

Hermite-Hadamard type inequality for r_1 -convex function and r_2 -convex function using Sugeno integrals

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ABSTRACT. In this paper we prove a fuzzy integral inequality for r_1 -convex function and r_2 -convex function. Some examples satisfying results are also given.

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1. INTRODUCTION

In 1974 theory of fuzzy measures and fuzzy integrals was introduced by M. Sugeno in his Ph.D. thesis [11]. In [12, 13, 14], Hadamard inequalities for convex function is given. Since then many authors have worked on fuzzy integrals inequalities [2, 3, 4, 7, 8, 9, 10].

Motivated from the above results in this paper we present Hermite-Hadamard type inequality for r_1 -convex function and r_2 -convex function using Sugeno integrals.

2. PRELIMINARIES

Now we give some basic definitions and properties of the fuzzy integral given in [11, 16]. Suppose that \wp is a σ -algebra of subsets of X and $\mu : \wp \rightarrow [0, \infty)$ be a non-negative, extended real valued set function. We say that μ is a fuzzy measure, if

- (i) $\mu(\emptyset) = 0$,
- (ii) $E, F \in \wp$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$,
- (iii) $\{E_n\} \subset \wp, E_1 \subset E_2 \subset \dots$, imply $\lim_{n \rightarrow \infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$,
- (iv) $\{E_n\} \subset \wp, E_1 \supset E_2 \supset \dots, \mu(E_1) < \infty$, imply $\lim_{n \rightarrow \infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$.

If f is non-negative real-valued function defined on X , we denote the set $\{x \in X : f(x) \geq \alpha\} = \{x \in X : f \geq \alpha\}$ by F_α for $\alpha \geq 0$, where if $\alpha \leq \beta$, then $F_\beta \subset F_\alpha$.

Let (X, \wp, μ) be a fuzzy measure space, we denote M^+ the set of all non-negative measurable functions with respect to \wp .

Definition 2.1 ([11], Sugeno). Let (X, \wp, μ) be a fuzzy measure space, $f \in M^+$ and $A \in \wp$, the Sugeno integral (or fuzzy integral) of f on A , with respect to the fuzzy measure μ , is defined as:

$$(2.1) \quad (s) \int_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap F_\alpha)],$$

when $A = X$,

$$(2.2) \quad (s) \int_X f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(F_\alpha)],$$

where \bigvee and \wedge denote the operations sup and inf on $[0, \infty)$, respectively.

Some of the properties of fuzzy integral are as follows.

Proposition 2.2 ([15]). Let (X, \wp, μ) be fuzzy measure space, $A, B \in \wp$ and $f, g \in M^+$ then

- (1) $(s) \int_A f d\mu \leq \mu(A)$;
- (2) $(s) \int_A k d\mu = k \wedge \mu(A)$, k for non-negative constant,
- (3) $(s) \int_A f d\mu \leq (s) \int_A g d\mu$, for $f \leq g$,
- (4) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \implies (s) \int_A f d\mu \geq \alpha$,
- (5) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \implies (s) \int_A f d\mu \leq \alpha$,
- (6) $(s) \int_A f d\mu > \alpha \iff$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$,
- (7) $(s) \int_A f d\mu < \alpha \iff$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$.

Consider the distribution function F associated to f on A , that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then from (4) and (5) of Proposition 2.2, we have $F(\alpha) = \alpha \implies (s) \int_A f d\mu = \alpha$. Fuzzy integral can be obtained by solving the equation $F(\alpha) = \alpha$.

Definition 2.3 ([1]). a positive function f is called r -convex on $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$,

$$(2.3) \quad f(tx + (1-t)y) \leq \begin{cases} [tf^r(x) + (1-t)f^r(y)]^{\frac{1}{r}}, & r \neq 0, \\ [f(x)]^t [f(y)]^{1-t}, & r = 0. \end{cases}$$

It is obvious 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions. Note that if f is r -convex $[a, b]$, then f^r is a convex function ($r > 0$).

3. MAIN RESULTS

The Hermite-Hadamard type inequality for r -convex function and s -convex function obtained in [5].

Theorem 3.1. Let $f, g : [a, b] \rightarrow (0, \infty)$ be r -convex and s -convex function respectively on $[a, b]$ and $r, s > 0$. Then the following inequality holds

$$(3.1) \quad \frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{2} \left(\frac{r}{r+2} \right) \frac{f^{r+2}(b) - f^{r+2}(a)}{f^r(b) - f^r(a)} + \frac{1}{2} \left(\frac{s}{s+2} \right) \frac{g^{s+2}(b) - g^{s+2}(a)}{g^s(b) - g^s(a)},$$

$f(a) \neq f(b), g(a) \neq g(b)$.

Now we give an example.

Example 3.2. Consider $X = [0, 1]$ and let μ be the Lebesgue measure on X . If we take $r = s = \frac{7}{8}$ -convex functions, $f(x) = \frac{x^2}{2}$ and $g(x) = \frac{x^2}{2}$ on $[0, 1]$, we have f and g are r -convex and s -convex functions respectively, from simple calculation we get

$$(3.2) \quad (s) \int_0^1 \frac{x^4}{4} d\mu = 0.1380,$$

and the other hand

$$\frac{1}{2} \left(\frac{r}{r+2} \right) \frac{f^{r+2}(b) - f^{r+2}(a)}{f^r(b) - f^r(a)} + \frac{1}{2} \left(\frac{s}{s+2} \right) \frac{g^{s+2}(b) - g^{s+2}(a)}{g^s(b) - g^s(a)} = 0.076.$$

This proves that the inequality (3.1) is not satisfied for Sugeno integral.

Now we give Hermite-Hadamard type inequality for Sugeno integral using r_1 -convex and r_2 -convex functions.

Theorem 3.3. Let $f, g : [0, 1] \rightarrow [0, \infty)$ be the r_1 -convex and r_2 -convex functions respectively. Let $r_1, r_2 > 0$ and μ be the Lebesgue measure on \mathbf{R} with $f(0) \neq f(1)$ and $g(0) \neq g(1)$.

Case 1. If $f(1) > f(0)$ and $g(1) > g(0)$, then

$$(3.3) \quad (s) \int_0^1 fg d\mu \leq \min\{\beta, 1\},$$

where β is given by

$$(3.4) \quad 1 - \left(\frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)} \right) - \left(\frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)} \right) + \left(\frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)} \right) \cdot \left(\frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)} \right) = \beta.$$

Case 2. If $f(0) > f(1)$ and $g(0) > g(1)$, then

$$(3.5) \quad (s) \int_0^1 fg d\mu \leq \min\{\beta, 1\},$$

where β is satisfying the following equation

$$(3.6) \quad \beta(f^{r_1}(1) - f^{r_1}(0))(g^{r_2}(1) - g^{r_2}(0)) - (\beta^{r_1} - f^{r_1}(0))(\beta^{r_2} - g^{r_2}(0)) = 0.$$

Proof. Since f and g are r_1 -convex and r_2 -convex function respectively, we have

$$\begin{aligned} f(x) &= f(x.1 + (1 - x).0) \\ &\leq [x.f^{r_1}(1) + (1 - x)f^{r_1}(0)]^{1/r_1} = h_1(x), \\ g(x) &= g(x.1 + (1 - x).0) \\ &\leq [x.g^{r_2}(1) + (1 - x)g^{r_2}(0)]^{1/r_2} = h_2(x). \end{aligned}$$

Then from Proposition 2.2, we have

$$\begin{aligned} (s) \int_0^1 fg d\mu &= (s) \int_0^1 f(x.1 + (1 - x).0).g(x.1 + (1 - x).0) d\mu \\ &\leq (s) \int_0^1 [x.f^{r_1}(1) + (1 - x)f^{r_1}(0)]^{1/r_1} . [x.g^{r_2}(1) + (1 - x)g^{r_2}(0)]^{1/r_2} d\mu \\ (3.7) \quad &= (s) \int_0^1 h_1(x)h_2(x) d\mu. \end{aligned}$$

Now to obtain right hand side of (3.7), we consider the distribution function F given by:

$$\begin{aligned} F(\beta) &= \mu([0, 1] \cap \{x|h_1(x)h_2(x) \geq \beta\}) \\ &= \mu([0, 1] \cap \{x|h_1(x) \geq \beta\}) . \mu([0, 1] \cap \{x|h_2(x) \geq \beta\}) \\ &= (\mu([0, 1] \cap \{x|[x.f^{r_1}(1) + (1 - x)f^{r_1}(0)]^{1/r_1} \geq \beta\})) \\ (3.8) \quad &\times (\mu([0, 1] \cap \{x|[x.g^{r_2}(1) + (1 - x)g^{r_2}(0)]^{1/r_2} \geq \beta\})). \end{aligned}$$

Case 1. If $f(1) > f(0)$ and $g(1) > g(0)$, then from (3.8), we have

$$\begin{aligned} F(\beta) &= \mu\left([0, 1] \cap \left\{x|x \geq \frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)}\right\}\right) . \mu\left([0, 1] \cap \left\{x|x \geq \frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)}\right\}\right) \\ &= \mu\left(\frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)}, 1\right) . \mu\left(\frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)}, 1\right) \\ (3.9) \quad &= \left(1 - \left(\frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)}\right)\right) . \left(1 - \left(\frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)}\right)\right) \end{aligned}$$

and solution of (3.9) is $F(\beta) = \beta$ given in (3.4). From Proposition 2.2, we have

$$(s) \int_0^1 fg d\mu \leq \min\{\beta, 1\}.$$

Case 2. If $f(0) > f(1)$ and $g(0) > g(1)$, then from (3.8), we have

$$\begin{aligned} F(\beta) &= \mu\left([0, 1] \cap \left\{x|x \leq \frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)}\right\}\right) . \mu\left([0, 1] \cap \left\{x|x \leq \frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)}\right\}\right) \\ &= \mu\left(0, \frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)}\right) . \mu\left(0, \frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)}\right) \\ (3.10) \quad &= \left(\frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)}\right) . \left(\frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)}\right) \end{aligned}$$

and solution of (3.10) is $F(\beta) = \beta$, given in (3.6). From Proposition 2.2, we have

$$(s) \int_0^1 fg d\mu \leq \min\{\beta, 1\}.$$

□

Remark 3.4. In case if we put $f(0) = f(1)$ and $g(0) = g(1)$ in Theorem 3.3 then

$$(s) \int_0^1 f(x)g(x)d\mu \leq (s) \int_0^1 f(0)g(0)d\mu = f(0)g(0) \wedge 1.$$

Example 3.5. Consider $X = [0, 1]$ and let μ be the Lebesgue measure on X . If we take $f(x) = \frac{x^3}{3}$ and $g(x) = \frac{x^3}{3}$ then $f(x), g(x)$ are $r_1 = r_2 = \frac{7}{8}$ -convex function then from Theorem 3.3 we have

$$(3.11) \quad 0.0712 = (s) \int_0^1 \frac{x^6}{9} d\mu \leq \min\{0.1781, 1\} = 0.1781.$$

Now we give the following theorem which is general case of Theorem 3.3.

Theorem 3.6. Let $r_1, r_2 > 0$ and let μ be the Lebesgue measure on \mathbf{R} . Let $f, g : [a, b] \rightarrow [0, \infty)$ be r_1 -convex and r_2 -convex function with $f(a) \neq f(b)$ and $g(a) \neq g(b)$.

Case 1. If $f(b) > f(a)$ and $g(b) > g(a)$, then

$$(s) \int_a^b fg d\mu \leq \min\{\beta, b - a\},$$

where β is given by:

$$(3.12) \quad (b - a)^2 - (b - a)^2 \left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)} \right) - (b - a)^2 \left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)} \right) + (b - a)^2 \left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)} \right) \left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)} \right) = \beta.$$

Case 2. If $f(a) > f(b)$ and $g(a) > g(b)$, then

$$(s) \int_a^b fg d\mu \leq \min\{\beta, b - a\},$$

where β satisfies the following equation

$$(3.13) \quad \beta(f^{r_1}(b) - f^{r_1}(a))(g^{r_2}(b) - g^{r_2}(a)) - (b - a)^2(\beta^{r_1} - f^{r_1}(a))(\beta^{r_2} - g^{r_2}(a)) = 0.$$

Proof. As f and g are r_1 -convex and r_2 -convex functions, $x \in [a, b]$, we have

$$\begin{aligned} f(x) &= f\left(\left(1 - \frac{x-a}{b-a}\right).a + \frac{x-a}{b-a}.b\right) \\ &\leq \left[\left(1 - \frac{x-a}{b-a}\right).f^{r_1}(a) + \frac{x-a}{b-a}.f^{r_1}(b)\right]^{\frac{1}{r_1}} = h_1(x), \\ g(x) &= g\left(\left(1 - \frac{x-a}{b-a}\right).a + \frac{x-a}{b-a}.b\right) \\ &\leq \left[\left(1 - \frac{x-a}{b-a}\right).g^{r_2}(a) + \frac{x-a}{b-a}.g^{r_2}(b)\right]^{\frac{1}{r_2}} = h_2(x). \end{aligned}$$

Hence by (3) of Proposition 2.2, we have

$$\begin{aligned} (s) \int_a^b fg d\mu &= (s) \int_a^b f\left(\left(1 - \frac{x-a}{b-a}\right).a + \frac{x-a}{b-a}.b\right).g\left(\left(1 - \frac{x-a}{b-a}\right).a + \frac{x-a}{b-a}.b\right) d\mu \\ &\leq (s) \int_a^b \left[\left(1 - \frac{x-a}{b-a}\right).f^{r_1}(a) + \frac{x-a}{b-a}.f^{r_1}(b)\right]^{\frac{1}{r_1}} \\ &\quad \left[\left(1 - \frac{x-a}{b-a}\right).g^{r_2}(a) + \frac{x-a}{b-a}.g^{r_2}(b)\right]^{\frac{1}{r_2}} d\mu \\ &= (s) \int_a^b h_1(x)h_2(x) d\mu. \end{aligned}$$

To obtain right hand side of above inequality, we consider the distribution function F as

$$\begin{aligned} F(\beta) &= \mu([a, b] \cap \{x|h_1(x)h_2(x) \geq \beta\}) \\ &= \mu([a, b] \cap \{x|h_1(x) \geq \beta\}) \cdot \mu([a, b] \cap \{x|h_2(x) \geq \beta\}) \\ &= \mu\left([a, b] \cap \left\{x \mid \left[\left(1 - \frac{x-a}{b-a}\right).f^{r_1}(a) + \frac{x-a}{b-a}.f^{r_1}(b)\right]^{\frac{1}{r_1}} \geq \beta\right\}\right) \\ (3.14) \quad &\mu\left([a, b] \cap \left\{x \mid \left[\left(1 - \frac{x-a}{b-a}\right).g^{r_2}(a) + \frac{x-a}{b-a}.g^{r_2}(b)\right]^{\frac{1}{r_2}} \geq \beta\right\}\right). \end{aligned}$$

Case 1. If $f(b) > f(a)$ and $g(b) > g(a)$, then from (3.14), we have

$$\begin{aligned} F(\beta) &= \mu\left([a, b] \cap \left\{x \mid x \geq (b-a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right) + a\right\}\right) \\ &\quad \mu\left([a, b] \cap \left\{x \mid x \geq (b-a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right) + a\right\}\right) \\ &= \mu\left((b-a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right) + a, b\right) \cdot \mu\left((b-a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right) + a, b\right) \\ (3.15) \quad &= \left((b-a) - (b-a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right)\right) \cdot \left((b-a) - (b-a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right)\right) \end{aligned}$$

and the solution of (3.15) is $F(\beta) = \beta$, given in (3.12). From (1) of Proposition 2.2, we have

$$(s) \int_0^1 fgd\mu \leq \min\{\beta, b - a\}.$$

Case 2. If $f(a) > f(b)$ and $g(a) > g(b)$, then from (3.14), we have

$$\begin{aligned} F(\beta) &= \mu\left([a, b] \cap \left\{x \mid x \leq (b - a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right) + a\right\}\right) \\ &\quad \mu\left([a, b] \cap \left\{x \mid x \leq (b - a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right) + a\right\}\right) \\ &= \mu\left(a, (b - a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right) + a\right) \cdot \mu\left(a, (b - a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right) + a\right) \\ (3.16) \quad &= \left((b - a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right)\right) \cdot \left((b - a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right)\right) \end{aligned}$$

and the solution of (3.16) is $F(\beta) = \beta$, given in (3.13). By (1) of Proposition 2.2, we have

$$(s) \int_0^1 fgd\mu \leq \min\{\beta, b - a\}.$$

□

Remark 3.7. If we put $f(a) = f(b)$ and $g(a) = g(b)$, then from Theorem 3.6, we have

$$(s) \int_a^b f(x)g(x)d\mu \leq (s) \int_a^b f(a)g(a)d\mu = f(a)g(a) \wedge b - a.$$

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