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Generalized upper and lower continuous multifunctions

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Abstract. In this paper, we introduce the concept of fuzzy $(\alpha, \beta, \theta, \delta, I)$-continuous multifunctions. There is a unification of several characterizations and properties of some types of modifications of upper and lower semi-continuous multifunctions. We deduce a generalized form of upper and lower continuous multifunctions, namely upper and lower $\eta\eta^*$-continuous multifunctions.

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1. Introduction

Multifunctions or multi-valued mappings have many applications in mathematical programming, probability, statistics, different inclusions, fixed point theorems and many branches, and continuity of multifunctions have been generalized in many ways. Many Mathematicians, see [10, 12, 13, 14, 21, 22], devoted a wide research work on studying the generalized continuity of multifunctions. We will study the case of multifunctions that maps each point in a fuzzy topological space in the sense of Šostak ([16]) into an ordinary topological space.

In this paper, we introduce the concepts of upper and lower $(\alpha, \beta, \theta, \delta, I)$-continuous multifunctions and prove that if α, β are operators on the fuzzy topological space (X, τ) in Šostak sense and θ, θ^*, δ are operators on the classical topological space (Y, T), and I is a proper fuzzy ideal on X, then a multifunction $F : X \to Y$ is upper (resp. lower) $(\alpha, \beta, \theta \sqcap \theta^*, \delta, I)$-continuous multifunction iff F is both of upper (resp. lower) $(\alpha, \beta, \theta, \delta, I)$-continuous and upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, I)$-continuous multifunction. Many generalized forms of upper and lower continuity have been studied in the literature. The properties which are mentioned may be determined by certain structures, such as the case in supra fuzzy topology [6]. Ideal topological and fuzzy topological spaces have been studied by Acharjee and Tripathy [2], Tripathy and Acharjee [19], Tripathy and Ray [17, 18], Tripathy and Sarma [20] and Abbas and Ibedou in [1].

2. Preliminaries

Throughout this paper, X refers to an initial universe, 2^X denotes the power set of X, I^X denotes the set of all fuzzy sets of X, $\lambda^c(x) = 1 - \lambda(x)\ \forall x \in X$ (where $I = [0,1], I_0 = (0,1]$). Two fuzzy sets $\lambda, \mu \in I^X$ are called quasi-coincident, denoted by $\lambda\, q\, \mu$, if there exists a $x \in X$ such that $\lambda(x) + \mu(x) > 1$.

As applications, $\alpha, \beta, id_X : I^X \times I_0 \to I^X$ are fuzzy operators on (X, τ) and $\theta, \delta, id_Y : 2^Y \to 2^Y$ are operators on (Y, T).

Recall that a fuzzy ideal I on X is a map $I : I^X \to I$ that satisfies the following conditions:

(1) $\lambda \leq \mu \Rightarrow I(\lambda) \geq I(\mu)$,
(2) $I(\lambda \vee \mu) \geq I(\lambda) \wedge I(\mu)$.

Also, I is called proper if $I(\overline{1}) = 0$ and there exists $\mu \in I^X$ such that $I(\mu) > 0$.
Define the fuzzy ideal I° by

$$I^\circ(\mu) = \begin{cases} 1 & \text{at } \mu = \overline{0}, \\ 0 & \text{otherwise} \end{cases}$$

Let (X, τ) be a fuzzy topological space due to Šostak [16] and (Y, T) a classical topological space. The fuzzy closure and the fuzzy interior of any fuzzy set λ in (X, τ) will be denoted by $\mathrm{cl}_\tau(\lambda, r)$ and $\mathrm{int}_\tau(\lambda, r)$ for any $\lambda \in I^X$, $r \in I_0$ while the closure and the interior of any set $A \in 2^Y$ will be denoted by T-$\mathrm{cl}(A)$ and T-$\mathrm{int}(A)$ respectively.

A fuzzy set $\mu \in I^X$ is called r-fuzzy strongly semi-open [8] (resp. r-fuzzy preopen [8] and r-fuzzy preclosed [8]) if and only if

$$\mu \leq \mathrm{int}_\tau(\mathrm{cl}_\tau(\mathrm{int}_\tau(\mu, r), r), r) \text{ (resp. } \mu \leq \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(\mu, r), r), r) \text{ and } \mu \geq \mathrm{cl}_\tau(\mathrm{int}_\tau(\mu, r), r))$$

while

$$\mathrm{s\,s\,cl}_\tau(\mu, r) = \bigvee\{\lambda : \lambda \leq \mu \text{ and } \lambda \text{ is } r-\text{fuzzy strongly semi-open}\},$$

$$\mathrm{s\,pre\,int}_\tau(\mu, r) = \bigvee\{\lambda : \lambda \leq \mu \text{ and } \lambda \text{ is } r-\text{fuzzy semi-preopen}\}$$

and

$$\mathrm{pre\,cl}_\tau(\mu, r) = \bigwedge\{\lambda : \lambda \geq \mu \text{ and } \lambda \text{ is } r-\text{fuzzy preclosed}\}.$$

Also, $A \subseteq Y$ is semi-closed ([9]) if $A \supseteq T$-$\mathrm{int}(T$-$\mathrm{cl}(A))$ while T-$\mathrm{s\,cl}(A) = \bigcap\{B : A \subseteq B \text{ and } B \text{ is semi-closed}\}$.

Let us define the fuzzy difference between two fuzzy sets as given in [7]:

$$(\lambda \mathbin{\bar{\wedge}} \mu) = \begin{cases} \overline{0} & \text{if } \lambda \leq \mu, \\ \lambda \wedge \mu^c & \text{otherwise.} \end{cases}$$

Consider the family Ω denotes the set of all fuzzy subsets of a given set X satisfying the following condition: $\forall \lambda, \mu \in \Omega,\ \ \lambda \leq \mu$ or $\mu \leq \lambda$ ([7]).

Note that: For each $\lambda, \mu, \nu \in \Omega$, we have:

(1) $\nu \bar{\wedge} (\lambda \wedge \mu) = (\nu \bar{\wedge} \lambda) \vee (\nu \bar{\wedge} \mu)$,
(2) $(\lambda \vee \mu) \bar{\wedge} \nu = (\lambda \bar{\wedge} \nu) \vee (\mu \bar{\wedge} \nu)$.

A mapping $F : X \to Y$ is called a multifunction if for each $x \in X$, $F(x)$ is a subset in Y. The Upper and Lower inverse of a set $A \subseteq Y$ are denoted by $F^u(A)$ and $F^l(A)$, respectively. That is ([3]),

$$F^u(A) = \{x \in X : F(x) \subseteq A\} \quad \text{and} \quad F^l(A) = \{x \in X : (F(x) \cap A) \neq \emptyset\}.$$

Also, a multifunction $F : X \to Y$ is called Upper (resp. Lower) semi-continuous if $F^u(B)$ (resp. $F^l(B)$) is open in X for every open set B in Y ([3]).

A mapping $F : X \to Y$ is called a fuzzy multifunction if for each $x \in X$, $F(x)$ is a fuzzy set in Y. The fuzzy upper inverse $F^+(\lambda)$ and the fuzzy lower inverse $F^-(\lambda)$ of $\lambda \in I^Y$ are defined as follows:

$$F^+(\lambda) = \{x \in X : F(x) \leq \lambda\} \quad \text{and} \quad F^-(\lambda) = \{x \in X : F(x)\, q\, \lambda\}.$$

For $A \subseteq X$, $F(A) = \bigvee\{F(x) : x \in A\}$ and $F^-(\lambda^c) = X - F^+(\lambda)$ for each $\lambda \in I^Y$ ([13]).

Also, a fuzzy multifunction $F : X \to Y$ is called fuzzy lower (resp. upper) semi-continuous if $F^-(\lambda)$ $(F^+(\lambda))$ is open in X for every $\lambda \in I^Y$ with $\tau(\lambda) \geq r$; $r \in I_0$ [15].

Now, for a multifunction $F : X \to Y$ and $\lambda \in I^X$, let us define $F(\lambda)$ as follows:

$$F(\lambda) = \bigcup\{F(x) \mid \lambda(x) > 0\}.$$

For any $A \in 2^Y$, the upper inverse $F^+(A)$ and the lower inverse $F^-(A)$ are defined as follows:

$$F^+(A) = \{\lambda \in I^X : F(\lambda) \subseteq A\} \quad \text{and} \quad F^-(A) = \{\lambda \in I^X \mid F(\lambda) \cap A \neq \emptyset\}.$$

For any $A \subseteq Y$, $F^-(A^c) = \bar{1} \bar{\wedge} F^+(A)$.

Also, a multifunction $F : X \to Y$ is called upper (resp. lower) semi-continuous if

$$F^+(A) \leq \text{int}_\tau(F^+(A), r) \text{ (resp. } F^-(A) \leq \text{int}_\tau(F^-(A), r)) \text{ for each } A \subseteq Y.$$

3. Upper and lower $(\alpha, \beta, \theta, \delta, I)$-continuous multifunctions

The idea of this section is based on the results and definitions in [23]

Definition 3.1. A mapping $F : (X, \tau) \to (Y, T)$ is said to be upper (resp. lower) $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction if for every $A \in 2^Y$ with $A \in T$, for some $r \in I_0$

$$I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)] \geq r$$

$$\text{(resp. } I[\alpha(F^-(\delta(A)), r) \bar{\wedge} \beta(F^-(\theta(A)), r)] \geq r).$$

We can see that the above definition generalizes the concept of fuzzy upper (resp. lower) semi-continuous multifunction when we choose α = fuzzy identity operator on X, β = fuzzy interior operator on X, δ, θ = identity operators on Y and $I = I^\circ$.

Let us give a historical justification of the definition:

(1) The concept of upper (resp. lower) almost continuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then
$F^+(A) \leq \text{int}_\tau(F^+(T\text{-int}(T\text{-cl}(A))), r)$ (resp. $F^-(A) \leq \text{int}_\tau(F^-(T\text{-int}(T\text{-cl}(A))), r)$).
Here, α = fuzzy identity operator, β = fuzzy interior operator, δ = identity operator, θ = interior closure operator and $I = I^\circ$.

(2) The concept of upper (resp. lower) weakly continuous multifunction as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then
$F^+(A) \leq \text{int}_\tau(F^+(T\text{-cl}(A)), r)$ (resp. $F^-(A) \leq \text{int}_\tau(F^-(T\text{-cl}(A)), r)$).
Here, α = fuzzy identity operator, β = fuzzy interior operator, δ = identity operator, θ = closure operator and $I = I^\circ$.

(3) The concept of upper (resp. lower) almost weakly continuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then
$F^+(A) \leq \text{int}_\tau(\text{cl}_\tau(F^+(T\text{-cl}(A)), r), r)$ (resp. $F^-(A) \leq \text{int}_\tau(\text{cl}_\tau(F^-(T\text{-cl}(A)), r), r)$).
Here, α = fuzzy identity operator, β = fuzzy interior closure operator, δ = identity operator, θ = closure operator and $I = I^\circ$.

(4) The concept of upper (resp. lower) strongly semi-continuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then

$$F^+(A) \leq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(F^+(A), r), r), r)$$

$$(\text{resp. } F^-(A) \leq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(F^-(A), r), r), r)).$$

Here, α = fuzzy identity operator, β = fuzzy interior closure interior operator, δ = identity operator, θ = identity operator and $I = I^\circ$.

(5) The concept of upper (resp. lower) almost strongly semi-continuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then
$F^+(A) \leq \text{s s int}_\tau(F^+(T\text{-s cl}(A)), r)$ (resp. $F^-(A) \leq \text{s s int}_\tau(F^-(T\text{-s cl}(A)), r)$).
Here, α = fuzzy identity operator, β = fuzzy strongly semi-interior operator, δ = identity operator, θ = semi-closure operator and $I = I^\circ$.

(6) The concept of upper (resp. lower) weakly strongly semi-continuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then
$F^+(A) \leq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(F^+(T\text{-cl}(A)), r), r), r)$ (resp. $F^-(A) \leq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(F^-(T\text{-cl}(A)), r), r), r)$).
Here, α = fuzzy identity operator, β = fuzzy interior closure interior operator, δ = identity operator, θ = closure operator and $I = I^\circ$.

(7) The concept of upper (resp. lower) semi-precontinuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then

$$F^+(A) \leq \text{cl}_\tau(\text{int}_\tau(\text{cl}_\tau(F^+(A), r), r), r)$$

$$(\text{resp. } F^-(A) \leq \text{cl}_\tau(\text{int}_\tau(\text{cl}_\tau(F^-(A), r), r), r)).$$

Here, α = fuzzy identity operator, β = fuzzy closure interior closure operator, δ = identity operator, θ = identity operator and $I = I^\circ$.

(8) The concept of upper (resp. lower) almost semi-precontinuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then
$F^+(A) \leq \text{s pre int}_\tau(F^+(T\text{-s cl}(A)), r)$ (resp. $F^-(A) \leq \text{s pre int}_\tau(F^-(T\text{-s cl}(A)), r)$).

Here, α = fuzzy identity operator, β = fuzzy semi-preinterior operator, δ = identity operator, θ = semi-closure operator and $I = I^{\circ}$.

(9) The concept of upper (resp. lower) weakly semi-precontinuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then $F^+(A) \leq \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(F^+(T\text{-}\mathrm{cl}(A)), r), r), r)$ (resp. $F^-(A) \leq \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(F^-(T\text{-}\mathrm{cl}(A)), r), r), r)$).

Here, α = fuzzy identity operator, β = fuzzy closure interior closure operator, δ = identity operator, θ = closure operator and $I = I^{\circ}$.

(10) The concept of upper (resp. lower) precontinuous multifunction is defined as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then

$$F^+(A) \leq \mathrm{int}_\tau(\mathrm{cl}_\tau(F^+(A), r), r) \quad (\text{resp. } F^-(A) \leq \mathrm{int}_\tau(\mathrm{cl}_\tau(F^-(A), r), r)).$$

Here, α = fuzzy identity operator, β = fuzzy interior closure operator, δ = identity operator, θ = identity operator and $I = I^{\circ}$.

(11) The concept of upper (resp. lower) strongly precontinuous multifunction as: For every $A \in 2^Y, r \in I_0$ with $A \in T$, then

$$F^+(A) \leq \mathrm{int}_\tau(\mathrm{pre\,cl}_\tau(F^+(A), r), r) \quad (\text{resp. } F^-(A) \leq \mathrm{int}_\tau(\mathrm{pre\,cl}_\tau(F^-(A), r), r)).$$

Here, α = fuzzy identity operator, β = fuzzy interior preclosure operator, δ = identity operator, θ = identity operator and $I = I^{\circ}$.

Definition 3.2. A mapping $F : (X, \tau) \to (Y, T)$ is called an upper (resp. lower) P-continuous multifunction iff for every $A \in 2^Y$, $r \in I_0$, with $A \in T$, we have

$$\tau(F^+(A)) \geq r \quad (\text{resp. } \tau(F^-(A) \geq r)) \quad \text{such that} \quad A \text{ satisfies the property } P.$$

Let $\theta_P : 2^Y \to 2^Y$ be an operator defined as:

$$\theta_P(A) = \begin{cases} A & \text{if } A \in T \text{ and } A \text{ satisfies the property } P, \\ Y & \text{otherwise} \end{cases}$$

Theorem 3.3. *A map $F : (X, \tau) \to (Y, T)$ is upper (resp. lower) P-continuous multifunction* iff *it is upper (resp. lower) $(id_X, \mathrm{int}_\tau, \theta_P, id_Y, I^{\circ})$-continuous multifunction.*

Proof. Suppose that F is an upper P-continuous multifunction and let $A \in 2^Y, r \in I_0$ with $A \in T$.

Case 1. If A satisfies the property P, $\theta_P(A) = A$, and then by hypothesis $\tau(F^+(A)) \geq r$ and $F^+(A) \leq \mathrm{int}_\tau(F^+(A), r) = \mathrm{int}_\tau(F^+(\theta_P(A)), r)$.

Case 2. A does not satisfy the property P, then $\theta_P(A) = Y$, and thus $id_X(F^+(A), r) \leq \overline{1} = \mathrm{int}_\tau(F^+(\theta_P(A)), r)$. That is, F is upper $(id_X, \mathrm{int}_\tau, \theta_P, id_Y, I^{\circ})$-continuous multifunction.

Conversely, suppose that $id_X(F^+(A), r) \leq \mathrm{int}_\tau(F^+(\theta_P(A)), r)$ for each $A \in 2^Y$, $r \in I_0$ with $A \in T$. Taking A satisfying the property P, then $\theta_P(A) = A$, and thus $F^+(A) \leq \mathrm{int}_\tau(F^+(\theta_P(A)), r) = \mathrm{int}_\tau(F^+(A))$. We conclude that $\tau(F^+(A)) \geq r$ and thus F is an upper P-continuous multifunction.

For lower P-continuous multifunction, the proof is similar. □

Definition 3.4. If γ and γ^* are operators on Y, then the operator $\gamma \sqcap \gamma^*$ is defined as follows:

$$(\gamma \sqcap \gamma^*)(A) \; = \; \gamma(A) \cap \gamma^*(A) \;\; \forall A \in 2^Y.$$

The operators γ and γ^* are said to be mutually dual if $\gamma \sqcap \gamma^*$ is the identity operator.

Theorem 3.5. *Let (X, τ) be a fuzzy topological space, (Y, T) a topological space and I a proper fuzzy ideal on X. Let $\alpha, \beta, \beta^* : I^X \times I_0 \to \Omega$ be fuzzy operators on (X, τ) and δ, θ, θ^* be operators on (Y, T). Then $F : X \to Y$ is:*

(1) *upper (resp. lower) $(\alpha, \beta, \theta \sqcap \theta^*, \delta, I)$-continuous multifunction iff it is both upper (resp. lower) $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction and upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, I)$-continuous multifunction provided that for all $\lambda, \mu \in I^X$, $r \in I_0$, we have $\beta((\lambda \wedge \mu), r) = \beta(\lambda, r) \wedge \beta(\mu, r)$.*

(2) *upper (resp. lower) $(\alpha, \beta \sqcap \beta^*, \theta, \delta, I)$-continuous multifunction iff it is both upper (resp. lower) $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction and upper (resp. lower) $(\alpha, \beta^*, \theta, \delta, I)$-continuous multifunction.*

Proof. (1) If F is both upper $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction and upper $(\alpha, \beta, \theta^*, \delta, I)$-continuous multifunction, then, for every $A \in 2^Y$, $r \in I_0$, with $A \in T$, we have

$$I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)] \geq r \;\text{ and }\; I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta^*(A)), r)] \geq r,$$

and then

$$I[(\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)) \;\vee\; (\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta^*(A)), r))] \;\geq r.$$

But

$$(\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)) \;\vee\; (\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta^*(A)), r))$$

$$\begin{cases} = & \alpha(F^+(\delta(A)), r) \bar{\wedge} (\beta(F^+(\theta(A)), r) \;\wedge\; \beta(F^+(\theta^*(A)), r)) \\ = & \alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A) \wedge \theta^*(A)), r) \\ = & \alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta \sqcap \theta^*(A)), r). \end{cases}$$

That is, F is fuzzy upper $(\alpha, \beta, \theta \sqcap \theta^*, \delta, I)$-continuous multifunction.
Conversely; if F is upper $(\alpha, \beta, \theta \sqcap \theta^*, \delta, I)$-continuous multifunction, then

$$I(\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta \sqcap \theta^*(A)), r)) \geq r.$$

Now, by the above equalities, we get that

$$I[(\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)) \;\vee\; (\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta^*(A)), r))] \;\geq r,$$

which implies that

$$I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)] \geq r \;\text{ and }\; I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta^*(A)), r)] \geq r,$$

which means that F is both upper $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction and upper $(\alpha, \beta, \theta^*, \delta, I)$-continuous multifunction.
(2) Similar to the proof in (1).
The proof for lower continuity is typical. □

Let Φ be the set of all fuzzy operators on the fuzzy topological space (X, τ). Then a partial order could be defined by the relation:

$$\alpha \sqsubseteq \beta \quad \text{iff} \quad \alpha(\lambda, r) \leq \beta(\lambda, r) \text{ for all } \lambda \in I^X,\ r \in I_0.$$

Theorem 3.6. *Let (X, τ) be a fuzzy topological space, (Y, T) a topological space and I a proper fuzzy ideal on X. Let $\alpha, \alpha^*, \beta, \beta^* : I^X \times I_0 \to I^X$ be fuzzy operators on (X, τ) and $\delta, \theta, \theta^* : 2^Y \to 2^Y$ are operators on (Y, T) and $F : X \to Y$ is a multifunction.*

(1) *If β is a monotone, $\theta \sqsubseteq \theta^*$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction, then F is upper (resp. lower) $(\alpha, \beta, \theta^*, \delta, I)$-continuous multifunction,*

(2) *If $\alpha^* \sqsubseteq \alpha$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction, then F is upper (resp. lower) $(\alpha^*, \beta, \theta, \delta, I)$-continuous multifunction,*

(3) *If $\beta \sqsubseteq \beta^*$ and F is upper (resp. lower) $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction, then F is upper (resp. lower) $(\alpha, \beta^*, \theta^*, \delta, I)$-continuous multifunction.*

Proof. (1) Since F is upper $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction, then for every $A \in 2^Y,\ r \in I_0$ with $A \in T$, it happens that

$$I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r) \ \geq r].$$

We know that $\theta \sqsubseteq \theta^*$, and then $\theta(A) \subseteq \theta^*(A)$, and thus $F^+(\theta(A)) \leq F^+(\theta^*(A))$ and $\beta(F^+(\theta(A)), r) \leq \beta(F^+(\theta^*(A)), r)$. That is,

$$\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r) \ \geq\ \alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta^*(A)), r).$$

Therefore,

$$I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)] \ \leq\ I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta^*(A)), r)],$$

which means that F is upper $(\alpha, \beta, \theta^*, \delta, I)$-continuous multifunction.

(2) and (3) are similar.

The case of lower continuity is similar. □

Definition 3.7. An operator Δ on a topological space (Y, T) induces another operator $(\text{int}_T\Delta)$ defined as follows:

$$(\text{int}_T\Delta)(A) \ =\ \text{int}_T(\Delta(A))\ \forall A \in 2^Y. \quad \text{Clearly, } \text{int}_T\Delta \ \sqsubseteq\ \Delta.$$

Theorem 3.8. *Let $\alpha, \beta : I^X \times I_0 \to I^X$ be fuzzy operators on (X, τ) and $\delta, \theta : 2^Y \to 2^Y$ are operators on (Y, τ) and I a proper fuzzy ideal on X. If $F : X \to Y$ is an upper (resp. lower) $(\alpha, \beta, \theta, \delta, I)$-continuous multifunction and*

$$\beta(F^+(A), r) \ \leq\ \beta(F^+(\text{int}_T(A)), r) \ \ (\textit{resp. } \ \beta(F^-(A), r) \ \leq\ \beta(F^-(\text{int}_T(A)), r)),$$

for every $A \in 2^Y,\ r \in I_0$. Then F is upper (resp. lower) $(\alpha, \beta, \text{int}_T\theta, \delta, I)$-continuous multifunction.

Proof. Let $A \in 2^Y,\ r \in I_0$ with $A \in T$. Then, we have that

$$I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)] \ \geq r.$$

Since $\beta(F^+(A), r) \ \leq\ \beta(F^+(\text{int}_T(A)), r)$, then $\beta(F^+(\theta(A)), r) \ \leq\ \beta(F^+(\text{int}_T\theta(A)), r)$. Hence,

$$I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\text{int}_T\theta(A)), r)] \ \geq\ I[\alpha(F^+(\delta(A)), r) \bar{\wedge} \beta(F^+(\theta(A)), r)],$$

and thus, F is upper $(\alpha, \beta, \text{int}_T\theta, \delta, I)$-continuous multifunction. □

Definition 3.9. Let (X, T) be a topological space. Then, an ordinary set $K \in 2^X$ is called θ-compact if for each family $\{B_j \in 2^X : B_j \in T\}$ with $K \subseteq \bigcup_{j\in J}(B_j)$, there exists a finite subset $J_0 \subseteq J$ such that $K \subseteq \bigcup_{j\in J_0}(\theta(B_j))$.

Theorem 3.10. *Let (X, τ) be a fuzzy topological space, (Y, T) a topological space, $\alpha : I^X \times I_0 \to I^X$ a fuzzy operator on (X, τ) with $\lambda \leq \alpha(\lambda, r)\ \forall \lambda \in I^X$, $r \in I_0$ and $\delta, \theta : 2^Y \to 2^Y$ are operators on (Y, T) with $A \subseteq \delta(A)\ \ \forall A \in 2^Y$. If $F : X \to Y$ is upper (resp. lower) $(\alpha, \mathrm{int}_\tau, \theta, \delta, I^\circ)$-continuous multifunction and λ is a fuzzy compact fuzzy subset of X, then, $F(\lambda)$ is θ-compact in 2^Y.*

Proof. Suppose that the family $\{B_j \in 2^Y : j \in J,\ B_j \in T\}$ satisfies that $F(\lambda) \subseteq \bigcup_{j\in J} B_j$. From F is upper $(\alpha, \mathrm{int}_\tau, \theta, \delta, I^\circ)$-continuous multifunction, then for each $j \in J$, we have

$$\alpha(F^+(\delta(B_j)), r) \ \leq\ \mathrm{int}_\tau(F^+(\theta(B_j)), r) \ \leq\ F^+(\theta(B_j)).$$

Then there exists $\mu_j \in I^X$ with $\tau(\mu_j) \geq r$ such that

$$\alpha(F^+(\delta(B_j)), r) \ \leq\ \mu_j \ \leq\ F^+(\theta(B_j)).$$

Since $F^+(\delta(B_j)) \ \leq\ \alpha(F^+(\delta(B_j)), r)$ and $B_j \leq \delta(B_j)$, then

$$\lambda \ \leq\ F^+(F(\lambda)) \ \leq\ \bigvee_{j\in J} F^+(B_j) \ \leq\ \bigvee_{j\in J} \mu_j.$$

From the fuzzy compactness of λ, there exists a finite subset J_0 of J such that $\lambda \leq \bigvee_{j\in J_0} \mu_j$. Then $F(\lambda) \subseteq \bigcup_{j\in J_0} F(\mu_j) \subseteq \bigcup_{j\in J_0} F(F^+(\theta(B_j))) \subseteq \bigcup_{j\in J_0} \theta(B_j)$, which means that $F(\lambda)$ is θ-compact. □

Corollary 3.11. *Let (X, τ) be a fuzzy topological space and (Y, T) a topological space. Let $F : X \to Y$ be an upper (resp. lower) weakly continuous multifunction and λ a compact fuzzy subset of X, then $F(\lambda)$ is an almost compact set in 2^Y.*

Proof. Take α = fuzzy identity operator, $\beta = \mathrm{int}_\tau$, δ = identity operator, θ = closure operator and $I = I^\circ$. Then the result is fulfilled directly from Theorem 3.10. □

Corollary 3.12. *Let (X, τ) be a fuzzy topological space and (Y, T) a topological space. Let $F : X \to Y$ be an upper (resp. lower) almost continuous multifunction and λ a compact fuzzy subset of X, then $F(\lambda)$ is a nearly compact set in 2^Y.*

Proof. Take α = fuzzy identity operator, $\beta = \mathrm{int}_\tau$, δ = identity operator, θ = closure operator on Y and $I = I^\circ$. Then the result follows from Theorem 3.10. □

4. Upper and lower $\eta\eta^*$-continuous multifunctions

Let X and Y be nonempty sets and $\eta^* \subseteq 2^Y$ be any collection of subsets of Y and $\eta : I^X \to I$ any function.

Definition 4.1. A function $F : X \to Y$ is said to be upper (resp. lower) $\eta\eta^*$-continuous multifunction if for $r \in I_0$, $\eta(F^+(A)) \geq r$ (resp. $\eta(F^-(A)) \geq r$) whenever $A \in 2^Y$ with $A \subseteq \mathrm{int}_{\eta^*}(A)$.

Remark 4.2. A generalized topology on a set Y ([5]) is a collection η^* of subsets of Y such that $\emptyset \in \eta^*$ and η^* is closed under arbitrary unions. Also, a generalized fuzzy topology on a set X ([5]) is a function $\eta : I^X \to I$ such that $\eta(\overline{0}) = 1$ and $\eta(\bigvee_{j\in J} \mu_j) \geq \bigwedge_{j\in J}(\eta(\mu_j))\ \forall \mu_j \in I^X$. Observe that if Definition 4.1, η and η^* are fuzzy generalized topology on X and generalized topology on Y respectively, then we just obtain the notion of upper (resp. lower) $\eta\eta^*$-continuous multifunctions. In [11], Maki et al., introduced the notion of minimal structure on a set Y, as the collection m_Y of subsets of Y such that $\emptyset \in m_Y$ and $Y \in m_Y$. Also, in [24], Yoo et al., introduced the notion of fuzzy minimal structure on a set X, as $m_X : I^X \to I$ such that $m_X(\overline{0}) = m_X(\overline{1}) = 1$. Now, if in Definition 4.1, $\eta = m_X$ and $\eta^* = m_Y$, we obtain the notion of upper (resp. lower) $m_X m_Y$-continuous multifunctions.

Any collection η^* of subsets of a set Y and any function $\eta : I^X \to I$ determine in a natural form an operator $\theta_{\eta^*} : 2^Y \to 2^Y$ and a fuzzy operator $\theta_\eta : I^X \times I_0 \to I^X$ respectively, so that

$$\theta_{\eta^*}(A) = \begin{cases} A & \text{if } A \in \eta^* \\ Y & \text{otherwise} \end{cases}$$

and

$$\theta_\eta(\mu, r) = \begin{cases} \mu & \text{if } \mu \in I^X, r \in I_0 \text{ with } \eta(\mu) \geq r \\ \overline{1} & \text{otherwise} \end{cases}$$

In the case that η is a generalized fuzzy topology on X and η^* is a generalized topology on Y, we obtain other operators (see [5]) that are important for its applications:

$$\begin{aligned} \text{int}_{\eta^*}(A) &= \bigcup\{B : B \subseteq A \text{ and } B \in \eta^*\}, \\ \text{cl}_{\eta^*}(A) &= \bigcap\{B : A \subseteq B \text{ and } X - B \in \eta^*\}, \\ \text{int}_\eta(\lambda, r) &= \bigvee\{\mu : \mu \leq \lambda \text{ and } \eta(\mu) \geq r\}, \\ \text{cl}_\eta(\lambda, r) &= \bigwedge\{\mu : \lambda \leq \mu \text{ and } \eta(\overline{1} - \mu) \geq r\}. \end{aligned}$$

Note that: $\text{int}_{\eta^*} \sqsubseteq id_Y \sqsubseteq \theta_{\eta^*}$ and $\text{int}_\eta \sqsubseteq id_X \sqsubseteq \theta_\eta$. Similarly, in the case of a fuzzy minimal structure m_X (see [24]) and a minimal structure m_Y (see [4]), we have

$$\begin{aligned} \text{int}_{m_Y}(A) &= \bigcup\{B : B \subseteq A \text{ and } B \in m_Y\}, \\ \text{cl}_{m_Y}(A) &= \bigcap\{B : A \subseteq B \text{ and } X - B \in m_Y\}, \\ \text{int}_{m_X}(\mu, r) &= \bigvee\{\nu : \nu \leq \mu \text{ and } m_X(\nu) \geq r\}, \\ \text{cl}_{m_X}(\mu, r) &= \bigwedge\{\nu : \mu \leq \nu \text{ and } m_X(\overline{1} - \nu) \geq r\}. \end{aligned}$$

Note that: $\text{int}_{m_X} \sqsubseteq id_X \sqsubseteq \theta_\eta$ and $\text{int}_{m_Y} \sqsubseteq id_Y \sqsubseteq \theta_{\eta^*}$. Also, $\text{int}_{m_Y}(A) = A$ if $A \in m_Y$ while $\text{int}_{m_Y}(A) \in m_Y$ whenever m_Y is a minimal structure with the Maki property [11]. $\text{int}_{m_X}(\lambda, r) = \lambda$ if $m_X(\lambda) \geq r$ while $m_X(\text{int}_{m_X}(\lambda, r)) \geq r$ whenever m_X is a fuzzy minimal structure with the Yoo property [24].

The following results give the relationship between upper (resp. lower) $\eta\eta^*$-continuous multifunctions and upper (resp. lower) $(\alpha,\beta,\theta,\delta,I)$-continuous multifunctions. We obtain some interesting properties of upper (resp. lower) $\eta\eta^*$-continuous multifunctions.

Theorem 4.3. *Let X and Y be nonempty sets, $\eta : I^X \to I$, $\eta^* \subseteq 2^X$. If $Y \in \eta^*$, then $F : X \to Y$ is upper (resp. lower) $\eta\eta^*$-continuous multifunction* iff *$F : X \to Y$ is upper (resp. lower) $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, I^\circ)$-continuous multifunction.*

Proof. Suppose that $F : X \to Y$ is an upper $\eta\eta^*$-continuous multifunction. Let $A \in 2^Y$, $r \in I_0$, we have two cases:
Case 1. If $A \in \eta^*$, then $\theta_{\eta^*}(A) = A$ and $\theta_\eta(F^+(A), r) = F^+(A)$. This follows that $\theta_\eta(F^+(id_Y(A)), r) = F^+(A) = id_X(F^+(\theta_{\eta^*}(A)), r)$, and consequently

$$\theta_\eta(F^+(id_Y(A)), r) \ \le\ id_X(F^+(\theta_{\eta^*}(A)), r).$$

Case 2. If $A \notin \eta^*$, $\theta_{\eta^*}(A) = Y$, then
$\theta_\eta(F^+(id_Y(A)), r) \le \overline{1} = F^+(Y) = id_X(F^+(\theta_{\eta^*}(A)), r)$. Hence,

$$\theta_\eta(F^+(id_Y(A)), r) \,\bar{\wedge}\, id_X(F^+(\theta_{\eta^*}(A)), r) \ =\ \overline{0}$$

for all $A \in 2^Y, r \in I_0$. Thus, F is an upper $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, I^\circ)$-continuous multifunction.

Necessity; suppose that F is upper $(\theta_\eta, id_X, \theta_{\eta^*}, id_Y, I^\circ)$-continuous multifunction, then $\theta_\eta(F^+(id_Y(A)), r) \,\bar{\wedge}\, id_X(F^+(\theta_{\eta^*}(A)), r) \ =\ \overline{0}$ for all $A \in 2^Y, r \in I_0$ with $A \in \eta^*$. This implies that $\theta_\eta(F^+(A), r) \ \le\ F^+(\theta_{\eta^*}(A))$. Assume that there is $B \in 2^Y, r \in I_0$ such that $B \in \eta^*$ and $\eta(F^+(B)) = 0$. Then we obtain $\overline{1} \le F^+(B)$. So, $F^+(B) = \overline{1}$. Now our hypothesis $Y \in \eta^*$ implies that $\eta(F^+(B)) \ge r, r \in I_0$, and a contradiction. Therefore, $\eta(F^+(A)) \ge r$ whenever $A \in 2^Y, r \in I_0$ with $A \in \eta^*$, and thus $F : X \to Y$ is an upper $\eta\eta^*$-continuous multifunction. □

In the case that η is a generalized fuzzy topology, then the following result is obtained.

Theorem 4.4. *If η is a generalized fuzzy topology such that $\eta(\overline{1}) \ge r, r \in I_0$ and $\eta^* \subseteq 2^Y$ is a family of subsets. Then $F : X \to Y$ is upper (resp. lower) $\eta\eta^*$-continuous multifunction* iff *$F : X \to Y$ is upper (resp. lower) $(id_X, \mathrm{int}_\eta, \theta_{\eta^*}, id_Y, I^\circ)$-continuous multifunction.*

Proof. Suppose that $F : X \to Y$ is upper $\eta\eta^*$-continuous multifunction. Let $A \in I^Y$, $r \in I_0$. Then consider two cases:
Case 1. If $A \in \eta^*$, then $\theta_{\eta^*}(A) = A$ and $id_X(F^+(A), r) = F^+(A) = \mathrm{int}_\eta(F^+(A), r)$. This follows that $id_X(F^+(id_Y(A)), r) = F^+(A) = \mathrm{int}_\eta(F^+(\theta_{\eta^*}(A)), r)$, and consequently

$$id_X(F^+(id_Y(A)), r) \ \le\ \mathrm{int}_\eta(F^+(\theta_{\eta^*}(A)), r).$$

Case 2. If $A \notin \eta^*$, $\theta_{\eta^*}(A) = Y$, since $Y \in \eta^*$, then

$$id_X(F^+(id_Y(A)), r) \ \le\ \overline{1} \ =\ F^+(Y) = \mathrm{int}_\eta(F^+(\theta_{\eta^*}(A)), r).$$

So,

$$id_X(F^+(id_Y(A)), r) \,\bar{\wedge}\, \mathrm{int}_\eta(F^+(\theta_{\eta^*}(A)), r) \ =\ \overline{0}$$

for all $A \in 2^Y, r \in I_0$. Hence, F is a fuzzy upper $(id_X, \text{int}_\eta, \theta_{\eta^*}, id_Y, I^\circ)$-continuous multifunction.

Necessity; suppose that F is upper $(id_X, \text{int}_\eta, \theta_{\eta^*}, id_Y, I^\circ)$-continuous multifunction. Then

$$id_X(F^+(id_Y(A)), r) \bar{\wedge} \text{int}_\eta(F^+(\theta_{\eta^*}(A)), r) = \overline{0}$$

for every $A \in 2^Y, r \in I_0$ with $A \in \eta^*$. This implies that

$$F^+(A) \leq \text{int}_\eta(F^+(\theta_{\eta^*}(A)), r) = \text{int}_\eta(F^+(A), r).$$

Assume that there is $B \in 2^Y, r \in I_0$ such that $B \in \eta^*$ and $\eta(F^+(B)) = 0$. Then we obtain $F^+(B) \leq \text{int}_\eta(F^+(B), r)$, and thus $F^+(B) = \text{int}_\eta(F^+(B), r)$, and $\eta(F^+(B)) \geq r$, which is a contradiction. Therefore, $\eta(F^+(A)) \geq r$ whenever $A \in 2^Y, r \in I_0$ with $A \in \eta^*$, that is, $F : X \to Y$ is an upper $\eta\eta^*$-continuous multifunction. □

The following corollaries are direct results.

Corollary 4.5. *Let $F : X \to Y$ be a multifunction. If F is upper (resp. lower) $m_X m_Y$-continuous multifunction, then F is upper (resp. lower) $(id_X, \text{int}_{m_X}, \theta_{m_Y}, id_Y, I)$-continuous multifunction whenever m_Y has the Maki property.*

Corollary 4.6. *Let η be a generalized fuzzy topology on X and η^* a generalized topology on Y such that $Y \in \eta^*$. Then, $F : X \to Y$ is upper (resp. lower) $\eta\eta^*$-continuous multifunction* iff *F is upper (resp. lower) $(id_X, \text{int}_\eta, \text{int}_{\eta^*}, id_Y, I^\circ)$-continuous multifunction.*

Corollary 4.7. *Let $F : (X, T_1) \to (Y, \tau)$ be fuzzy upper (resp. lower) semi-continuous multifunction and $G : (Y, \tau) \to (Z, T_2)$ be upper (resp. lower) semi-continuous multifunction. Then the composition $G \circ F : (X, T_1) \to (Z, T_2)$ is an Upper (resp. Lower) semi-continuous multifunction.*

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