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Variants of compatible mappings in fuzzy metric spaces

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Abstract. In this paper, we introduce the notions of compatible mappings of type(R), type (K) and type (E) in Fuzzy metric spaces and prove some common fixed point theorems for these mappings.In fact, we call these maps as variants of compatible mappings.

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1. Introduction

It proved a turning point in the development of fuzzy mathematics when the notion of fuzzy set was introduced by Zadeh [20]. Fuzzy set theory has many applications in applied science such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. There are many view points of the notion of the metric space in fuzzy topology, see, e.g., Erceg [2], Jungck [8], Deng [1], Kaleva and Seikkala [9], Kramosil and Michalek [10], George and Veermani [4], Sessa [16]. In this paper, we are considering the Fuzzy metric space in the sense of Kramosil and Michalek [10].

Definition 1.1. A binary operation $*$ on $[0,1]$ is a t-norm, if it satisfies the following conditions:
(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for every $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.
Basics examples of t-norm are t-norm Δ_L, $\Delta_L(a,b) = \max(a+b-1,0)$, t-norm Δ_P, $\Delta_P(a,b) = ab$ and t-norm $\Delta_M, \Delta_M(a,b) = \min\{a,b\}$.

Definition 1.2. The 3-tuple (X, M, Δ) is called a fuzzy metric space (in the sence of Kramosil and Michalek), if X is an arbitrary set, Δ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(i) $M(x, y, 0) = 0$, $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, z, t + s) \geq \Delta(M(x, y, t), M(y, z, s))$,
(v) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with $t = 0$.

Definition 1.3. A sequence $\{x_n\}$ in (X, M, Δ) is said to be:

(i) convergent with limit x, if $\lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$,

(ii) Cauchy sequence in X, if given $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\varepsilon, \lambda)$ such that $M(x_n, x_m, \varepsilon) > 1 - \lambda$, for all $n, m \geq N(\varepsilon, \lambda)$,

(iii) complete ,if every Cauchy sequence in X is convergent in X.

Fixed point theory in fuzzy metric space has been developing since the paper of Grabiec [3]. Subramanyam [18] gave a generalization of Jungck [6] theorem for commuting mapping in the setting of fuzzy metric space.

In 1996, Jungck [7] introduced the notion of weakly compatible as follows:

Definition 1.4. Two maps f and g are said to be weakly compatible, if they commute at their coincidence points.

In 1999, Vasuki [19] introduced the notion of R-weakly commutingas follows:

Definition 1.5. Two self-mapping f and g of a fuzzy metric space (X, M, Δ) are said to be R-weakly commuting, if $M(fgx, gfx, t) \geq M(fx, gx, t/R)$, for each $x \in X$ and for each $t > 0$.

In 1994, Mishra [11] generalized the notion of weakly commuting to compatible mappings in fuzzy metric space akin to the concept of compatible mapping in metric space.

Definition 1.6 ([14])**.** Let f and g be self-mappings from a fuzzy metric space (X, M, Δ) into itself. A pair of map $\{f, g\}$ is said to be compatible, if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$ and for all $t > 0$.

In 1994, Pant [12] introduced the concept of R-weakly commuting maps in metric space. Later on, Vasuki [19] initiated the concept of non-compatible mapping in fuzzy metric space and introduced the notion of R-weakly commuting mapping in fuzzy metric space and proved some common fixed point theorems for these mappings.

Definition 1.7 ([12])**.** Let f and g be self-mapping from a fuzzy metric space (X, M, Δ) into itself. Then the mappings f and g are said to be non-compatible, if

$\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$ and for all $t > 0$.

In 1999, Pant [13] introduced a new continuity condition, known as reciprocal continuity as follows:

Definition 1.8 ([13])**.** Two self-maps f and g of a fuzzy metric space (X, M, Δ) are called reciprocally continuous, if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

If f and g are both continuous then they are obviously reciprocally continuous but the converse is need not be true.

Recently, Pant et al. [13] generalized the notion of reciprocal continuity to weak reciprocal continuity as follows:

Definition 1.9 ([13])**.** Two self-maps f and g of a fuzzy metric space (X, M, Δ) are called weakly reciprocally continuous, if $\lim_{n\to\infty} fgx_n = fz$ or $\lim_{n\to\infty} gfx_n = gz$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

If f and g are reciprocally continuous, then they are obviously weak reciprocally continuous but the converse is not true.

In 2004, Rohan et al. [15] introduced the concept of compatible mappings of type (R) in a metric space as follows:

Definition 1.10 ([15])**.** Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (R), if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0, \text{ and } \lim_{n\to\infty} d(ffx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

In 2007, Singh and Singh et al. [17] introduced the concept of compatible mappings of type (E) in a metric space as follows:

Definition 1.11 ([17])**.** Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (E), if $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt$ and $\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

In 2014, Jha et al. [5] introduced the concept of compatible mappings of type (K) in a metric space as follows:

Definition 1.12 ([5])**.** Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (K), if

$$\lim_{n\to\infty} d(ffx_n, gt) = 0, \text{ and } \lim_{n\to\infty} d(ggx_n, ft) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

2. Properties of compatible mappings of types

Now we introduce the notions of compatible mappings of types in the setting of a Fuzzy metric space as follows:

Definition 2.1. Let f and g be self-mapping on Fuzzy metric space (X, M, Δ). Then f and g are called:

(i) compatible of type (R), if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, and $\lim_{n\to\infty} M(ffx_n, ggx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$ and for all $t > 0$,

(ii) compatible of type (K), if $\lim_{n\to\infty} M(ffx_n, gx, t) = 1$, and $\lim_{n\to\infty} M(ggx_n, fx, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$, for some $x \in X$ and for all $t > 0$,

(iii) compatible of type (E), if $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = gt$ and $\lim_{n\to\infty} ggx_n = \lim_{n\to\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$, for some $x \in X$.

Now we give some properties related to compatible mappings of type (R) and type (E).

Proposition 2.2. *Let f and g be compatible mappings of type (R) of a Fuzzy metric space (X, M, Δ) into itself. If $fx = gx$, for some $x \in X$, then $fgx = ffx = ggx = gfx$.*

Proposition 2.3. *Let f and g be compatible mappings of type (R) of Fuzzy metric space (X, M, Δ) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X. Then*

(1) *$\lim_{n\to\infty} gfx_n = fx$, if f is continuous at x,*

(2) *$\lim_{n\to\infty} fgx_n = gx$, if g is continuous at x,*

(3) *$fgx = gfx$ and $fx = gx$ if f and g are continuous at x.*

Proposition 2.4. *Let f and g be compatible mappings of type (E) of a Fuzzy metric space (X, M, Δ) into itself. Let one of f and g be continuous. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$. Then*

(1) *$fx = gx$ and $\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n$,*

(2) *if there exists $u \in X$ such that $fu = gu = x$, then we have $fgu = gfu$.*

Lemma 2.5 ([13])**.** *Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, Δ). If there exists $q \in (0, 1)$ such that $M(y_{n+2}, y_{n+1}, qt) \geq M(y_{n+1}, y_n, t)$, $t > 0$, $n \in N$, then $\{y_n\}$ is a Cauchy sequence in X.*

Lemma 2.6 ([13])**.** *Let (X, M, Δ) be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) \geq M(x, y, t)$ for all $x, y \in X$, and $t > 0$, then $x = y$.*

3. Main Results

Rohan et al. [15], Singh and Singh [17] and Jha [5] proved the common fixed point theorems in a complete metric space.

Now we prove the same in Fuzzy metric space.

Theorem 3.1. *Let P, R, S and T be mappings of a complete Fuzzy metric space (X, M, Δ) into itself satisfying the conditions:*

(3.1) $T(X) \subset P(X), S(X) \subset R(X)$,

(3.2) $M(Sz, Tw, kt) \geq \min \begin{Bmatrix} (M(Pz, Rw, t), M(Pz, Sz, t), M(Rw, Tw, t), \\ M(Sz, Rw, \alpha t), M(Pz, Tw, (2-\alpha)t)) \end{Bmatrix}$

hold for all z, w in X, where $\alpha \in (0, 2)$, $t > 0$,

(3.3) *one of the mappings P, R, S and T is continuous.*

Assume that the pairs P, S and R, T are compatible of type (R). Then P, R, S and T have a unique common fixed point in X.

Proof. Let us consider a point $z_0 \in X$. Since $S(X) \subset R(X)$, there exists $z_1 \in X$ such that $Sz_0 = Rz_1 = w_0$, for this point z_1, there exists $z_2 \in X$ such that $Tz_1 = Pz_2 = w_1$. Continuing in this way, we can define a sequence $\{w_n\}$ in X such that

$$w_{2n} = Sz_{2n} = Rz_{2n+1}; \quad w_{2n+1} = Tz_{2n+1} = Pz_{2n+2}\,.$$

Now we prove that $\{w_n\}$ is Cauchy sequence in X.

Putting $z = z_{2n}$, $w = z_{2n+1}$, $\alpha = 1 - \beta$ with $\beta \in (0, 1)$ in inequality (3.2), we have

$$M(Sz_{2n}, Tz_{2n+1}, kt) \geq \min \begin{Bmatrix} (M(Pz_{2n}, Rz_{2n+1}, t), M(Pz_{2n}, Sz_{2n}, t), \\ M(Rz_{2n+1}, Tz_{2n+1}, t), M(Sz_{2n}, Rz_{2n+1}, (1-\beta)t), \\ M(Pz_{2n}, Tz_{2n+1}, (1+\beta)t)) \end{Bmatrix}$$

$$M(w_{2n}, w_{2n+1}, kt) \geq \min \begin{Bmatrix} (M(w_{2n-1}, w_{2n}, t), M(w_{2n-1}, w_{2n}, t), \\ M(w_{2n}, w_{2n+1}, t), M(w_{2n}, w_{2n}, (1-\beta)t), \\ M(w_{2n-1}, w_{2n+1}, (1+\beta)t)) \end{Bmatrix}$$

$$\begin{aligned} M(w_{2n}, w_{2n+1}, kt) &\geq \min \begin{Bmatrix} (M(w_{2n-1}, w_{2n}, t), M(w_{2n-1}, w_{2n}, t), \\ M(w_{2n}, w_{2n+1}, t), 1, \\ M(w_{2n-1}, w_{2n+1}, (1+\beta)t)) \end{Bmatrix} \\ &\geq \min \begin{Bmatrix} (M(w_{2n-1}, w_{2n}, t), M(w_{2n}, w_{2n+1}, t), \\ M(w_{2n-1}, w_{2n+1}, (1+\beta)t)) \end{Bmatrix} \\ &\geq \min \begin{Bmatrix} (M(w_{2n-1}, w_{2n}, t), M(w_{2n}, w_{2n+1}, t), \\ M(w_{2n-1}, w_{2n}, (t), M(w_{2n}, w_{2n+1}, (\beta t)) \end{Bmatrix} \\ &\geq \min \begin{Bmatrix} (M(w_{2n-1}, w_{2n}, t), M(w_{2n}, w_{2n+1}, t), \\ M(w_{2n}, w_{2n+1}, (\beta t)) \end{Bmatrix} \end{aligned}$$

As Δ is continuous, letting $\beta \to 1$, we get

$$\begin{aligned} M(w_{2n}, w_{2n+1}, kt) &\geq \min \begin{Bmatrix} (M(w_{2n-1}, w_{2n}, t), M(w_{2n}, w_{2n+1}, t), \\ M(w_{2n}, w_{2n+1}, (t)) \end{Bmatrix} \\ &= \min\{M(w_{2n-1}, w_{2n}, t), M(w_{2n}, w_{2n+1}, t)\}. \end{aligned}$$

Then $M(w_{2n}, w_{2n+1}, kt) \geq \min\{M(w_{2n-1}, w_{2n}, t), M(w_{2n}, w_{2n+1}, t)\}$.

Similarly,

$$M(w_{2n+1}, w_{2n+2}, kt) \geq \min\{M(w_{2n}, w_{2n+1}, t), M(w_{2n+1}, w_{2n+2}, t\}.$$

Thus, for all n even or odd, we have

$$M(w_n, w_{n+1}, kt) \geq \min\{M(w_{n-1}, w_n, t), M(w_n, w_{n+1}, t)\}.$$

Consequently,

$$M(w_n, w_{n+1}, t) \geq \min\left\{M\left(w_{n-1}, w_n, \frac{t}{k}\right), M\left(w_n, w_{n+1}, \frac{t}{k}\right)\right\}.$$

By repeated application of above inequality, we get

$$M(w_n, w_{n+1}, t) \geq \min\left\{M\left(w_{n-1}, w_n, \frac{t}{k}\right), M\left(w_n, w_{n+1}, \frac{t}{k^m}\right)\right\}.$$

Since $M(w_n, w_{n+1}, \frac{t}{k^m}) \to 1$ as $m \to \infty$, it follows that

$$M(w_n, w_{n+1}, kt) \geq M(w_{n-1}, w_n, t) \text{ for all } n \in N \text{ for all } t > 0.$$

So by Lemma 2.5, $\{w_n\}$ is a Cauchy sequence in X and hence it converges to some point $u \in X$. Consequently, the subsequence $\{Sz_{2n}\}$, $\{Rz_{2n+1}\}$, $\{Tz_{2n+1}\}$ and $\{Pz_{2n}\}$ of $\{w_n\}$ also converges to u.

Now, suppose that P is continuous. Since P and S are compatible of type (R), by Proposition 2.3, PPz_{2n} and SPz_{2n} converges to Pu as $n \to \infty$.

We claim that $u = Pu$. Putting $z = Pz_{2n}$ and $w = z_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$M(SPz_{2n}, Tz_{2n+1}, kt) \geq \min\left\{\begin{matrix}(M(PPz_{2n}, Rz_{2n+1}, t), M(PPz_{2n}, SPz_{2n}, t), \\ M(Rz_{2n+1}, Tz_{2n+1}, t), M(SPz_{2n}, Rz_{2n+1}, t), \\ M(PPz_{2n}, Tz_{2n+1}, t))\end{matrix}\right\}$$

Letting $n \to \infty$. Then we have

$$M(Pu, u, kt) \geq \min\left\{\begin{matrix}(M(Pu, u, t), M(Pu, Pu, t), \\ M(u, u, t), M(Pu, u, t), M(Pu, u, t))\end{matrix}\right\}$$

$$M(Pu, u, kt) \geq M(Pu, u, t).$$

By Lemma 2.6, $Au = u$.

Next we claim that $u = Su$. Putting $z = u$ and $w = z_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$M(Su, Tz_{2n+1}, kt) \geq \min\left\{\begin{matrix}(M(Pu, Rz_{2n+1}, t), M(Pu, Su, t), M(Rz_{2n+1}, Tz_{2n+1}, t), \\ M(Su, Rz_{2n+1}, t), M(Pu, Tz_{2n+1}, t))\end{matrix}\right\}$$

Letting $n \to \infty$. Then we have

$$M(Su, u, kt) \geq \min\left\{\begin{matrix}(M(u, u, t), M(u, Su, t), \\ M(u, u, t), M(u, u, t), M(u, u, t))\end{matrix}\right\}.$$

Thus $M(Su, u, kt) \geq M(Su, u, t)$. By Lemma 2.6, $Su = u$. Since $S(X) \subset R(X)$, there exists a point $v \in X$ such that $u = Su = Rv$.

We claim that $u = Tv$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$M(u, Tv, kt) = M(Su, Tv, kt) \geq \min\left\{\begin{matrix}(M(Pu, Rv, t), M(Pu, Su, t), M(Rv, Tv, t), \\ M(Su, Rv, t), M(Pu, Tv, t))\end{matrix}\right\}$$

$$M(u, Tv, kt) \geq \min\{M(u, u, t), M(u, u, t), M(u, Tv, t), M(u, u, t), M(u, Tv, t)\}.$$

Then $M(u, Tv, kt) \geq M(u, Tv, t)$. By Lemma 2.6, $u = Tv$. Since R and T are compatible of type (R) and $Rv = Tv = u$, by Proposition 2.2, $RTv = TRv$ and hence $Ru = RTv = TRv = Tu$. Also, we have

$$\begin{aligned} M(u, Ru, kt) &= M(Su, Tu, kt) \\ &\geq \min \begin{Bmatrix} (M(Pu, Ru, t), M(Pu, Su, t), M(Ru, Tu, t), \\ M(Su, Ru, t), M(Pu, Tu, t)) \end{Bmatrix} \\ &= \min \begin{Bmatrix} (M(u, Ru, t), M(u, u, t), M(Ru, Ru, t), \\ M(u, Ru, t), M(u, Ru, t)) \end{Bmatrix}. \end{aligned}$$

Thus $M(u, Ru, kt) \geq M(u, Ru, t)$. By Lemma 2.6, $u = Ru$. So $u = Ru = Tu = Pu = Sz$. Hence u is a common fixed point of P, S, R and T.

Similarly, we can complete the proof when R is continuous.

Next, suppose that S is continuous. Since P and S are compatible of type (R), by Proposition 2.3, SSz_{2n} and Sz_{2n} converges to Su as $n \to \infty$.

We claim that $u = Su$. Putting $z = Sz_{2n}$ and $w = z_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$M(SSz_{2n}, Tz_{2n+1}, kt) \geq \min \begin{Bmatrix} (M(PSz_{2n}, Rz_{2n+1}, t), M(PSz_{2n}, SSz_{2n}, t), \\ M(Rz_{2n+1}, Tz_{2n+1}, t), M(SSz_{2n}, Rz_{2n+1}, t), \\ M(PSz_{2n}, Tz_{2n+1}, t)) \end{Bmatrix}.$$

Letting $n \to \infty$. Then we have

$$M(Su, u, kt) \geq \min \begin{Bmatrix} (M(Su, u, t), M(Su, Su, t), M(u, u, t), \\ M(Su, u, t), M(Su, u, t)) \end{Bmatrix}.$$

Thus $M(Su, u, kt) \geq M(Su, u, t)$. By Lemma 2.6, $Su = u$. Since $S(X) \subset R(X)$, there exists a point $p \in X$ such that $u = Su = Rp$.

We claim that $u = Tp$. Putting $z = Sz_{2n}$ and $w = p$, $\alpha = 1$ in inequality (3.2), we have

$$M(SSz_{2n}, Tp, kt) \geq \min \begin{Bmatrix} (M(PSz_{2n}, Rp, t), M(PSz_{2n}, Sz_{2n}, t), M(Rp, Tp, t), \\ M(SSz_{2n}, Rp, t), M(PSz_{2n}, Tp, t)) \end{Bmatrix}.$$

Letting $n \to \infty$. Then we have

$$M(u, Tp, kt) \geq \min \begin{Bmatrix} (M(u, u, t), M(u, u, t), M(u, Tp, t), \\ M(u, u, t), M(u, Tp, t)) \end{Bmatrix}.$$

Thus $M(u, Tv, kt) \geq M(u, Tv, t)$. By Lemma 2.6, $u = Tp$. Since R and T are compatible of type (R) and $Rp = Tp = u$, by Proposition 2.3, $RTp = TRp$. So $Ru = RTp = TRp = Tu$.

We claim that $u = Tu$. Putting $z = z_{2n}$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$M(Sz_{2n}, Tu, kt) \geq \min \begin{Bmatrix} (M(Pz_{2n}, Ru, t), M(Pz_{2n}, Sz_{2n}, t), M(Ru, Tu, t), \\ M(Sz_{2n}, Ru, t), M(Pz_{2n}, Tu, t)) \end{Bmatrix}.$$

Letting $n \to \infty$. Then we have

$$M(u, Tu, kt) \geq \min \begin{Bmatrix} (M(u, Tu, t), M(u, u, t), M(Tu, Tu, t), \\ M(u, Tu, t), M(u, Tu, t)) \end{Bmatrix}.$$

Thus $M(u, Tu, kt) \geq M(u, Tu, t)$. By Lemma 2.6, $Tu = u$. Since $T(X) \subset P(X)$, there exists a point $x \in X$ such that $u = Su = Ax$.

We claim that $u = Sx$. Putting $z = x$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} M(Sx, u, kt) &= M(Sx, Tu, kt) \\ &\geq \min \begin{Bmatrix} (M(Px, Ru, t), M(Px, Sx, t), M(Ru, Tu, t) \\ M(Sx, Rx, t), M(Px, Tu, t)) \end{Bmatrix} \\ &= \min \begin{Bmatrix} (M(u, u, t), M(u, Sx, t), M(Tu, Tu, t), \\ M(Sx, u, t), M(u, u, t)) \end{Bmatrix}. \end{aligned}$$

Then $M(Sx, u, kt) \geq M(u, Sx, t)$. By Lemma 2.6, $u = Sx$. Since P and S are compatible of type (R) and $Sx = Px = u$, by Proposition 2.3, $PSx = SPx$. Thus $Pu = PSx = SPx = Su$. So $u = Ru = Tu = Pu = Su$. Hence u is a common fixed point of P, S, R and T.

Similarly, we can complete the proof when T is continuous.

Uniqueness: If possible let u and v be two fixed point of the mappings P, R, S and T.

Finally, we claim that $u = v$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} M(Su, Tv, kt) &= M(u, v, kt) \\ &\geq \min \begin{Bmatrix} (M(Pu, Rv, t), M(Pu, Su, t), M(Rv, Tv, t), \\ M(Su, Rv, t), M(Pu, Tv, t)) \end{Bmatrix} \\ &= \min \begin{Bmatrix} (M(u, v, t), M(u, u, t), M(v, v, t), \\ M(u, v, t), M(u, v, t)) \end{Bmatrix}. \end{aligned}$$

Then $M(u, v, kt) \geq M(u, v, t)$. By Lemma 2.6, $u = v$. Thus P, S, R and T have a unique common fixed point. □

Next we prove the following theorem for compatible mappings of type (K).

Theorem 3.2. *Let P, S, R and T be mappings of a complete Fuzzy metric space (X, M, Δ) into itself satisfying the conditions (3.1), (3.2) of Theorem 3.1. Suppose that the pairs P, S and R, T are reciprocally continuous. Assume that the pairs P, S and R, T are compatible of type (K). Then P, S, R and T have a unique common fixed point in X.*

Proof. Now from the proof of Theorem 3.1, we can easily prove that $\{w_n\}$ is Cauchy sequence in X and thus it converges to some point $u \in X$. Consequently, the subsequence $\{Sz_{2n}\}$, $\{Rz_{2n+1}\}$, $\{Tz_{2n+1}\}$ and $\{Pz_{2n}\}$ of $\{w_n\}$ also converges to u. Since the pairs P, S and R, T are compatible of type (K), we have $PPz_{2n} \to Su$, $SSz_{2n} \to Pu$ and $RRz_{2n} \to Tu$, $TTz_{2n+1} \to Ru$ as $n \to \infty$.

We claim that $Ru = Pu$. Putting $z = Sz_{2n}$ and $w = Tz_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$M(SSz_{2n}, TTz_{2n+1}, kt) \geq \min \begin{Bmatrix} (M(PSz_{2n}, RTz_{2n+1}, t), M(PSz_{2n}, SSz_{2n}, t), \\ M(RTz_{2n+1}, TTz_{2n+1}, t), M(SSz_{2n}, RTz_{2n+1}, t), \\ M(RSz_{2n}, Tz_{2n+1}, t)) \end{Bmatrix}.$$

Letting $n \to \infty$ and using reciprocal continuity of the pairs P, S and R, T, we have

$$M(Pu, Ru, kt) \geq \min \left\{ \begin{matrix} (M(Au, Ru, t), M(Pu, Pu, t), M(Ru, Ru, t), \\ M(Pu, Ru, t), M(Pu, Ru, t)) \end{matrix} \right\}.$$

we get $M(Pu, Ru, kt) \geq M(Pu, Ru, t)$. By Lemma 2.6, $Pu = Ru$.

Next we claim that $Ru = Su$. Putting $z = u$ and $w = Tz_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$M(Su, TTz_{2n+1}, kt) \geq \min \left\{ \begin{matrix} (M(Pu, RTz_{2n+1}, t), M(Pu, Su, t), \\ M(RTz_{2n+1}, TTz_{2n+1}, t), M(Su, RTz_{2n+1}, t), \\ M(Pu, TTz_{2n+1}, t)) \end{matrix} \right\}.$$

Letting $n \to \infty$ and using reciprocal continuity of the pairs P, S and R, T, we have

$$M(Su, Ru, kt) \geq \min \left\{ \begin{matrix} (M(Ru, Ru, t), M(Ru, Su, t), M(Ru, Ru, t), \\ M(Su, Ru, t), M(Ru, Ru, t)) \end{matrix} \right\}.$$

Thus $M(Su, Ru, kt) \geq M(Su, Ru, t)$. By Lemma 2.6, $Su = Ru$.

We claim that $Su = Tu$. Putting $z = u$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$M(Su, Tu, kt) \geq \min \left\{ \begin{matrix} (M(Pu, Ru, t), M(Pu, Su, t), M(Ru, Tu, t), \\ M(Su, Ru, t), M(Pu, Tu, t)) \end{matrix} \right\}$$

$$M(Su, Tu, kt) \geq \min \left\{ \begin{matrix} (M(Ru, Ru, t), M(Pu, Pu, t), M(Su, Tu, t), \\ M(Su, Su, t), M(Su, Tu, t)) \end{matrix} \right\}.$$

So $M(Su, Tu, kt) \geq M(Su, Tu, t)$. By Lemma 2.6, $Su = Tu$.

We claim that $u = Tu$. Putting $z = z_{2n}$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$M(Sz_{2n}, Tu, kt) \geq \min \left\{ \begin{matrix} (M(Pz_{2n}, Ru, t), M(Pz_{2n}, Sz_{2n}, t), \\ M(Ru, Tu, t), M(Sz_{2n}, Ru, t), M(Rz_{2n}, Tu, t)) \end{matrix} \right\}.$$

Letting $n \to \infty$. Then we have

$$M(u, Tu, kt) \geq \min \left\{ \begin{matrix} (M(u, Tu, t), M(u, u, t), M(u, Tu, t) \\ M(u, Tu, t), M(u, Tu, t)) \end{matrix} \right\}.$$

Thus $M(u, Tu, kt) \geq M(u, Tu, t)$. By Lemma 2.6, $u = Tu$. So $u = Bu = Tu = Au = Su$. Hence u is a common fixed point of P, S, R and T.

Uniqueness: If possible let u and v be two fixed point of the mappings P, R, S and T.

Finally, we claim that $u = v$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} M(Su, Tv, kt) &= M(u, v, kt) \\ &\geq \min \left\{ \begin{matrix} (M(Pu, Rv, t), M(Pu, Su, t), M(Rv, Tv, t) \\ M(Su, Rv, t), M(Pu, Tv, t)) \end{matrix} \right\} \\ &= \min\{M(u, v, t), M(u, u, t), M(v, v, t), M(u, v, t), M(u, v, t)\}. \end{aligned}$$

Then $M(u, v, kt) \geq M(u, v, t)$. By Lemma 2.6, $u = v$. Thus P, S, R and T have a unique common fixed point. □

Now we prove the following theorem for compatible mappings of type (E).

Theorem 3.3. *Let P, R, S and T be mappings of a complete Fuzzy metric space (X, M, Δ) into itself satisfying the conditions (3.1), (3.2) of Theorem 3.1. Suppose that one of P and S is continuous and one of R and T is continuous. Assume that the pairs P, S and R, T are compatible of type (E). Then P, R, S and T have a unique common fixed point in X.*

Proof. Now from the proof of Theorem 3.1, we can easily prove that $\{w_n\}$ is Cauchy sequence in X and hence it converges to some point $u \in X$. Consequently, the subsequence $\{Sz_{2n}\}$, $\{Rz_{2n+1}\}$, $\{Tz_{2n+1}\}$ and $\{Pz_{2n}\}$ of $\{w_n\}$ also converges to u.

Now, suppose that one of the mappings P and S is continuous. Since P and S are compatible of type (E), by Proposition 2.4, $Pu = Su$. Since $S(X) \subset R(X)$, there exists a point $v \in X$ such that $Su = Rv$.

We claim that $Su = Tv$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} M(Su, Tv, kt) &\geq \min \left\{ \begin{matrix} (M(Pu, Rv, t), M(Pu, Su, t), M(Rv, Tv, t), \\ M(Su, Rv, t), M(Pu, Tv, t)) \end{matrix} \right\} \\ &= \min \left\{ \begin{matrix} (M(Pu, Su, t), M(Su, Su, t), M(Su, Tv, t), \\ M(Su, Rv, t), M(Su, Tv, t)) \end{matrix} \right\}. \end{aligned}$$

Then $M(Su, Tv, kt) \geq M(Su, Tv, t)$. By Lemma 2.6, $Su = Tv$. Thus $Pu = Su = Tv = Rv$.

We claim that $Su = u$. Putting $z = u$ and $w = z_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} M(Su, Tz_{2n+1}, kt) &\geq \min \left\{ \begin{matrix} (M(Pu, Rz_{2n+1}, t), M(Pu, Su, t), M(Rz_{2n+1}, Tz_{2n+1}, t), \\ M(Su, Rz_{2n+1}, t), M(Pu, Tz_{2n+1}, t)) \end{matrix} \right\} \\ &= \min \left\{ \begin{matrix} (M(Su, u, t), M(u, u, t), M(u, u, t), \\ M(Su, u, t), M(Su, u, t)) \end{matrix} \right\}. \end{aligned}$$

Thus $M(Su, u, kt) \geq M(Su, u, t)$. By Lemma 2.6, $u = Su$. So $u = Ru = Tu = Pu = Su$. Hence u is a common fixed point of P, S, R and T.

Again, suppose R and T are compatible of type (E) and one of the mappings R and T is continuous. Then we get $Rv = Tv = u$. By Proposition 2.4, $RRv = RTv = TRv = TTv$. Thus $Ru = Tu$.

We claim that $u = Tu$. Putting $z = z_{2n}$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$M(Sz_{2n}, Tu, kt) \geq \min \left\{ \begin{matrix} (M(Pz_{2n}, Ru, t), M(Pz_{2n}, Sz_{2n}, t), M(Ru, Tu, t), \\ M(Sz_{2n}, Ru, t), M(Pz_{2n}, Tu, t)) \end{matrix} \right\}.$$

Letting $n \to \infty$. Then we have

$$M(Su, u, kt) \geq \min \left\{ \begin{matrix} (M(Su, u, t), M(Su, Su, t), M(u, u, t), \\ M(Su, u, t), M(Su, u, t)) \end{matrix} \right\}.$$

Thus $M(Su, u, kt) \geq M(Su, u, t)$. By Lemma 2.6, $Su = u$. Since $S(X) \subset R(X)$, there exists a point $p \in X$ such that $u = Su = Rp$.

We claim that $u = Tp$. Putting $z = Sz_{2n}$ and $w = p$, $\alpha = 1$ in inequality (3.2), we have

$$M(SSz_{2n}, Tp, kt) \geq \min \begin{Bmatrix} (M(PSz_{2n}, Rp, t), M(PSz_{2n}, Sz_{2n}, t), M(Rp, Tp, t), \\ M(SSz_{2n}, Rp, t), M(Pu, Tp, t)) \end{Bmatrix}.$$

Letting $n \to \infty$. Then we have

$$M(u, Tp, kt) \geq \min \begin{Bmatrix} (M(u, u, t), M(u, u, t), M(u, Tp, t), \\ M(u, u, t), M(u, Tp, t)) \end{Bmatrix}.$$

Thus $M(u, Tp, kt) \geq M(u, Tp, t)$. By Lemma 2.6, $u = Tp$. Since R and T are compatible of type (R) and $Rp = Tp = u$, by Proposition 2.4, $RTp = TRp$. So $Ru = RTp = TRp = Tu$.

We claim that $u = Tu$. Putting $z = z_{2n}$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$M(Sz_{2n}, Tu, kt) \geq \min \begin{Bmatrix} (M(Pz_{2n}, Ru, t), M(Pz_{2n}, Sz_{2n}, t), M(Ru, Tu, t), \\ M(Sz_{2n}, Ru, t), M(Pz_{2n}, Tu, t)) \end{Bmatrix}.$$

Letting $n \to \infty$. Then we have

$$M(u, Tu, kt) \geq \min \begin{Bmatrix} (M(u, Tu, t), M(u, u, t), M(Tu, Tu, t), \\ M(u, Tu, t), M(u, Tu, t)) \end{Bmatrix}.$$

Thus $M(u, Tu, kt) \geq M(u, Tu, t)$. By Lemma 2.6, $Tu = u$. So $u = Ru = Tu$. Hence u is a common fixed point of P, S, R and T.

Similarly, we can complete the proof when T is continuous.

Uniqueness: If possible let u and v be two fixed point of the mappings P, R, S and T.

Finally, we claim that $u = v$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} M(Su, Tv, kt) &= M(u, v, kt) \\ &\geq \min \begin{Bmatrix} (M(Pu, Rv, t), M(Pu, Su, t), M(Rv, Tv, t), \\ M(Su, Rv, t), M(Pu, Tv, t)) \end{Bmatrix} \\ &= \min \begin{Bmatrix} (M(u, v, t), M(u, u, t), M(v, v, t), \\ M(u, v, t), M(u, v, t)) \end{Bmatrix}. \end{aligned}$$

Then $M(u, v, kt) \geq M(u, v, t)$. By Lemma 2.6, $u = v$. Thus P, S, R and T have a unique common fixed point. □

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