakly Volterra space and (λ_k)'s ($k = 1$ to N) be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2), by Proposition 3.4, $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since f is a somewhat pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy dense sets in (Y, S_1, S_2), by Theorem 2.18, $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Then $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space, $\wedge_{k=1}^N\left(f^{-1}(\lambda_k)\right) \neq 0$, where $\left(f^{-1}(\lambda_k)\right)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). By Lemma 2.14, $\wedge_{k=1}^N\left(f^{-1}(\lambda_k)\right) = f^{-1}\left(\wedge_{k=1}^N(\lambda_k)\right)$. Thus $f^{-1}\left(\wedge_{k=1}^N(\lambda_k)\right) \neq 0$. Now $f\left(f^{-1}\left(\wedge_{k=1}^N(\lambda_k)\right)\right) \neq f(0) = 0$. That is, $f\left(f^{-1}\left(\wedge_{k=1}^N(\lambda_k)\right)\right) \neq 0......(A)$. Since f is onto, $f\left(f^{-1}\left(\wedge_{k=1}^N(\lambda_k)\right)\right) = \wedge_{k=1}^N(\lambda_k)$. From (A), $\wedge_{k=1}^N(\lambda_k) \neq 0$. So $\wedge_{k=1}^N(\lambda_k) \neq 0$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2), implies that (Y, S_1, S_2) is a pairwise fuzzy weakly Volterra space. □

Proposition 3.13. *If $f : (X, T_1, T_2) \to (Y, S_1, S_2)$ is a somewhat pairwise fuzzy continuous, 1-1 and pairwise fuzzy open function from a fuzzy bitopological space (X, T_1, T_2) onto another fuzzy bitopological space (Y, S_1, S_2), then (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space if and only if (Y, S_1, S_2) is a pairwise fuzzy weakly Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy weakly Volterra space and (λ_k)'s ($k = 1$ to N) be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Then, $\wedge_{k=1}^{N}(\lambda_k) \neq 0$ ......(A). Since f is a pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2), by Proposition 3.5, $(f(\lambda_k))$'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a somewhat pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2), by Theorem 2.17, $(f(\lambda_k))$'s are pairwise fuzzy dense sets in (Y, S_1, S_2). Then $(f(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2).

Now it has to be proved that $\wedge_{k=1}^{N}(f(\lambda_k)) \neq 0$. Assume the contrary. Suppose that $\wedge_{k=1}^{N}(f(\lambda_k)) = 0$. Then $f^{-1}\left(\wedge_{k=1}^{N}(f(\lambda_k))\right) = f^{-1}(0) = 0$. But by Lemma 2.14, $f^{-1}\left(\wedge_{k=1}^{N}(f(\lambda_k))\right) = \wedge_{k=1}^{N}\left(f^{-1}(f(\lambda_k))\right)$. That is, $\wedge_{k=1}^{N}\left(f^{-1}(f(\lambda_k))\right) = 0$. Since f is a 1-1 function, $f^{-1}f(\lambda_k) = \lambda_k$ and thus $\wedge_{k=1}^{N}(\lambda_k) = 0$, a contradiction [since (X, T_1, T_2) being a pairwise fuzzy weakly Volterra space, $\wedge_{k=1}^{N}(\lambda_k) \neq 0$ from (A)]. So $\wedge_{k=1}^{N}(f(\lambda_k)) \neq 0$, where $(f(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2), implies that (Y, S_1, S_2) is a pairwise fuzzy weakly Volterra space.

Conversely, let (Y, S_1, S_2) be a pairwise fuzzy weakly Volterra space and (λ_k)'s ($k = 1$ to N) be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since f is a pairwise fuzzy open function and (λ_k)'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2), by Proposition 3.5, $(f(\lambda_k))$'s are pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since f is a somewhat pairwise fuzzy continuous function and (λ_k)'s are pairwise fuzzy dense sets in (X, T_1, T_2), by Theorem 2.17, $(f(\lambda_k))$'s are pairwise fuzzy dense sets in (Y, S_1, S_2). Then $(f(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Since (Y, S_1, S_2) is a pairwise fuzzy weakly Volterra space, $\wedge_{k=1}^{N}(f(\lambda_k)) \neq 0$, where $(f(\lambda_k))$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (Y, S_1, S_2). Thus $f^{-1}\left(\wedge_{k=1}^{N}(f(\lambda_k))\right) \neq f^{-1}(0) = 0$. But $f^{-1}\left(\wedge_{k=1}^{N}(f(\lambda_k))\right) = \wedge_{k=1}^{N}\left(f^{-1}(f(\lambda_k))\right)$ [by lemma 2.14]. That is, $\wedge_{k=1}^{N}\left(f^{-1}(f(\lambda_k))\right) \neq 0$. Since f is a 1-1 function, $f^{-1}f(\lambda_k) = \lambda_k$. So $\wedge_{k=1}^{N}(\lambda_k) \neq 0$. Hence, $\wedge_{k=1}^{N}(\lambda_k) \neq 0$, where (λ_k)'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2), implies that (X, T_1, T_2) is a pairwise fuzzy weakly Volterra space. □

4. Conclusions

In this paper, the results concerning pairwise fuzzy functions such as pairwise fuzzy continuous, somewhat pairwise fuzzy continuous, pairwise fuzzy open and somewhat pairwise fuzzy open functions that preserves pairwise fuzzy Volterraness, pairwise fuzzy weakly Volterraness in the context of images and pre-images in fuzzy bitopological spaces are obtained.

References

[1] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.
[2] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[3] David H. Foster, Fuzzy topological groups, J. Math. Anal. Appl. 67 (1979) 549–564.
[4] David Gauld and Zbigniew Piotrowski, On Volterra spaces, Far East J. Math. Sci. 1 (2) (1993) 209–214.
[5] David Gauld, Sina Greenwood and Zbigniew Piotrowski, On Volterra spaces - II, Papers on General Topology and its Applications, Ann. New York Acad. Sci. 806 (1996) 169–173.
[6] David Gauld, Sina Greenwood and Zbigniew Piotrowski, On Volterra spaces - III, Topological Operations, Topology Proc. 23 (1998) 167–182.
[7] G.Gruenhage and D.Lutzer, Baire and Volterra spaces , Proc. Amer. Math. Soc. 128 (10) (2000) 3115–3124.
[8] Jiling Cao and David Gauld, On Volterra spaces revisited, J. Aust. Math. Soc. 79 (2005) 61–76.
[9] A. Kandil, Biproximities and fuzzy bitopological spaces, Simen Stevin 63 (1989) 45–66.
[10] S. S. Thakur and R. Malviya, Semi-open sets and semi-continuity in fuzzy bitopological spaces, Fuzzy sets and systems 79 (1996) 251–256.
[11] G. Thangaraj, On pairwise fuzzy resolvable and fuzzy irresolvable spaces, Bull. Cal. Math. Soc. 102 (1) (2010) 59–68.
[12] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math. 11 (2) (2003) 725–736.
[13] G. Thangaraj and S. Sethuraman, A note on pairwise fuzzy Baire spaces, Ann. Fuzzy Math. Inform. 8(5) (2014) 729–737.
[14] G. Thangaraj and S. Sethuraman, Some remarks on pairwise fuzzy Baire spaces, Ann. Fuzzy Math. Inform. 9 (5) (2015) 683–691.
[15] G. Thangaraj and V. Chandiran, On pairwise fuzzy Volterra spaces, Ann. Fuzzy Math. Inform. 7 (6) (2014) 1005–1012.
[16] M. K. Uma, E. Roja and G. Balasubramanian, On somewhat pairwise fuzzy continuous functions, East Asian Math. J. 23 (1) (2007) 83–101.
[17] R. H. Warren, Neighborhoods bases and continuity in fuzzy topological spaces, Rocky Mountain J. Math. 8 (3) (1978) 459–470.
[18] T. H. Yalvac, Semi-interior and semi-closure of a fuzzy set, J. Math. Anal, Appl. 132 (1988) 356–364.
[19] L .A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338–353.

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