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Fuzzy topological concepts via ideals and grills

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Abstract. In this paper, we have introduced various types of r-fuzzy ideal continuity based on a fuzzy ideal $\mathcal{I}$ on a fuzzy topological space (X,τ). According to various types of r-fuzzy ideal openness, many implications between these types of r-fuzzy ideal continuity are illustrated. Fuzzy ideal openness and fuzzy ideal β-continuity are the core of these types of continuity. Fuzzy grills are investigated, and it is shown that studying concepts in view of fuzzy ideals is equivalent to studying the same concepts in view of fuzzy grills.

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1. Introduction and Preliminaries

Using a fuzzy ideal $\mathcal{I}$ defined on a fuzzy topological space (X,τ), it is generated a fuzzy ideal topological space $(X,\tau,\mathcal{I})$. It is a way of generalization of many notions and results in fuzzy topological spaces. The main definition of fuzzy topology was defined by Sŏstak in [8]. The notion of fuzzy ideal was given in [7], and various types of fuzzy continuity were defined and studied in [1, 2, 4, 5, 6, 7]. The notion of fuzzy grill was given in [3]. Tripathy and et. in [9, 10, 11, 12, 13], introduced many research studies on fuzzy topological spaces, fuzzy ideal topological spaces and several types of fuzzy continuity.

In this paper, several types of r-fuzzy ideal openness and r-fuzzy ideal continuity are introduced and studied. It is proved many implications in between these notions of r-fuzzy ideal continuity itself in fuzzy ideal topological spaces, and also between these notions of r-fuzzy ideal continuity and the notions of usual r-fuzzy continuity in fuzzy topological spaces. Fuzzy grill notion is introduced and it is proved that there is a one-to-one correspondence between the fuzzy ideal notion and the fuzzy grill notion. From that correspondence, any topological fuzzy property was generalized to the fuzzy ideal topological spaces could be generalized to the fuzzy grill topological spaces, and the converse is also true. As a conclusion, adding a fuzzy ideal $\mathcal{I}$ on a fuzzy topological space (X,τ) gives us a generalization of fuzzy topological properties equivalent to the generalization has been made by adding a fuzzy grill $\mathcal{G}$ on the space (X,τ). r-fuzzy ideal compactness and fuzzy grill compactness are introduced using the fuzzy ideal $\mathcal{I}$ and the fuzzy grill $\mathcal{G}$ on X respectively, giving a generalization of r-fuzzy compactness. This is a short study on fuzzy ideal compactness, just to illustrate that studying one of the fuzzy topological properties based on fuzzy ideals or based on fuzzy grills is identical. Throughout the paper, X refers to an initial universe, I^X is the set of all fuzzy sets on X (where $I=[0,1], I_0=(0,1], \lambda^c(x)=1-\lambda(x)\ \forall x\in X$ and for all $t\in I, \bar{t}(x)=t\ \forall x\in X$). (X,τ) is a fuzzy topological space as in [8].

A map $\mathcal{I}: I^X \to I$ is called a fuzzy ideal ([7]) on X if it satisfies the following conditions:

(i) $\mathcal{I}(\overline{0})=1$,
(ii) $\lambda\le\mu \Rightarrow \mathcal{I}(\lambda)\ge\mathcal{I}(\mu)$ for all $\lambda,\mu\in I^X$,
(iii) $\mathcal{I}(\lambda\vee\mu)\ge\mathcal{I}(\lambda)\wedge\mathcal{I}(\mu)$ for all $\lambda,\mu\in I^X$.

If $\mathcal{I}_1$ and $\mathcal{I}_2$ are fuzzy ideals on X, we have $\mathcal{I}_1$ is finer than $\mathcal{I}_2$ ($\mathcal{I}_2$ is coarser than $\mathcal{I}_1$), denoted by $\mathcal{I}_1\le\mathcal{I}_2$ iff $\mathcal{I}_1(\lambda)\le\mathcal{I}_2(\lambda)\ \forall\lambda\in I^X$. The triple $(X,\tau,\mathcal{I})$ is called a fuzzy ideal topological space. Also, $\mathcal{I}$ is called proper if $\mathcal{I}(\overline{1})=0$. Define the fuzzy ideal $\mathcal{I}^\circ$ by $\mathcal{I}^\circ(\mu)=1$ at $\mu=\overline{0}$ and $\mathcal{I}^\circ(\mu)=0$ otherwise.

Let us define the fuzzy difference between two fuzzy sets as follows:

$$(\lambda \bar{\wedge} \mu) = \begin{cases} \overline{0} & \text{if } \lambda\le\mu, \\ \lambda\wedge\mu^c & \text{otherwise.} \end{cases}$$

Consider the family Ω denotes the set of all fuzzy subsets of a given set X satisfying the following condition: $\forall\lambda,\mu\in\Omega,\quad \lambda\le\mu$ or $\mu\le\lambda$.

Note that: For each $\lambda,\mu,\nu\in\Omega$, we have:

(1) $\nu\bar{\wedge}(\lambda\wedge\mu) = (\nu\bar{\wedge}\lambda)\vee(\nu\bar{\wedge}\mu)$,
(2) $(\lambda\vee\mu)\bar{\wedge}\nu = (\lambda\bar{\wedge}\nu)\vee(\mu\bar{\wedge}\nu)$.

Definition 1.1. Let $(X,\tau,\mathcal{I})$ be a fuzzy ideal topological space and $\lambda\in I^X$. Then, the r-fuzzy open local function $\lambda^*_r(\tau,\mathcal{I})$ of λ is defined by:

$$\lambda^*_r(\tau,\mathcal{I}) = \bigwedge\{\mu\in I^X : \mathcal{I}(\lambda\bar{\wedge}\mu)\ge r,\ \tau(\mu^c)\ge r\}.$$

Occasionally, we will write λ^*_r or $\lambda^*_r(\mathcal{I})$ for $\lambda^*_r(\tau,\mathcal{I})$ and it will be no ambiguity.

Example 1.2. Let $(X,\tau,\mathcal{I})$ be a fuzzy ideal topological space. The simplest fuzzy ideal on X is the ideal $\mathcal{I}^\circ$. If $\mathcal{I}=\mathcal{I}^\circ$ then, for each $\lambda\in I^X$, $r\in I_0$, we have $\lambda^*_r=\mathrm{cl}_\tau(\lambda,r)$.

Proposition 1.3. *Let $(X,\tau,\mathcal{I})$ be a fuzzy ideal topological space and $\mathcal{I}_1,\mathcal{I}_2$ be fuzzy ideals on X. Then*

(1) $\lambda\le\mu$ *implies* $\lambda^*_r\le\mu^*_r$,
(2) *if* $\mathcal{I}_1\le\mathcal{I}_2$, *then* $\lambda^*_r(\mathcal{I}_1)\ge\lambda^*_r(\mathcal{I}_2)$,
(3) $\lambda^*_r=\mathrm{cl}_\tau(\lambda^*_r,r)\le\mathrm{cl}_\tau(\lambda,r)$ *and* $(\lambda^*_r)^*_r\le\lambda^*_r$.

(4) $\lambda^*_r \vee \mu^*_r \leq (\lambda \vee \mu)^*_r$, and $\lambda^*_r \wedge \mu^*_r \geq (\lambda \wedge \mu)^*_r$.
(5) *if* $\mathcal{I}(\mu) \geq r$, *then* $(\lambda \vee \mu)^*_r \geq \lambda^*_r$.

Proof. (1) Suppose $\lambda^*_r \not\leq \mu^*_r$, then there exists $\nu \in I^X$ with $\mathcal{I}(\mu \bar{\wedge} \nu) \geq r$, for each $\tau(\nu^c) \geq r$ such that $\lambda^*_r > \nu \geq \mu^*_r$. Since $\lambda \leq \mu$, $\lambda \bar{\wedge} \nu \leq \mu \bar{\wedge} \nu$ and $\mathcal{I}(\lambda \bar{\wedge} \nu) \geq \mathcal{I}(\mu \bar{\wedge} \nu) \geq r$, for each $\tau(\nu^c) \geq r$. Thus $\lambda^*_r \leq \nu$ and so we arrive at a contradiction. Hence $\lambda^*_r \leq \mu^*_r$.

(2) Suppose $\lambda^*_r(\mathcal{I}_1) \not\geq \lambda^*_r(\mathcal{I}_2)$, then there exists $\nu \in I^X$ with $\mathcal{I}_1(\lambda \bar{\wedge} \nu) \geq r$, for each $\tau(\nu^c) \geq r$ such that $\lambda^*_r(\mathcal{I}_1) \leq \nu < \lambda^*_r(\mathcal{I}_2)$. Since ($\mathcal{I}_1$ is finer than $\mathcal{I}_2$) $\mathcal{I}_2(\lambda \bar{\wedge} \nu) \geq \mathcal{I}_1(\lambda \bar{\wedge} \nu) \geq r$, for each $\tau(\nu^c) \geq r$, $\lambda^*_r(\mathcal{I}_2) \leq \nu$. Which is a contradiction. Thus $\lambda^*_r(\mathcal{I}_1) \geq \lambda^*_r(\mathcal{I}_2)$.

(3) Suppose $\lambda^*_r \not\leq \mathrm{cl}_\tau(\lambda, r)$. Then, there exists $\nu \in I^X$ with $\lambda \leq \nu$, $\tau(\nu^c) \geq r$ such that $\lambda^*_r > \nu \geq \mathrm{cl}_\tau(\lambda, r)$. Since $\lambda \leq \nu$, $\mathcal{I}(\lambda \bar{\wedge} \nu) \geq r$ with $\tau(\nu^c) \geq r$. Thus $\lambda^*_r \leq \nu$. It is a contradiction. So $\lambda^*_r = \mathrm{cl}_\tau(\lambda^*_r, r) \leq \mathrm{cl}_\tau(\lambda, r)$. Hence from (3), we have $(\lambda^*_r)^*_r = \mathrm{cl}_\tau((\lambda^*_r)^*_r, r) \leq \mathrm{cl}_\tau(\lambda^*_r, r) = \lambda^*_r$.

(4) Since $\lambda, \mu \leq \lambda \vee \mu$. By (1), we have $\lambda^*_r \leq (\lambda \vee \mu)^*_r$, $\mu^*_r \leq (\lambda \vee \mu)^*_r$. Then $\lambda^*_r \vee \mu^*_r \leq (\lambda \vee \mu)^*_r$.

(5) Can be easily established using standard technique. □

Lemma 1.4. *Let $\tau : \Omega \to I$ be a fuzzy topology on X and $\mathcal{I} : \Omega \to I$ a fuzzy ideal on X. Then, for each $\lambda, \mu \in \Omega$, $r \in I_0$,*

(1) $(\lambda \vee \mu)^*_r = \lambda^*_r \vee \mu^*_r$,
(2) *If* $\mathcal{I}(\mu) \geq r$, *then* $(\lambda \vee \mu)^*_r = \lambda^*_r$.

Proof. (1) Already, we have $\lambda^*_r \vee \mu^*_r \leq (\lambda \vee \mu)^*_r$. Suppose $\lambda^*_r \vee \mu^*_r \not\geq (\lambda \vee \mu)^*_r$. Then, there exist $\nu_1, \nu_2 \in \Omega$, $\mathcal{I}(\lambda \bar{\wedge} \nu_1) \geq r$ with $\tau(\nu_1^c) \geq r$ and $\mathcal{I}(\mu \bar{\wedge} \nu_2) \geq r$ with $\tau(\nu_2^c) \geq r$ such that $\lambda^*_r \vee \mu^*_r \leq \nu_1 \vee \nu_2 < (\lambda \vee \mu)^*_r$. But $(\lambda \vee \mu) \bar{\wedge} (\nu_1 \vee \nu_2) = (\lambda \bar{\wedge} (\nu_1 \vee \nu_2)) \vee (\mu \bar{\wedge} (\nu_1 \vee \nu_2)) \leq (\lambda \bar{\wedge} \nu_1) \vee (\mu \bar{\wedge} \nu_2)$, and then $\mathcal{I}((\lambda \vee \mu) \bar{\wedge} (\nu_1 \vee \nu_2)) \geq r$ and $\tau((\nu_1 \vee \nu_2)^c) \geq r$. Thus $(\lambda \vee \mu)^*_r \leq \nu_1 \vee \nu_2$, which is a contradiction. So $\lambda^*_r \vee \mu^*_r \geq (\lambda \vee \mu)^*_r$.

(2) Clear. □

Proposition 1.5. *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\{\mu_j : j \in J\} \subseteq I^X$ a family. Then*

(1) $\bigvee((\mu_j)^*_r : j \in J) \leq (\bigvee(\mu_j) : j \in J)^*_r$,
(2) $\bigwedge((\mu_j)^*_r : j \in J) \geq (\bigwedge(\mu_j) : j \in J)^*_r$.

Proof. (1) Since $\mu_j \leq \bigvee \mu_j \ \forall j \in J$, and by (1) in Proposition 1.3, we have $(\bigvee(\mu_j))^*_r \geq (\mu_j)^*_r$, $j \in J$. Then (1) holds.

(2) Similar to the proof of (1). □

Definition 1.6. Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\mu \in I^X$. Then

$$\mathrm{cl}^*_\tau(\mu, r) = \mu \vee \mu^*_r \quad \text{and} \quad \mathrm{int}^*_\tau(\mu, r) = \mu \wedge ((\mu^c)^*_r)^c.$$

cl^*_τ is a fuzzy closure operator and $\tau^*(\mathcal{I})$ is a fuzzy topology on X generated by cl^*_τ, that is, $(\tau^*(\mathcal{I}))(\mu) = \bigvee\{r \in I_0 : \mathrm{cl}^*_\tau(\mu^c, r) = \mu^c\}$. Now, if $\mathcal{I} = \mathcal{I}^\circ$, then for each $\mu \in I^X, r \in I_0$, $\mathrm{cl}^*_\tau(\mu, r) = \mu \vee \mu^*_r = \mu \vee \mathrm{cl}_\tau(\mu, r) = \mathrm{cl}_\tau(\mu, r)$. So, $\tau^*(\mathcal{I}^\circ) = \tau$.

Proposition 1.7. *Let $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space and $\lambda, \mu \in I^X$, $r \in I_0$. Then*

(1) $\mathrm{int}^*_\tau(\lambda \vee \mu, r) \geq \mathrm{int}^*_\tau(\lambda, r) \vee \mathrm{int}^*_\tau(\mu, r)$,
(2) $\mathrm{int}_\tau(\lambda, r) \leq \mathrm{int}^*_\tau(\lambda, r) \leq \lambda \leq \mathrm{cl}^*_\tau(\lambda, r) \leq \mathrm{cl}_\tau(\lambda, r)$,

(3) $\mathrm{cl}^*_\tau(\lambda^c, r) = (\mathrm{int}^*_\tau(\lambda, r))^c$ and $\mathrm{int}^*_\tau(\lambda^c, r) = (\mathrm{cl}^*_\tau(\lambda, r))^c$,
(4) $\mathrm{int}^*_\tau(\lambda \wedge \mu, r) \le \mathrm{int}^*_\tau(\lambda, r) \wedge \mathrm{int}^*_\tau(\mu, r)$.

Proof. (1) From Proposition 1.3 (4), we have

$$\begin{aligned} \mathrm{int}^*_\tau(\lambda \vee \mu, r) &= (\lambda \vee \mu) \wedge ((((\lambda \vee \mu)^c)^*_r)^c) \\ &\ge (\lambda \vee \mu) \wedge (((\lambda^c)^*_r \wedge (\mu^c)^*_r)^c) \\ &\ge (\lambda \wedge (((\lambda^c)^*_r)^c)) \vee (\mu \wedge (((\mu^c)^*_r)^c)) \\ &= \mathrm{int}^*_\tau(\lambda, r) \vee \mathrm{int}^*_\tau(\mu, r). \end{aligned}$$

(2) Follows directly from definitions of cl^*_τ, int^*_τ and cl_τ.
(3) $\mathrm{cl}^*_\tau(\lambda^c, r) = (\lambda^c) \vee ((\lambda^c)^*_r) = (\lambda^c) \vee [((\lambda^c)^*_r)^c]^c = [\lambda \wedge ((\lambda^c)^*_r)^c]^c = [\mathrm{int}^*_\tau(\lambda, r)]^c$.
(4) From Proposition 1.3 (4), we have

$$\begin{aligned} \mathrm{int}^*_\tau(\lambda \wedge \mu, r) &= (\lambda \wedge \mu) \wedge ((((\lambda \wedge \mu)^c)^*_r)^c) \\ &\le (\lambda \wedge ((\lambda^c)^*_r)^c) \wedge (\mu \wedge ((\mu^c)^*_r)^c) \\ &= \mathrm{int}^*_\tau(\lambda, r) \wedge \mathrm{int}^*_\tau(\mu, r). \end{aligned}$$

□

Lemma 1.8. *Let $\tau : \Omega \to I$ be a fuzzy topology on X and $\mathcal{I} : \Omega \to I$ a fuzzy ideal on X. Then, for each $\lambda, \mu \in \Omega$, $r \in I_0$, the operator $\mathrm{int}^*_\tau : \Omega \times I_0 \to \Omega$ satisfies the following:*

$$\mathrm{int}^*_\tau((\lambda \wedge \mu), r) = \mathrm{int}^*_\tau(\lambda, r) \wedge \mathrm{int}^*_\tau(\mu, r).$$

Proof. From Proposition 1.7 (4), and from Lemma 1.4. □

Corollary 1.9. *Let $(X, \tau_1, \mathcal{I})$, $(X, \tau_2, \mathcal{I})$ be fuzzy ideal topological spaces and $\tau_1 \le \tau_2$. Then, for each $\lambda \in I^X$, $r \in I_0$, $\lambda^*_r(\tau_2, \mathcal{I}) \le \lambda^*_r(\tau_1, \mathcal{I})$ and $\tau^*_1(\mathcal{I}) \le \tau^*_2(\mathcal{I})$.*

Corollary 1.10. *Let $(X, \tau, \mathcal{I}_1)$, $(X, \tau, \mathcal{I}_2)$ be fuzzy ideal topological spaces and $\mathcal{I}_1 \le \mathcal{I}_2$. Then, for each $\lambda \in I^X$, $r \in I_0$, $\lambda^*_r(\tau, \mathcal{I}_1) \ge \lambda^*_r(\tau, \mathcal{I}_2)$ and $\tau^*(\mathcal{I}_1) \le \tau^*(\mathcal{I}_2)$.*

Proposition 1.11. *Let (X, τ) be a fuzzy topological space, and $\mathcal{I}_1, \mathcal{I}_2$ fuzzy ideals on X. Then, for each $\lambda \in I^X$, $r \in I_0$,*
(1) $\lambda^*_r(\tau, \mathcal{I}_1 \wedge \mathcal{I}_2) = \lambda^*_r(\tau, \mathcal{I}_1) \vee \lambda^*_r(\tau, \mathcal{I}_2)$,
(2) $\lambda^*_r(\tau, \mathcal{I}_1 \vee \mathcal{I}_2) = \lambda^*_r(\tau^*(\mathcal{I}_2), \mathcal{I}_1) \wedge \lambda^*_r(\tau^*(\mathcal{I}_1), \mathcal{I}_2)$.

Proof. (1) Suppose $\lambda^*_r(\tau, \mathcal{I}_1 \wedge \mathcal{I}_2) \not\le \lambda^*_r(\tau, \mathcal{I}_1) \vee \lambda^*_r(\tau, \mathcal{I}_2)$. Then, there exist $\nu_1, \nu_2 \in I^X$, $\mathcal{I}_1(\lambda \bar{\wedge} \nu_1) \ge r$ with $\tau(\nu_1^c) \ge r$ and $\mathcal{I}_2(\lambda \bar{\wedge} \nu_2) \ge r$ with $\tau(\nu_2^c) \ge r$ such that $\lambda^*_r(\tau, \mathcal{I}_1 \wedge \mathcal{I}_2) > \nu_1 \vee \nu_2 \ge \lambda^*_r(\tau, \mathcal{I}_1) \vee \lambda^*_r(\tau, \mathcal{I}_2)$. Since $(\mathcal{I}_1 \wedge \mathcal{I}_2)(\lambda \bar{\wedge} (\nu_1 \vee \nu_2)) \ge r$ and $\tau((\nu_1 \vee \nu_2)^c) \ge r$, $\lambda^*_r(\tau, \mathcal{I}_1 \wedge \mathcal{I}_2) \le \nu_1 \vee \nu_2$. Which is a contradiction. Thus, $\lambda^*_r(\tau, \mathcal{I}_1 \wedge \mathcal{I}_2) \le \lambda^*_r(\tau, \mathcal{I}_1) \vee \lambda^*_r(\tau, \mathcal{I}_2)$.

Conversely, since $\mathcal{I}_1 \wedge \mathcal{I}_2 \le \mathcal{I}_1, \mathcal{I}_2$, by Proposition 1.3 (2), we get that $\lambda^*_r(\tau, \mathcal{I}_1 \wedge \mathcal{I}_2) \ge \lambda^*_r(\tau, \mathcal{I}_1) \vee \lambda^*_r(\tau, \mathcal{I}_2)$. Thus $\lambda^*_r(\tau, \mathcal{I}_1 \wedge \mathcal{I}_2) = \lambda^*_r(\tau, \mathcal{I}_1) \vee \lambda^*_r(\tau, \mathcal{I}_2)$.

(2) Suppose $\lambda^*_r(\tau, \mathcal{I}_1 \vee \mathcal{I}_2) \not\ge \lambda^*_r(\tau^*(\mathcal{I}_2), \mathcal{I}_1) \wedge \lambda^*_r(\tau^*(\mathcal{I}_1), \mathcal{I}_2)$. Then, there exists $\nu \in I^X$, $(\mathcal{I}_1 \vee \mathcal{I}_2)(\lambda \bar{\wedge} \nu) \ge r$ with $\tau(\nu^c) \ge r$ such that

$$\lambda^*_r(\tau, \mathcal{I}_1 \vee \mathcal{I}_2) \le \nu < \lambda^*_r(\tau^*(\mathcal{I}_2), \mathcal{I}_1) \wedge \lambda^*_r(\tau^*(\mathcal{I}_1), \mathcal{I}_2).$$

Thus $\mathcal{I}_1(\lambda \bar{\wedge} \nu) \ge r$ or $\mathcal{I}_2(\lambda \bar{\wedge} \nu) \ge r$ with $\tau(\nu^c) \ge r$. But $\tau \le \tau^*$ implies $\tau^*(\mathcal{I}_1)(\nu^c) \ge r$ and $\tau^*(\mathcal{I}_2)(\nu^c) \ge r$. So $\lambda^*_r(\tau^*(\mathcal{I}_2), \mathcal{I}_1) \le \nu$ and $\lambda^*_r(\tau^*(\mathcal{I}_1), \mathcal{I}_2) \le \nu$, which is a contradiction.

Conversely, similarly, we get that $\lambda^*_r(\tau, \mathcal{I}_1 \vee \mathcal{I}_2) \leq \lambda^*_r(\tau^*(\mathcal{I}_2), \mathcal{I}_1) \wedge \lambda^*_r(\tau^*(\mathcal{I}_1), \mathcal{I}_2)$. □

$\tau^*(\mathcal{I})$ and $(\tau^*(\mathcal{I}))^*(\mathcal{I})$ (τ^{**}, for short) are equal for any fuzzy ideal on X.

Corollary 1.12. *Let $(X, \tau, \mathcal{I})$, be a fuzzy ideal topological space. For any $\lambda \in I^X$, $r \in I_0$, then $\lambda^*_r(\tau, \mathcal{I}) = \lambda^*_r(\tau^*, \mathcal{I})$ and $\tau^*(\mathcal{I}) = \tau^{**}$ (Putting $\mathcal{I}_1 = \mathcal{I}_2$ in Proposition 1.11).*

Corollary 1.13. *Let (X, τ) be a fuzzy topological space and $\mathcal{I}_1$, $\mathcal{I}_2$ fuzzy ideals on X. Then, (from Proposition 1.11),*

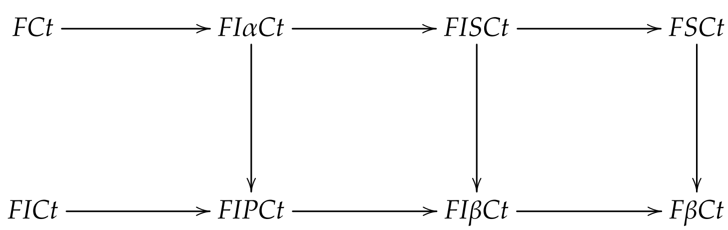
(1) $\tau^*(\mathcal{I}_1 \vee \mathcal{I}_2) = (\tau^*(\mathcal{I}_2))^*(\mathcal{I}_1) = (\tau^*(\mathcal{I}_1))^*(\mathcal{I}_2)$,
(2) $\tau^*(\mathcal{I}_1 \wedge \mathcal{I}_2) = \tau^*(\mathcal{I}_1) \wedge \tau^*(\mathcal{I}_2)$.

2. Continuity between fuzzy ideal topological spaces

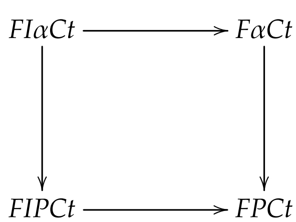
Definition 2.1. A map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called:

(i) fuzzy ideal continuous (*FICt*, for short), if $f^{-1}(\lambda) \leq \mathrm{int}_\tau((f^{-1}(\lambda)^*_r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r$, $r \in I_0$,
(ii) fuzzy ideal precontinuous (*FIPCt*, for short), if $f^{-1}(\lambda) \leq \mathrm{int}_\tau(\mathrm{cl}^*_\tau(f^{-1}(\lambda), r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r$, $r \in I_0$,
(iii) fuzzy ideal semi-continuous (*FISCt*, for short), if $f^{-1}(\lambda) \leq \mathrm{cl}^*_\tau(\mathrm{int}_\tau(f^{-1}(\lambda), r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r$, $r \in I_0$,
(iv) fuzzy ideal α-continuous (*FIαCt*, for short), if $f^{-1}(\lambda) \leq \mathrm{int}_\tau(\mathrm{cl}^*_\tau(\mathrm{int}_\tau(f^{-1}(\lambda), r), r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r$, $r \in I_0$,
(v) fuzzy ideal β-continuous (*FIβCt*, for short), if $f^{-1}(\lambda) \leq \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}^*_\tau(f^{-1}(\lambda), r), r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r$, $r \in I_0$.

The implications in the following diagrams are satisfied:



and



where *FCt*, *FSCt*, *FαCt*, *FβCt*, *FPCt* are the abbreviations of both of the notions of fuzzy continuity, fuzzy semi-continuity, fuzzy α-continuity, fuzzy β-continuity and fuzzy pre-continuity, respectively which are studied in details in [7, 4, 5, 6, 1, 2].

Example 2.2. Let X be a non-empty set. Define $\tau_i, \mathcal{I}_k : I^X \to I,\ i = 1,2,3,4,5,$ $k = 1,2$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.3 & \text{at } \lambda = \overline{0.4}, \overline{0.6} \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.3 & \text{at } \lambda = \overline{0.5} \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_3(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.3 & \text{at } \lambda = \overline{0.5}, \overline{0.7} \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_4(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.3 & \text{at } \lambda = \overline{0.8} \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_5(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.3 & \text{at } \lambda = \overline{0.2}, \overline{0.8} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_1(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0} \\ 0.3 & \text{at } \overline{0} < \lambda \leq \overline{0.4} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_2(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0} \\ 0.3 & \text{at } \overline{0} < \lambda \leq \overline{0.1} \\ 0 & \text{otherwise.} \end{cases}$$

(1) The identity function $id_X : (X, \tau_1, \mathcal{I}_1) \to (X, \tau_2)$ is *FPCt* but it is neither *FCt* nor *FICt*.
(2) The identity function $id_X : (X, \tau_3, \mathcal{I}_2) \to (X, \tau_4)$ is *FICt* but it is not *FCt*.
(3) The identity function $id_X : (X, \tau_5, \mathcal{I}_1) \to (X, \tau_2)$ is *FIβCt* but it is neither *FISCt* nor *FIαCt*.
(4) The identity function $id_X : (X, \tau_5, \mathcal{I}_1) \to (X, \tau_2)$ is *FIPCt* but it is not *FIαCt*.

Example 2.3. Let $X = \{a, b, c\},\ \tau, \tau^*, \mathcal{I} : I^X \to I$ be defined by:

$$\tau(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.5 & \text{at } \lambda = a_{0.3} \vee b_{0.4} \vee c_{0.3} \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau^*(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0}, \overline{1} \\ 0.33 & \text{at } \lambda = a_{0.4} \vee b_{0.5} \vee c_{0.7} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{I}(\lambda) = \begin{cases} 1 & \text{at } \lambda = \overline{0} \\ 0.5 & \text{at } \overline{0} < \lambda \leq \overline{0.3} \\ 0 & \text{otherwise.} \end{cases}$$

Then, the identity function $f : (X, \tau, \mathcal{I}) \to (X, \tau^*)$ is *FIβCt*, but it is neither *FIPCt* nor *FIαCt*.

Theorem 2.4. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function. Then, the following statements are equivalent:*

(1) *f is FIβCt,*
(2) *$f^{-1}(\lambda)$ is r-FIβ-closed* (i.e. $f^{-1}(\lambda) \geq \text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau(f^{-1}(\lambda), r), r), r))$, *for each $\lambda \in I^Y$ with $\sigma(\lambda^c) \geq r,\ r \in I_0$,*
(3) $\text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau(f^{-1}(\lambda), r), r), r) \leq f^{-1}(\text{cl}_\sigma(\lambda, r))$, *for each $\lambda \in I^Y,\ r \in I_0$,*
(4) $f(\text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau(\mu, r), r), r)) \leq \text{cl}_\sigma(f(\mu), r)$, *for each $\mu \in I^X,\ r \in I_0$.*

Proof. (1) ⇒ (2): Easy, so omitted.

(2) ⇒ (3): Let $\lambda \in I^Y,\ r \in I_0$. Since $\sigma((\text{cl}_\sigma(\lambda, r))^c) \geq r$, by (2), $f^{-1}(\text{cl}_\sigma(\lambda, r))$ is *r-FIβ*-closed and $(f^{-1}(\text{cl}_\sigma(\lambda, r)))^c$ is *r-FIβ*-open. Thus

$$\begin{aligned}(f^{-1}(\text{cl}_\sigma(\lambda, r)))^c &\leq \text{cl}_\tau(\text{int}_\tau(\text{cl}^*_\tau(((f^{-1}(\text{cl}_\sigma(\lambda, r)))^c), r), r), r)\\ &= (\text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau((f^{-1}(\text{cl}_\sigma(\lambda, r))), r), r), r))^c.\end{aligned}$$

So we obtain $\text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau((f^{-1}(\text{cl}_\sigma(\lambda, r))), r), r), r) \leq f^{-1}(\text{cl}_\sigma(\lambda, r))$.

(3) ⇒ (4): For any $\mu \in I^X,\ r \in I_0$, by (3), we have

$$\text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau(\mu, r), r), r) \leq \text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau(f^{-1}(f(\mu)), r), r), r) \leq f^{-1}(\text{cl}_\sigma(f(\mu), r)).$$

Then $f(\text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau(\mu, r), r), r)) \leq \text{cl}_\sigma(f(\mu), r)$.

(4) ⇒ (1): Let $\lambda \in I^Y,\ r \in I_0$ with $\sigma(\lambda) \geq r$. Then, by (4),

$$f(\text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau(f^{-1}(\lambda^c), r), r), r)) \leq \text{cl}_\sigma(f(f^{-1}(\lambda^c)), r) \leq \text{cl}_\sigma(\lambda^c, r) = \lambda^c,$$

which means $\text{int}_\tau(\text{cl}_\tau(\text{int}^*_\tau(f^{-1}(\lambda^c), r), r), r) \leq f^{-1}(\lambda^c) = (f^{-1}(\lambda))^c$. Thus we obtain that $f^{-1}(\lambda) \leq \text{cl}_\tau(\text{int}_\tau(\text{cl}^*_\tau(f^{-1}(\lambda), r), r), r)$. So $f^{-1}(\lambda)$ is *r-FIβ*-open. Hence f is *FIβCt*. □

Theorem 2.5. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function. Then, the following statements are equivalent:*

(1) *f is FIPCt,*
(2) *$f^{-1}(\lambda)$ is r-FI-preclosed* (i.e. $f^{-1}(\lambda) \geq \text{cl}_\tau(\text{int}^*_\tau(f^{-1}(\lambda), r), r))$, *for each $\lambda \in I^Y$ with $\sigma(\lambda^c) \geq r,\ r \in I_0$,*
(3) $\text{cl}_\tau(\text{int}^*_\tau(f^{-1}(\lambda), r), r) \leq f^{-1}(\text{cl}_\sigma(\lambda, r))$, *for each $\lambda \in I^Y,\ r \in I_0$,*
(4) $f(\text{cl}_\tau(\text{int}^*_\tau(\mu, r), r)) \leq \text{cl}_\sigma(f(\mu), r)$, *for each $\mu \in I^X,\ r \in I_0$.*

Proof. Can be established following Theorem 2.4. □

Theorem 2.6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function. Then, the following statements are equivalent:*

(1) *f is FISCt,*
(2) *$f^{-1}(\lambda)$ is r-FI-semi-closed* (i.e. $f^{-1}(\lambda) \geq \text{int}_\tau(\text{cl}^*_\tau(f^{-1}(\lambda), r), r))$, *for each $\lambda \in I^Y$ with $\sigma(\lambda^c) \geq r,\ r \in I_0$,*
(3) $\text{int}_\tau(\text{cl}^*_\tau(f^{-1}(\lambda), r), r) \leq f^{-1}(\text{cl}_\sigma(\lambda, r))$, *for each $\lambda \in I^Y,\ r \in I_0$,*
(4) $f(\text{int}_\tau(\text{cl}^*_\tau(\mu, r), r)) \leq \text{cl}_\sigma(f(\mu), r)$, *for each $\mu \in I^X,\ r \in I_0$.*

Proof. Can be established following Theorem 2.4. □

Theorem 2.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function. Then, the following statements are equivalent:*

(1) *f is FIαCt,*
(2) *$f^{-1}(\lambda)$ is r-FIα-closed* (i.e. $f^{-1}(\lambda) \geq \mathrm{cl}_\tau(\mathrm{int}^*_\tau(\mathrm{cl}_\tau(f^{-1}(\lambda), r), r), r)$), *for each $\lambda \in I^Y$ with $\sigma(\lambda^c) \geq r$, $r \in I_0$,*
(3) *$\mathrm{cl}_\tau(\mathrm{int}^*_\tau(\mathrm{cl}_\tau(f^{-1}(\lambda), r), r), r) \leq f^{-1}(\mathrm{cl}_\sigma(\lambda, r))$, for each $\lambda \in I^Y$, $r \in I_0$,*
(4) *$f(\mathrm{cl}_\tau(\mathrm{int}^*_\tau(\mathrm{cl}_\tau(\mu, r), r), r)) \leq \mathrm{cl}_\sigma(f(\mu), r)$, for each $\mu \in I^X$, $r \in I_0$.*

Proof. Can be established following Theorem 2.4. □

Corollary 2.8. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an FIαCt function. Then*

(1) *$f(\mathrm{cl}^*_\tau(\lambda, r)) \leq \mathrm{cl}_\sigma(f(\lambda), r)$, for each r-FI-preopen set $\lambda \in I^X$, $r \in I_0$,*
(2) *$\mathrm{cl}^*_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mathrm{cl}_\sigma(\mu, r))$, for each r-FI-preopen set $\mu \in I^Y$, $r \in I_0$.*

Proof. (1) Let $\lambda \in I^X$ be an r-*FI*-preopen set and $r \in I_0$. Then $\lambda \leq \mathrm{int}_\tau(\mathrm{cl}^*_\tau(\lambda, r), r)$. Thus, by Theorem 2.7, we obtain

$$\begin{aligned} f(\mathrm{cl}^*_\tau(\lambda, r)) &\leq f(\mathrm{cl}_\tau(\lambda, r)) \\ &\leq f(\mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}^*_\tau(\lambda, r), r), r)) \\ &\leq f(\mathrm{cl}_\tau(\mathrm{int}^*_\tau(\mathrm{cl}^*_\tau(\lambda, r), r), r)) \\ &\leq f(\mathrm{cl}_\tau(\mathrm{int}^*_\tau(\mathrm{cl}_\tau(\lambda, r), r), r)) \\ &\leq \mathrm{cl}_\sigma(f(\lambda), r). \end{aligned}$$

(2) Let $\mu \in I^Y$ be an r-*FI*-preopen set and $r \in I_0$. Then, by Theorem 2.7, we have

$$\begin{aligned} \mathrm{cl}^*_\tau(f^{-1}(\mu), r) &\leq \mathrm{cl}_\tau(f^{-1}(\mu), r) \\ &\leq \mathrm{cl}_\tau(f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}^*_\sigma(\mu, r), r)), r) \\ &\leq \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}^*_\tau(\mathrm{int}_\tau(f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}^*_\sigma(\mu, r), r)), r), r), r), r) \\ &\leq \mathrm{cl}_\tau(\mathrm{int}^*_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}^*_\sigma(\mu, r), r)), r), r), r) \\ &\leq f^{-1}(\mathrm{cl}_\sigma(\mathrm{int}_\sigma(\mathrm{cl}^*_\sigma(\mu, r), r), r)) \\ &\leq f^{-1}(\mathrm{cl}_\sigma(\mu, r)). \end{aligned}$$

□

Corollary 2.9. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is FIαCt iff it is FISCt and FIPCt.*

Corollary 2.10. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function, θ a fuzzy operator on X, and δ a fuzzy operator on Y. Then, f is (δ, θ)-continuous, if for each $\mu \in I^Y$ with $\sigma(\mu) \geq r$, $r \in I_0$, we have $f^{-1}(\mu) \leq \theta(f^{-1}(\delta(\mu, r)), r)$.*

We observe that the above definition generalizes the concepts of FICt (resp. FCt) when we choose $\theta = \mathrm{int}^*_\tau$ and $\delta = id_Y$ (resp. $\theta = \mathrm{int}_\tau$ and $\delta = id_Y$). Also,

(1) *if we take $\theta = \mathrm{int}_\tau \mathrm{cl}^*_\tau$ and $\delta = id_Y$, then f is FIPCt,*
(2) *if we take $\theta = \mathrm{cl}^*_\tau \mathrm{int}_\tau$ and $\delta = id_Y$, then f is FISCt,*
(3) *if we take $\theta = \mathrm{int}_\tau \mathrm{cl}^*_\tau \mathrm{int}_\tau$ and $\delta = id_Y$, then f is FIαCt,*
(4) *if we take $\theta = \mathrm{cl}_\tau \mathrm{int}_\tau \mathrm{cl}^*_\tau$ and $\delta = id_Y$, then f is FIβCt.*

3. Fuzzy grill topological spaces

A map $\mathcal{G} : I^X \to I$ is called a fuzzy grill ([3]) on X, if it satisfies the following conditions:

(i) $\mathcal{G}(\overline{0}) = 0$ and $\mathcal{G}(\overline{1}) = 1$,
(ii) $\lambda \le \mu \;\Rightarrow\; \mathcal{G}(\lambda) \le \mathcal{G}(\mu)$ for all $\lambda, \mu \in I^X$,
(iii) $\mathcal{G}(\lambda) \vee \mathcal{G}(\mu) \;\ge\; \mathcal{G}(\lambda \vee \mu)$ for all $\lambda, \mu \in I^X$.

The triple $(X, \tau, \mathcal{G})$ is called a fuzzy grill topological space. Let $\mathcal{G}(X)$ denote the set of all fuzzy grills on X.

Define the fuzzy grill $\mathcal{G}^\circ$ by $\mathcal{G}^\circ(\mu) = 0$ at $\mu = \overline{0}$ and $\mathcal{G}^\circ(\mu) = 1$, otherwise.

Definition 3.1. Let $(X, \tau, \mathcal{G})$ be a fuzzy grill topological space and $\lambda \in I^X$. Then, the r-fuzzy local function $\lambda^\bullet_r(\tau, \mathcal{G})$ of λ is defined by:

$$\lambda^\bullet_r(\tau, \mathcal{G}) = \bigwedge \{\mu \in I^X : \mathcal{G}(\lambda \bar{\wedge} \mu) < r,\ \tau(\mu^c) \ge r\}.$$

If $\mathcal{G} = \mathcal{G}^\circ$ then, for each $\lambda \in I^X$, $r \in I_0$, we have $\lambda^\bullet_r = \mathrm{cl}_\tau(\lambda, r)$.

Definition 3.2. Let $(X, \tau, \mathcal{G})$ be a fuzzy grill topological space and $\mu \in I^X$. Then,

$$\mathrm{cl}^\bullet_\tau(\mu, r) = \mu \vee \mu^\bullet_r \quad \text{and} \quad \mathrm{int}^\bullet_\tau(\mu, r) = \mu \wedge ((\mu^c)^\bullet_r)^c.$$

$\mathrm{cl}^\bullet_\tau$ is a fuzzy closure operator and $\tau^\bullet(\mathcal{G})$ is a fuzzy topology on X generated by $\mathrm{cl}^\bullet_\tau$, that is, $(\tau^\bullet(\mathcal{G}))(\mu) = \bigvee\{r \in I_0 : \mathrm{cl}^\bullet_\tau(\mu^c, r) = \mu^c\}$. Now, if $\mathcal{G} = \mathcal{G}^\circ$, then for each $\mu \in I^X, r \in I_0$, $\mathrm{cl}^\bullet_\tau(\mu, r) = \mu \vee \mu^\bullet_r = \mu \vee \mathrm{cl}_\tau(\mu, r) = \mathrm{cl}_\tau(\mu, r)$. So, $\tau^\bullet(\mathcal{G}^\circ) = \tau$.

Theorem 3.3. *Let X be a non-empty set and let $\mathcal{I}, \mathcal{G} : I^X \to I$ be two mappings satisfying the following conditions:*

$$\mathcal{I}_{\mathcal{G}}(\lambda) \;=\; \bigvee \{r \;:\; \mathcal{G}(\lambda) < r\,;\; r \in I_0\} \;\; \forall \lambda \in I^X, \tag{3.1}$$

$$\mathcal{G}_{\mathcal{I}}(\lambda) \;=\; \bigwedge \{r \;:\; \mathcal{I}(\lambda) \ge r\,;\; r \in I_0\} \;\; \forall \lambda \in I^X. \tag{3.2}$$

If $\mathcal{G}$ is a fuzzy grill on X, then $\mathcal{I}_{\mathcal{G}}$ is a fuzzy ideal on X generated by $\mathcal{G}$. Also, if $\mathcal{I}$ is a fuzzy ideal on X, then $\mathcal{G}_{\mathcal{I}}$ is a fuzzy grill on X generated by $\mathcal{I}$. This correspondence is given by (3.1) *and* (3.2)*.*

Proof. Let $\mathcal{G}$ be a fuzzy grill on X. Since $\mathcal{G}(\overline{0}) = 0 < r \;\forall r \in I_0$, $\mathcal{I}_{\mathcal{G}}(\overline{0}) = 1$.

Let $\mu \le \lambda \in I^X$, $\mathcal{I}_{\mathcal{G}}(\lambda) \ge r$; $r \in I_0$. Then, $\mathcal{G}(\lambda) < r$; $r \in I_0$, which implies $\mathcal{G}(\mu) \le \mathcal{G}(\lambda) < r$; $r \in I_0$, and thus $\mathcal{G}(\mu) < r$; $r \in I_0$. So $\mathcal{I}_{\mathcal{G}}(\mu) \ge r$; $r \in I_0$. That is, $\mathcal{I}_{\mathcal{G}}(\mu) \ge \mathcal{I}_{\mathcal{G}}(\lambda)$.

Let $\lambda, \mu \in I^X$, with $\mathcal{I}_{\mathcal{G}}(\lambda) \ge r$, $\mathcal{I}_{\mathcal{G}}(\mu) \ge s$; $r, s \in I_0$. Then $\mathcal{G}(\lambda) < r$, $\mathcal{G}(\mu) < s$; $r, s \in I_0$, which means $(r \vee s) > \mathcal{G}(\lambda) \vee \mathcal{G}(\mu) \ge \mathcal{G}(\lambda \vee \mu)$, that is, $\mathcal{I}_{\mathcal{G}}(\lambda \vee \mu) \ge (r \vee s)$. Thus $\mathcal{I}_{\mathcal{G}}(\lambda \vee \mu) \ge \mathcal{I}_{\mathcal{G}}(\lambda) \wedge \mathcal{I}_{\mathcal{G}}(\mu)$. So $\mathcal{I}_{\mathcal{G}}$ is a fuzzy ideal on X generated by the fuzzy grill $\mathcal{G}$.

Similarly, $\mathcal{G}_{\mathcal{I}}$ is a fuzzy grill on X generated by the fuzzy ideal $\mathcal{I}$. □

Remark 3.4. Similar results as found in Proposition 1.3, Proposition 1.11 and Corollary 1.9, Corollary 1.13 are also satisfied with respect to fuzzy grills in place of fuzzy ideals.

Corollary 3.5. *Let $(X, \tau, \mathcal{G})$ be a fuzzy grill topological space. Then*

(1) $\lambda^{\bullet}_r(\mathcal{G}) = \lambda^{*}_r(\mathcal{I}_{\mathcal{G}}),\ \forall \lambda \in I^X$ *and* $\lambda^{*}_r(\mathcal{I}) = \lambda^{\bullet}_r(\mathcal{G}_{\mathcal{I}})\ \forall \lambda \in I^X$,
(2) $\mathrm{cl}^{\bullet}_{\tau}(\lambda, r) = \mathrm{cl}^{*}_{\tau}(\lambda, r),\ \forall \lambda \in I^X$,
(3) $\tau^{\bullet}(\mathcal{G}) = \tau^{*}(\mathcal{I}_{\mathcal{G}}),\ \tau^{*}(\mathcal{I}) = \tau^{\bullet}(\mathcal{G}_{\mathcal{I}})$.

Corollary 3.6. *For all* $\mathcal{G} \in \mathcal{G}(X)$ *and for all* $\mathcal{I} \in \mathcal{I}(X)$, *we have* $\mathcal{G}_{\mathcal{I}_{\mathcal{G}}} = \mathcal{G},\ \mathcal{I}_{\mathcal{G}_{\mathcal{I}}} = \mathcal{I}$.

Proposition 3.7. *For* $\mathcal{I}(X)$ *and* $\mathcal{G}(X)$, *there is a one-to-one correspondence mapping.*

Proof. Let $h : \mathcal{G}(X) \to \mathcal{I}(X)$ be a mapping defined by $h(\mathcal{G}) = \mathcal{I}_{\mathcal{G}}$ for each fuzzy grill $\mathcal{G}$ on X. For $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{G}(X)$, we have: $\mathcal{G}_1 = \mathcal{G}_2$ implies $\mathcal{I}_{\mathcal{G}_1} = \mathcal{I}_{\mathcal{G}_2}$, and also $\mathcal{I}_{\mathcal{G}_1} = \mathcal{I}_{\mathcal{G}_2}$ implies that $\mathcal{G}_1 = \mathcal{G}_{\mathcal{I}_{\mathcal{G}_1}} = \mathcal{G}_{\mathcal{I}_{\mathcal{G}_2}} = \mathcal{G}_2$. That is, h is an injective function.

From Theorem 3.3, we get that for any $\mathcal{I} \in \mathcal{I}(X)$, there is a fuzzy grill $\mathcal{G}_{\mathcal{I}} \in \mathcal{G}(X)$ so that (from Corollary 3.6) $h(\mathcal{G}_{\mathcal{I}}) = \mathcal{I}_{\mathcal{G}_{\mathcal{I}}} = \mathcal{I}$, and thus h is a surjective function. Hence, h is a one-to-one correspondence between $\mathcal{I}(X)$ and $\mathcal{G}(X)$.

The same result could be proved by a map $k : \mathcal{I}(X) \to \mathcal{G}(X)$ defined by

$$k(\mathcal{I}) = \mathcal{G}_{\mathcal{I}} \text{ for each fuzzy ideal } \mathcal{I} \text{ on } X.$$

□

Several types of fuzzy continuity could be defined using the notion of fuzzy grills similar to Definition 2.1 as follows:

Definition 3.8. A map $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is called:

(i) fuzzy grill continuous (*FGCt*, for short), if $f^{-1}(\lambda) \leq \mathrm{int}_{\tau}((f^{-1}(\lambda)^{\bullet}_r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r,\ r \in I_0$,
(ii) fuzzy grill precontinuous (*FGPCt*, for short), if $f^{-1}(\lambda) \leq \mathrm{int}_{\tau}(\mathrm{cl}^{\bullet}_{\tau}(f^{-1}(\lambda), r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r,\ r \in I_0$,
(iii) fuzzy grill semi-continuous (*FGSCt*, for short), if $f^{-1}(\lambda) \leq \mathrm{cl}^{\bullet}_{\tau}(\mathrm{int}_{\tau}(f^{-1}(\lambda), r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r,\ r \in I_0$,
(iv) fuzzy grill α-continuous (*FGαCt*, for short) if $f^{-1}(\lambda) \leq \mathrm{int}_{\tau}(\mathrm{cl}^{\bullet}_{\tau}(\mathrm{int}_{\tau}(f^{-1}(\lambda), r), r), r)$ for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r,\ r \in I_0$.
(v) fuzzy grill β-continuous (*FGβCt*, for short), if $f^{-1}(\lambda) \leq \mathrm{cl}_{\tau}(\mathrm{int}_{\tau}(\mathrm{cl}^{\bullet}_{\tau}(f^{-1}(\lambda), r), r), r)$, for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r,\ r \in I_0$.

Also, the implications and diagrams of fuzzy ideal continuity are satisfied with respect to fuzzy grills.

Corollary 3.9. *For all* $\mathcal{G} \in \mathcal{G}(X)$ *and for all* $\mathcal{I} \in \mathcal{I}(X)$, *we have* $f : (X, \tau, \mathcal{I}_{\mathcal{G}}) \to (Y, \sigma)$ *is* *FI_GCt* (resp., *FI_GPCt, FI_GSCt, $FI_G\alpha Ct$, $FI_G\beta Ct$*) *if* $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ *is FGCt* (resp., *FGPCt, FGSCt, FGαCt, FGβCt*).

Conversely, $f : (X, \tau, \mathcal{G}_{\mathcal{I}}) \to (Y, \sigma)$ *is FG_ICt* (resp., *FG_IPCt, FG_ISCt, $FG_I\alpha Ct$, $FG_I\beta Ct$*) *if* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is FICt* (resp., *FIPCt, FISCt, FIαCt, FIβCt*).

Proof. Straightforward. □

From the correspondence proved in Proposition 3.7, we get that Definition 2.1 and Definition 3.8 are identical. Hence, fuzzy continuity based on fuzzy ideals or based on fuzzy grills are the same.

Here, we show the equivalence between fuzzy ideal compactness and fuzzy grill compactness.

Definition 3.10. $(X, \tau, \mathcal{I})$ be a fuzzy ideal topological space, $\lambda \in I^X, r \in I_0$. Then λ is said to be r-fuzzy ideal compact (r-*FI*-compact, for short), if for every family $\{\mu_j \in I^X : \tau(\mu_j) \geq r\ j \in J\}$ with $\lambda \leq \bigvee_{j\in J} \mu_j$, there exists a finite subset J_0 of J such that $\mathcal{I}(\lambda \bar{\wedge} (\bigvee_{j\in J_0} \mu_j)) \geq r$.

If $\mathcal{I} = \mathcal{I}^\circ$, then the concepts of r-fuzzy compact and r-*FI*-compact are equivalent.

Definition 3.11. Let $(X, \tau, \mathcal{G})$ be a fuzzy grill topological space, $\lambda \in I^X, r \in I_0$. Then λ is said to be r-fuzzy grill compact (r-*FG*-compact, for short), if for every family $\{\mu_j \in I^X : \tau(\mu_j) \geq r\ j \in J\}$ with $\lambda \leq \bigvee_{j\in J} \mu_j$, there exists a finite subset J_0 of J such that $\mathcal{G}(\lambda \bar{\wedge} (\bigvee_{j\in J_0} \mu_j)) < r$.

If $\mathcal{G} = \mathcal{G}^\circ$, then the concepts of r-fuzzy compact and r-*FG*-compact are equivalent.

Now, we prove that the topological properties are the same from the point of view of fuzzy ideals and fuzzy grills.

Theorem 3.12. *Let $(X, \tau, \mathcal{G})$ be a fuzzy grill topological space and $\lambda \in I^X$ is an r-FG-compact. Then, λ is an r-FI-compact with respect to $\mathcal{I}_\mathcal{G}$ as well.*

Conversely, if $(X, \tau, \mathcal{I})$ is a fuzzy ideal topological space and λ is an r-FI-compact, then λ is an r-FG-compact with respect to $\mathcal{G}_\mathcal{I}$.

Proof. Let $\{\mu_j \in I^X : \tau(\mu_j) \geq r,\ j \in J\}$ be a family with $\lambda \leq \bigvee_{j\in J} \mu_j$. Then by r-*FG*-compactness of λ, there exists a finite subset J_0 of J such that $\mathcal{G}(\lambda \bar{\wedge} (\bigvee_{j\in J_0} (\mu_j))) < r$. Thus from (3.1), $\mathcal{I}_\mathcal{G}(\lambda \bar{\wedge} (\bigvee_{j\in J_0} (\mu_j))) \geq r$. So λ is an r-*FI*-compact with respect to $\mathcal{I}_\mathcal{G}$.

Similarly, we can prove the converse. □

Corollary 3.13. *Let (X, τ) be a fuzzy topological space. Then*

(1) *If $\mathcal{G}$ is a fuzzy grill on X, and $(X, \tau, \mathcal{G})$ is an r-FG-compact space, then $(X, \tau, \mathcal{I}_\mathcal{G})$ is an r-FI-compact space,*

(2) *If $\mathcal{I}$ is a fuzzy ideal on X, and $(X, \tau, \mathcal{I})$ is an r-FI-compact space, then $(X, \tau, \mathcal{G}_\mathcal{I})$ is an r-FG-compact space.*

Proof. Obvious from Equations (3.1) and (3.2). □

4. Conclusion

Results already introduced and studied with fuzzy ideals are satisfied with respect to fuzzy grills from that correspondence between the two notions of fuzzy ideal and fuzzy grill. We have established the equivalence between fuzzy ideal compactness and fuzzy grill compactness.

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