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Neutrosophic falling shadows applied to subalgebras and ideals in BCK/BCI-algebras

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Abstract. As a combination of neutrosophic set and falling shadow, neutrosophic falling shadow is introduced, and applied to BCK/BCI-algebras. Falling neutrosophic subalgebra and falling neutrosophic ideal in BCK/BCI-algebras are introduced, and related properties are investigated. Relations between falling neutrosophic subalgebra and falling neutrosophic ideal are discussed. A characterization of falling neutrosophic ideal is established.

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Keywords: $(\in, \in)$-neutrosophic subalgebra, $(\in, \in)$-neutrosophic ideal, Neutrosophic random set, Neutrosophic falling shadow, Falling neutrosophic subalgebra, Falling neutrosophic ideal.

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1. Introduction

Neutrosophic set (NS) developed by Smarandache [12, 13, 14] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. In a neutrosophic set, an element has three associated defining functions such as truth membership function (T), indeterminate membership function (I) and false membership function (F) defined on a universe of discourse X. Neutrosophic set theory is applied to various part which is refered to the site

http://fs.gallup.unm.edu/neutrosophy.htm.

Nowadays, neutrosophic set theory is applied to a large area. Liu et al. [9] discussed neutrosophic uncertain linguistic number Heronian mean operators with application in multi-attribute group decision making. In algebraic envirnoment, Jun, Borumand Saeid and Öztürk studied neutrosophic subalgebras and neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [7] and [11]). Goodman [3] pointed out the equivalence of a fuzzy set and a class of random sets in the study of a unified treatment of uncertainty modeled by means of combining probability and fuzzy set theory. Wang and Sanchez [17] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. The mathematical structure of the theory of falling shadows is formulated in [18]. Tan et al. [15, 16] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [8] considered a fuzzy subalgebra and a fuzzy ideal as the falling shadow of the cloud of the subalgebra and ideal.

In this manuscript, we introduce the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, we also introduce the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in BCK/BCI-algebras, and investigate related properties. We discuss relations between falling neutrosophic subalgebra and falling neutrosophic ideal. We establish a characterization of falling neutrosophic ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [4] and [5]) and was extensively investigated by several researchers.

By a BCI-algebra, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-algebra. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a subalgebra of X, if $x * y \in S$, for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an ideal of X, if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \Rightarrow x \in I)\,. \tag{2.6}$$

We refer the reader to the books [6, 10] for further information regarding BCK/BCI-algebras.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [13]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$\begin{aligned} T_\in(A;\alpha) &:= \{x \in X \mid A_T(x) \geq \alpha\},\\ I_\in(A;\beta) &:= \{x \in X \mid A_I(x) \geq \beta\},\\ F_\in(A;\gamma) &:= \{x \in X \mid A_F(x) \leq \gamma\}. \end{aligned}$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are neutrosophic $\in$-subsets.

A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-neutrosophic subalgebra of X (see [7]) if the following assertions are valid.

$$(2.7)\quad (\forall x, y \in X)\left(\begin{array}{l} x \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y) \ \Rightarrow\ x * y \in T_\in(A;\alpha_x \wedge \alpha_y),\\ x \in I_\in(A;\beta_x),\ y \in I_\in(A;\beta_y) \ \Rightarrow\ x * y \in I_\in(A;\beta_x \wedge \beta_y),\\ x \in F_\in(A;\gamma_x),\ y \in F_\in(A;\gamma_y) \ \Rightarrow\ x * y \in F_\in(A;\gamma_x \vee \gamma_y) \end{array}\right)$$

for all $\alpha_x,\ \alpha_y,\ \beta_x,\ \beta_y \in (0,1]$ and $\gamma_x,\ \gamma_y \in [0,1)$.

A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-neutrosophic ideal of X (see [11]) if the following assertions are valid.

$$(2.8)\qquad (\forall x \in X)\left(\begin{array}{l} x \in T_\in(A;\alpha_x) \ \Rightarrow\ 0 \in T_\in(A;\alpha_x)\\ x \in I_\in(A;\beta_x) \ \Rightarrow\ 0 \in I_\in(A;\beta_x)\\ x \in F_\in(A;\gamma_x) \ \Rightarrow\ 0 \in F_\in(A;\gamma_x) \end{array}\right)$$

and

$$(2.9)\quad (\forall x, y \in X)\left(\begin{array}{l} x * y \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y) \ \Rightarrow\ x \in T_\in(A;\alpha_x \wedge \alpha_y)\\ x * y \in I_\in(A;\beta_x),\ y \in I_\in(A;\beta_y) \ \Rightarrow\ x \in I_\in(A;\beta_x \wedge \beta_y)\\ x * y \in F_\in(A;\gamma_x),\ y \in F_\in(A;\gamma_y) \ \Rightarrow\ x \in F_\in(A;\gamma_x \vee \gamma_y) \end{array}\right)$$

for all $\alpha_x, \alpha_y,\ \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

3. Neutrosophic falling shadows

In what follows, let X and $\mathcal{P}(X)$ denote a BCK/BCI-algebra and the power set of X, respectively, unless otherwise specified.

For each $x \in X$ and $D \in \mathcal{P}(X)$, let

$$(3.1)\qquad \bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\},$$

and

$$(3.2)\qquad \bar{D} := \{\bar{x} \mid x \in D\}.$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be a hyper-measurable structure on X if $\mathcal{B}$ is a σ-field in $\mathcal{P}(X)$ and $\bar{X} \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on X, a neutrosophic random set on X is defined to be a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which ξ_T, ξ_I and ξ_F are mappings from Ω to $\mathcal{P}(X)$ which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(3.3)\qquad (\forall C \in \mathcal{B}) \left(\begin{array}{l} \xi_T^{-1}(C) = \{\omega_T \in \Omega \mid \xi_T(\omega_T) \in C\} \in \mathcal{A} \\ \xi_I^{-1}(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\ \xi_F^{-1}(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A} \end{array} \right).$$

Given a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, consider functions:

$$\begin{aligned} &\tilde{H}_T : X \to [0,1],\ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I : X \to [0,1],\ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F : X \to [0,1],\ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)). \end{aligned}$$

Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a neutrosophic set on X, and we call it a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$, and $\xi := (\xi_T, \xi_I, \xi_F)$ is called a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

For example, consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic set in X. Then a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\begin{aligned} &\xi_T : [0,1] \to \mathcal{P}(X), \alpha \mapsto T_\in(\tilde{H}; \alpha), \\ &\xi_I : [0,1] \to \mathcal{P}(X), \beta \mapsto I_\in(\tilde{H}; \beta), \\ &\xi_F : [0,1] \to \mathcal{P}(X), \gamma \mapsto F_\in(\tilde{H}; \gamma) \end{aligned}$$

is a neutrosophic random set and $\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. We will call $\xi := (\xi_T, \xi_I, \xi_F)$ defined above as the neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

4. Neutrosophic subalgebras/ideals based on neutrosophic falling shadows

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of X for all ω_T, ω_I, $\omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a falling neutrosophic subalgebra (resp., falling neutrosophic ideal) of X.

Example 4.1. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1 (see [10]).

Table 1. Cayley table for the binary operation "$*$"

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X;*,0)$ is a BCK-algebra (see [10]). Let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0,0.3), \\ \{0,2\} & \text{if } t \in [0.3,0.7), \\ \{0,1,2\} & \text{if } t \in [0.7,0.8), \\ X & \text{if } t \in [0.8,1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in [0,0.5), \\ \{0,2\} & \text{if } t \in [0.5,0.7), \\ X & \text{if } t \in [0.7,1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9,1], \\ \{0,1\} & \text{if } t \in (0.7,0.9], \\ \{0,2\} & \text{if } t \in (0.5,0.7], \\ \{0,1,2\} & \text{if } t \in (0.3,0.5], \\ X & \text{if } t \in [0,0.3]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are subalgebras/ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic subalgebra/ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.6 & \text{if } x = 1, \\ 0.7 & \text{if } x = 2, \\ 0.2 & \text{if } x \in \{3,4\}, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = 2, \\ 0.3 & \text{if } x \in \{1,3,4\}, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.3 & \text{if } x \in \{1,2\}, \\ 0.7 & \text{if } x \in \{3,4\}. \end{cases}$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{4.1}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\,((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{4.2}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a BCK/BCI-algebra (see [8]) where θ is given as follows:

$$\theta : \Omega \to X, \ \omega \mapsto 0.$$

For any subset A of X and g_T, g_I, $g_F \in \mathcal{F}(X)$, consider the followings:

$$A_T^g := \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\},$$
$$A_I^g := \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\},$$
$$A_F^g := \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\}$$

and

$$\xi_T : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_T \mapsto \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in A\},$$
$$\xi_I : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_I \mapsto \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in A\},$$
$$\xi_F : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_F \mapsto \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in A\}.$$

Then A_T^g, A_I^g, $A_F^g \in \mathcal{A}$.

Assume that A is a subalgebra (resp., ideal) of X and let ω_T, ω_I, $\omega_F \in \Omega$. Since $\theta(\omega) = 0 \in A$ for $\omega \in \{\omega_T, \omega_I, \omega_F\}$, we know that $\theta \in \xi_T(\omega_T)$, $\theta \in \xi_I(\omega_I)$ and $\theta \in \xi_F(\omega_F)$. For any f_T, $g_T \in \mathcal{F}(X)$, if f_T, $g_T \in \xi_T(\omega_T)$, then

$$(f_T \circledast g_T)(\omega_T) = f_T(\omega_T) * g_T(\omega_T) \in A$$

and so $f_T \circledast g_T \in \xi_T(\omega_T)$. Thus $\xi_T(\omega_T)$ is a subalgebra of $\mathcal{F}(X)$ for all $\omega_T \in \Omega$. If $f_T \circledast g_T \in \xi_T(\omega_T)$ and $g_T \in \xi_T(\omega_T)$, then $f_T(\omega_T) * g_T(\omega_T) = (f_T \circledast g_T)(\omega_T) \in A$ and $g_T(\omega_T) \in A$. Since A is an ideal of X, it follows that $f_T(\omega_T) \in A$, i.e., $f_T \in \xi_T(\omega_T)$. Hence $\xi_T(\omega_T)$ is an ideal of $\mathcal{F}(X)$ for all $\omega_T \in \Omega$. By the similar way, we can verify that $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of $\mathcal{F}(X)$ for all $\omega_I, \omega_F \in \Omega$. Since

$$\xi_T^{-1}(\bar{g}_T) = \{\omega_T \in \Omega \mid g_T \in \xi_T(\omega_T)\} = \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\} = A_T^g \in \mathcal{A},$$
$$\xi_I^{-1}(\bar{g}_I) = \{\omega_I \in \Omega \mid g_I \in \xi_I(\omega_I)\} = \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\} = A_I^g \in \mathcal{A},$$
$$\xi_F^{-1}(\bar{g}_F) = \{\omega_F \in \Omega \mid g_F \in \xi_F(\omega_F)\} = \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\} = A_F^g \in \mathcal{A},$$

$\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic random set on $\mathcal{F}(X)$. Hence $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic subalgebra/ideal of $\mathcal{F}(X)$ where

$$\tilde{H}_T(g_T) = P(\omega_T \mid g_T(\omega_T) \in A),$$
$$\tilde{H}_I(g_I) = P(\omega_I \mid g_I(\omega_I) \in A),$$
$$\tilde{H}_F(g_F) = P(\omega_F \mid g_F(\omega_F) \in A).$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$. For $x \in X$, let

$$\Omega(x;\xi_T) := \{\omega_T \in \Omega \mid x \in \xi_T(\omega_T)\},$$
$$\Omega(x;\xi_I) := \{\omega_I \in \Omega \mid x \in \xi_I(\omega_I)\},$$
$$\Omega(x;\xi_F) := \{\omega_F \in \Omega \mid x \in \xi_F(\omega_F)\}.$$

Then $\Omega(x;\xi_T), \Omega(x;\xi_I), \Omega(x;\xi_F) \in \mathcal{A}$.

Proposition 4.2. *Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic subalgebra of X, then*

$$(\forall x, y \in X) \left(\begin{array}{l} \Omega(x;\xi_T) \cap \Omega(y;\xi_T) \subseteq \Omega(x*y;\xi_T) \\ \Omega(x;\xi_I) \cap \Omega(y;\xi_I) \subseteq \Omega(x*y;\xi_I) \\ \Omega(x;\xi_F) \cap \Omega(y;\xi_F) \subseteq \Omega(x*y;\xi_F) \end{array} \right). \tag{4.3}$$

If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then

$$(\forall x, y \in X) \left(x \leq y \;\Rightarrow \left\{ \begin{array}{l} \Omega(y;\xi_T) \subseteq \Omega(x;\xi_T) \\ \Omega(y;\xi_I) \subseteq \Omega(x;\xi_I) \\ \Omega(y;\xi_F) \subseteq \Omega(x;\xi_F) \end{array} \right. \right), \tag{4.4}$$

$$(\forall x, y \in X) \left(\begin{array}{l} \Omega(x*y;\xi_T) \cap \Omega(y;\xi_T) \subseteq \Omega(x;\xi_T) \\ \Omega(x*y;\xi_I) \cap \Omega(y;\xi_I) \subseteq \Omega(x;\xi_I) \\ \Omega(x*y;\xi_F) \cap \Omega(y;\xi_F) \subseteq \Omega(x;\xi_F) \end{array} \right). \tag{4.5}$$

If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic subalgebra/ideal of a BCK-algebra X, then

$$(\forall x \in X) \left(\begin{array}{l} \Omega(x;\xi_T) \subseteq \Omega(0;\xi_T) \\ \Omega(x;\xi_I) \subseteq \Omega(0;\xi_I) \\ \Omega(x;\xi_F) \subseteq \Omega(0;\xi_F) \end{array} \right). \tag{4.6}$$

If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of a BCK-algebra X, then

$$(\forall x, y \in X) \left(\begin{array}{l} \Omega(x;\xi_T) \subseteq \Omega(x*y;\xi_T) \\ \Omega(x;\xi_I) \subseteq \Omega(x*y;\xi_I) \\ \Omega(x;\xi_F) \subseteq \Omega(x*y;\xi_F) \end{array} \right). \tag{4.7}$$

Proof. Assume that $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic subalgebra of X. If $\omega_T \in \Omega(x;\xi_T) \cap \Omega(y;\xi_T)$ for any $x, y \in X$, then $x \in \xi_T(\omega_T)$ and $y \in \xi_T(\omega_T)$. Since $\xi_T(\omega_T)$ is a subalgebra of X, it follows that $x*y \in \xi_T(\omega_T)$, that is, $\omega_T \in \Omega(x*y;\xi_T)$. Similarly we can verify that if $\omega_I \in \Omega(x;\xi_I) \cap \Omega(y;\xi_I)$ for any $x, y \in X$, then $\omega_I \in \Omega(x*y;\xi_I)$. Now let $\omega_F \in \Omega(x;\xi_F) \cap \Omega(y;\xi_F)$ for any $x, y \in X$. Then $x \in \xi_F(\omega_F)$ and $y \in \xi_F(\omega_F)$, which imply that $x*y \in \xi_F(\omega_F)$ since $\xi_F(\omega_F)$ is a subalgebra of X. Hence $\omega_F \in \Omega(x*y;\xi_F)$, and therefore (4.3) is valid. Suppose that $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X and let $x, y \in X$ be such that $x \leq y$. Then $x*y = 0$. If $\omega_T \in \Omega(y;\xi_T)$, then $y \in \xi_T(\omega_T)$ and $x*y = 0 \in \xi_T(\omega_T)$. Thus $x \in \xi_T(\omega_T)$ since $\xi_T(\omega_T)$ is an ideal of X. Hence $\omega_T \in \Omega(x;\xi_T)$, and so $\Omega(y;\xi_T) \subseteq \Omega(x;\xi_T)$. By the similar way, we know that $\Omega(y;\xi_I) \subseteq \Omega(x;\xi_I)$. Let $\omega_F \in \Omega(y;\xi_F)$. Then $y \in \xi_F(\omega_F)$ and $x*y = 0 \in \xi_F(\omega_F)$, which imply that $x \in \xi_F(\omega_F)$ since $\xi_F(\omega_F)$ is an ideal of X. Hence $\omega_F \in \Omega(x;\xi_F)$ which shows that $\Omega(y;\xi_F) \subseteq \Omega(x;\xi_F)$. If $\omega_I \in \Omega(x*y;\xi_I) \cap \Omega(y;\xi_I)$ for any $x, y \in X$, then $x*y \in \xi_I(\omega_I)$ and $y \in \xi_I(\omega_I)$. Since $\xi_I(\omega_I)$ is an ideal of X, it follows that $x \in \xi_I(\omega_I)$ and so that $\omega_I \in \Omega(x;\xi_I)$. Thus $\Omega(x*y;\xi_I) \cap \Omega(y;\xi_I) \subseteq \Omega(x;\xi_I)$. The inclusions $\Omega(x*y;\xi_T) \cap \Omega(y;\xi_T) \subseteq \Omega(x;\xi_T)$ and $\Omega(x*y;\xi_F) \cap \Omega(y;\xi_F) \subseteq \Omega(x;\xi_F)$ are obtained by the similarly way . Note that $0 \leq x$ and $x*y \leq x$ in a BCK-algebra. Hence the result (4.4) induces (4.6) and (4.7). □

Theorem 4.3. *If we consider a probability space* $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$*, then every* $(\in, \in)$*-neutrosophic subalgebra (resp.,* $(\in, \in)$*-neutrosophic ideal) is a falling neutrosophic subalgebra (resp., falling neutrosophic ideal).*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be an $(\in, \in)$-neutrosophic subalgebra (resp., $(\in, \in)$-neutrosophic ideal) of X. Then $T_\in(\tilde{H};\alpha)$, $I_\in(\tilde{H};\beta)$ and $F_\in(\tilde{H};\gamma)$ are subalgebras (resp., ideals) of X for all α, β, $\gamma \in [0,1]$. Hence a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\xi_T : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto T_\in(\tilde{H};\alpha),$$
$$\xi_I : [0,1] \to \mathcal{P}(X),\ \beta \mapsto I_\in(\tilde{H};\beta),$$
$$\xi_F : [0,1] \to \mathcal{P}(X),\ \gamma \mapsto F_\in(\tilde{H};\gamma)$$

is a neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$, and so $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic subalgebra (resp., neutrosophic ideal) of X. □

The converse of Theorem 4.3 is not true as seen in the following example.

Example 4.4. Consider a set $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 2.

TABLE 2. Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X; *, 0)$ is a BCI-algebra (see [10]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in [0, 0.2), \\ \{0, a\} & \text{if } t \in [0.2, 0.7), \\ \{0, b\} & \text{if } t \in [0.7, 0.8), \\ X & \text{if } t \in [0.8, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0, a\} & \text{if } t \in [0, 0.3), \\ \{0, b\} & \text{if } t \in [0.4, 0.6), \\ \{0, c\} & \text{if } t \in [0.6, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.8, 1], \\ \{0, a\} & \text{if } t \in (0.7, 0.8], \\ \{0, b\} & \text{if } t \in (0.5, 0.7], \\ \{0, c\} & \text{if } t \in [0, 0.5]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are subalgebras/ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic subalgebra/ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.7 & \text{if } x = a, \\ 0.3 & \text{if } x = b, \\ 0.2 & \text{if } x = c, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.3 & \text{if } x = a, \\ 0.2 & \text{if } x = b, \\ 0.4 & \text{if } x = c, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.9 & \text{if } x = a, \\ 0.8 & \text{if } x = b, \\ 0.5 & \text{if } x = c. \end{cases}$$

Note that $a \in F_\in(\tilde{H}; 0.2)$ and $b \in F_\in(\tilde{H}; 0.3)$, but $a * b = c \notin F_\in(\tilde{H}; 0.3)$. Also $a \in I_\in(\tilde{H}; 0.25)$ and $b * a = c \in I_\in(\tilde{H}; 0.35)$ but $b \notin I_\in(\tilde{H}; 0.25)$. Thus $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not an $(\in, \in)$-neutrosophic subalgebra/ideal of X.

Theorem 4.5. *If we consider a probability space* $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$*, then every falling neutrosophic ideal is a falling neutrosophic subalgebra in a* BCK*-algebra.*

Proof. Since every ideal is a subalgebra in a BCK-algebra, it is straightforward. □

The following example shows that Theorem 4.5 is not true in a BCI-algebra.

Example 4.6. Let X be the set of all nonzero rational numbers. If we take a binary operation $*$ on X defined by division as general, then $(X; *, 1)$ is a BCI-algebra (see [2]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} X & \text{if } t \in (0.6, 1], \\ \mathbb{Z}^* & \text{if } t \in [0, 0.6], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} X & \text{if } t \in [0.3, 1], \\ \mathbb{Z}^* & \text{if } t \in [0, 0.3), \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} X & \text{if } t \in [0, 0.7), \\ \mathbb{Z}^* & \text{if } t \in [0.7, 1], \end{cases}$$

where $\mathbb{Z}^*$ is the set of all nonzero integers. Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic ideal of X, but it is not a falling neutrosophic subalgebra of X because $\xi_T(0.5) = \mathbb{Z}^*$, $\xi_I(0.2) = \mathbb{Z}^*$ and/or $\xi_F(0.8) = \mathbb{Z}^*$ are not subalgebras of X since $2 \in \mathbb{Z}^*$ and $3 \in \mathbb{Z}^*$ but $2*3 \notin \mathbb{Z}^*$.

We provide conditions for a falling neutrosophic subalgebra to be a falling neutrosophic ideal in BCI-algebras.

Theorem 4.7. *Given a BCI-algebra X, assume that the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic subalgebra of X. Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X if and only if for every $x, y \in X$ and ω_T, ω_I, $\omega_F \in \Omega$, the following is valid:*

$$\begin{array}{l} x \in \xi_T(\omega_T),\ y \notin \xi_T(\omega_T) \ \Rightarrow\ y * x \notin \xi_T(\omega_T), \\ x \in \xi_I(\omega_I),\ y \notin \xi_T(\omega_I) \ \Rightarrow\ y * x \notin \xi_I(\omega_I), \\ x \in \xi_F(\omega_F),\ y \notin \xi_T(\omega_F) \ \Rightarrow\ y * x \notin \xi_F(\omega_F). \end{array} \tag{4.8}$$

Proof. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are ideals of X for all ω_T, ω_I, $\omega_F \in \Omega$. Let $x, y \in X$ be such that $x \in \xi_T(\omega_T)$ and $y \notin \xi_T(\omega_T)$. If $y * x \in \xi_T(\omega_T)$, then $y \in \xi_T(\omega_T)$ since $\xi_T(\omega_T)$ is an ideal of X. Hence $y * x \notin \xi_T(\omega_T)$. Similarly, if $x \in \xi_I(\omega_I)$ and $y \notin \xi_I(\omega_I)$ (resp., $x \in \xi_F(\omega_F)$ and $y \notin \xi_F(\omega_F)$), then $y * x \notin \xi_I(\omega_I)$ (resp., $y * x \notin \xi_F(\omega_F)$).

Conversely, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a falling neutrosophic subalgebra of X that satisfies the condition (4.8). Then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras of X for all ω_T, ω_I, $\omega_F \in \Omega$. Hence 0 is contained in $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$. Let x, y, a, b, c, $d \in X$ be such that $x * y \in \xi_T(\omega_T)$, $y \in \xi_T(\omega_T)$, $a * b \in \xi_I(\omega_I)$, $b \in \xi_I(\omega_I)$, $c * d \in \xi_F(\omega_F)$ and $d \in \xi_F(\omega_F)$. If $x \notin \xi_T(\omega_T)$ (resp., $a \notin \xi_I(\omega_I)$ and $c \notin \xi_T(\omega_T)$), then $x * y \notin \xi_T(\omega_T)$ (resp., $a * b \notin \xi_I(\omega_I)$ and $c * d \notin \xi_F(\omega_F)$) by (4.8). This is a contradiction, and therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X. □

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