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A characterization of generalized cyclic group

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Abstract. This paper demonstrates that a generalized cyclic group can be characterized in terms of the distributivity of its L-subgroup lattice wherein the join structure of a pair of L-subgroups is formulated with the help of the notion of tip extended pair of L-subgroups. Also using this join structure, the modularity of the lattice of normal L-subgroups of a group G is established. In the last section, we establish our main theorems by an application of subdirect product theorems.

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1. Introduction

The relationship between the properties of subgroup lattices and the properties of corresponding underlying groups has drawn attention of many researchers in the past. It is found that groups having isomorphic subgroup lattices have many properties in common. For example, it is well known that, all groups whose subgroup lattices are isomorphic with subgroup lattices of Abelian groups are necessarily metaabelian. Moreover, certain groups such as cyclic groups of prime order and generalized cyclic groups are characterized in terms of properties of their subgroup lattices. Here we carry over such studies to the lattices of L-subgroups.

The construction of various types of lattices and sublattices of fuzzy subgroups in a systematic and organized way was initiated by N. Ajmal and K. V. Thomas. Modularity of the lattice of all fuzzy normal subgroups was established in a systematic and stepwise manner in the papers [3, 4, 5, 6]. The join structure of two fuzzy algebraic substructures with identical tips plays a key role in the development of most of the fuzzy algebraic substructures. The join structure of two fuzzy subgroups with identical tips in terms of their set product was formulated by N. Ajmal [1]. However, when the tips differ even then the set product of the tip extended pair μ^{t_η} and η^{t_μ} of fuzzy subgroups μ and η provides the join of μ and η as demonstrated by T. Head in his erratum [17]. A similar technique is used by I. Jahan [18] for constructing the join of two L-ideals of a ring. Moreover, N. Ajmal and K. V. Thomas modified, simplified and utilized the construction of the generated fuzzy subalgebras in groups, rings and lattices in a series of papers [3, 4, 5, 6, 7, 8, 9, 10, 11]. The most economical construction of a generated fuzzy subgroup appeared so far, is due to N. Sultana and N. Ajmal [21] and later by N. Ajmal and A. Jain [12]. These results have been extended to L-setting by N. Ajmal and I. Jahan [13]. In this paper, firstly, we formulate the join of a pair of L-subgroups by using the notion of tip extended pair of L-subgroups, then we characterize generalized cyclic groups in terms of the distributivity of its L-subgroup lattice. Also, we use the construction of an L-subgroup generated by an L-subset by N. Ajmal and I. Jahan [13] to develop the join of a pair of L-subgroups and the apply it to the characterization of generalized cyclic groups in terms of distributivity of its L-subgroups lattice. As an immediate corollary to our main theorem (Theorem 3.3), we obtain that a group is generalized cyclic if and only if its lattice of fuzzy subgroups is distributive. This result puts into shape the main result (Theorem 4.3) of the paper [23]. Moreover, we establish that the lattice of normal L-subgroups of G is modular. In the last section, it has been demonstrated that these two main theorems can be obtained as simple corollaries to Tom Head's subdirect product theorem [16] and a subdirect product theorem established in [2].

2. Preliminaries

We recall here some basic concepts and results. For details, we refer to [14, 15, 16, 19, 20, 22, 24].

In our work, the system $\langle L, \leq, \vee, \wedge \rangle$ denotes a completely distributive lattice, where '$\leq$' denotes the partial ordering of L, the join (sup) and the meet (inf) of the elements of L are denoted by '$\vee$' and '$\wedge$', respectively. Also, we write 1 and 0 for the maximal and the minimal elements of L, respectively. The definition of a completely distributive lattice is well known in literature and can be found in any standard text on the subject. We recall the following:

Let $\{J_i : i \in I\}$ be any family of subsets of a complete lattice L and let F denotes the set of choice functions for J_i, i.e. functions $f : I \to \prod_{i \in I} J_i$ such that $f(i) \in J_i$ for each i. Then, we say that L is a completely distributive lattice, if

$$\bigwedge_{i \in I} \left\{ \bigvee J_i \right\} = \bigvee_{f \in F} \left\{ \bigwedge_{i \in I} f(i) \right\}.$$

The above law is known as the complete distributive law. Moreover, a lattice L is said to be infinitely meet distributive, if for every subset $\{b_\beta : \beta \in B\}$ of L, we have:

$$a \wedge \left\{ \bigvee_{\beta \in B} b_\beta \right\} = \bigvee_{\beta \in B} \{a \wedge b_\beta\},$$

provided L is join complete. The above law is known as the infinitely meet distributive law. The definition of infinitely join distributive lattice is dual to the above definition, i.e., a lattice L is said to be infinitely join distributive, if for every subset $\{b_\beta : \beta \in B\}$ of L, we have:

$$a \vee \left\{ \bigwedge_{\beta \in B} b_\beta \right\} = \bigwedge_{\beta \in B} \{a \vee b_\beta\},$$

provided L is meet complete. The above law is known as the infinitely join distributive law.

Clearly, both these laws follow from the definition of a completely distributive lattice. Here we also mention that the dual of complete distributive law is valid in a completely distributive lattice whereas the infinitely meet and join distributive laws are independent from each other.

A complete lattice which satisfies infinitely meet distributive law is known as a complete Heyting algebra or a frame.

Note that a completely distributive lattice is always a complete Heyting algebra.

An L-subset of a non empty set X is a function from X into L. The set of L-subsets of X is called the L-power set of X and is denoted by L^X. For $\mu \in L^X$ the set $\{\mu(x) : x \in X\}$, denoted by $Im\mu$ is called the image of μ and the tip of μ, denoted by $\sup \mu$, is defined as $\bigvee_{x \in X} \mu(x)$. We say that an L-subset μ of X is contained in an L-subset ν of X if $\mu(x) \leq \nu(x)$ for every $x \in X$ and is denoted by $\mu \subseteq \nu$. For a family $\{\mu_i : i \in I\}$ of L-subsets of X, where I is a non empty index set, the union $\bigcup_{i \in I} \mu_i$ and the intersection $\bigcap_{i \in I} \mu_i$ of $\{\mu_i : i \in I\}$ are, respectively, defined by :

$$\bigcup_{i \in I} \mu_i(x) = \bigvee_{i \in I} \mu_i(x) \text{ and } \bigcap_{i \in I} \mu_i(x) = \bigwedge_{i \in I} \mu_i(x),$$

for each $x \in X$. If $\mu \in L^X$ and $a \in L$, then the level subset μ_a of μ is defined by:

$$\mu_a = \{x \in X : \mu(x) \geq a\}.$$

Proposition 2.1. *Let $\eta, \theta \in L^X$.*
(1) *If $a \leq b$, then $\eta_b \subseteq \eta_a$.*
(2) *$(\eta \cup \theta)_a = \eta_a \cup \theta_a$, for each $a \in L$, provided L is a chain.*
(3) *$(\eta \cap \theta)_a = \eta_a \cap \theta_a$, for each $a \in L$.*

The set product $\mu \circ \nu$ of $\mu, \nu \in L^S$, where S is a groupoid, is an L-subset of S defined by:

$$\mu \circ \nu(x) = \bigvee_{x = yz} \{\mu(y) \wedge \nu(z)\}.$$

Recall that if x cannot be factored as $x = yz$ in S, then $\mu \circ \nu(x)$, being the least upper bound of the empty set, is zero.

It is well known that the L-power set L^X constitutes a completely distributive lattice under the ordering of L-set inclusion "$\subseteq$" for a completely distributive lattice L. The join '$\vee$' and the meet '$\wedge$' of an arbitrary family of L-subsets $\{\mu_i : i \in I\}$ of X are $\bigcup_{i \in I} \mu_i$ and $\bigcap_{i \in I} \mu_i$, respectively. The least and the greatest elements of the lattice L^X are 0_X and 1_X, respectively. Here 0_X and 1_X are L-subsets of X which map each element of X to 0 and 1, respectively. Moreover, the lattice L can be isomorphically embedded into the lattice L^X.

Throughout this paper G denotes a group with the identity element 'e'.

Definition 2.2. Let $\mu \in L^G$. Then μ is called an L-subgroup of G, if
(i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$,
(ii) $\mu(x^{-1}) = \mu(x)$, for each $x, y \in G$.

The set of L-subgroups of G is denoted by $L(G)$. Clearly, the tip of an L-subgroup is attained at the identity element of G.

Theorem 2.3. *Let $\mu \in L^G$ with tip a_0. Then $\mu \in L(G)$ if and only if each non empty level subset μ_a is a subgroup of G.*

Liu and Wu [19, 24] introduced the notion of a fuzzy normal subgroup of a group during 1981-82.

Definition 2.4. Let $\mu \in L(G)$. Then μ is called a normal L-subgroup of G if for all $x, y \in G$, $\mu(xy) = \mu(yx)$.

The set of normal L-subgroups of G is denoted by $NL(G)$.

Theorem 2.5. *Let $\mu \in L(G)$. Then $\mu \in NL(G)$ if and only if each non empty level subset μ_a is a normal subgroup of G.*

It is well known that the intersection of any arbitrary family of L-subgroups of a group is an L-subgroup of the given group.

Definition 2.6. Let $\mu \in L^G$. Then the L-subgroup of G generated by μ is defined as the smallest L-subgroup of G which contains μ. It is denoted by $\langle \mu \rangle$, i.e.,

$$\langle \mu \rangle = \bigcap \{\nu \in L(G) : \mu \subseteq \nu\}.$$

Further, we recall the definition of a tip extended pair of L-subgroups.

Definition 2.7. Let $\theta, \phi \in L(G)$ and $t_\phi = t_\theta = \theta(e) \vee \phi(e)$. Define L-subsets θ^{t_ϕ} and ϕ^{t_θ} of G as follows:

$$\theta^{t_\varphi}(x) = \theta(x) \text{ and } \phi^{t_\theta}(x) = \phi(x) \text{ for all } x \neq e,$$

and

$$\theta^{t_\phi}(e) = \phi^{t_\theta}(e) = \theta(e) \vee \phi(e).$$

Clearly, $\theta \subseteq \theta^{t_\phi}$ and $\phi \subseteq \phi^{t_\theta}$. Also, $\theta^{t_\phi}, \phi^{t_\theta} \in L(G)$ and the pair θ^{t_ϕ}, ϕ^{t_θ} is known as the tip-extended pair of L-subgroups θ and ϕ.

The following result is immediate:

Theorem 2.8. *Let $\theta, \phi \in L(G)$ and $\theta \circ \phi \in L(G)$. Then $\theta^{t_\phi} \circ \phi^{t_\theta}$ is an L-subgroup of G and*

$$\theta^{t_\phi} \circ \phi^{t_\theta} = \langle \theta \cup \phi \rangle,$$

where $\langle \theta \cup \phi \rangle$ is an L-subgroup of G generated by the union $\theta \cup \phi$. Moreover, if $\theta, \phi \in NL(G)$, then their tip extended pair θ^{t_ϕ}, $\phi^{t_\theta} \in NL(G)$.

Theorem 2.9. *Let $\theta, \phi \in NL(G)$. Then*
(1) $\theta \circ \phi \in NL(G)$,
(2) $\theta^{t_\phi} \circ \phi^{t_\theta} \in NL(G)$.

Here we recall the following construction of an L-subgroup generated by an L-subset of G from [13].

Theorem 2.10. *Let $\eta \in L^G$. Define the following L-subset $\hat{\eta}$ of G by:*

$$\hat{\eta}(x) = \bigvee \{a : x \in \langle \eta_a \rangle \text{ and } a \leq \sup \eta\}.$$

Then, $\hat{\eta} \in L(G)$ and $\hat{\eta} = \langle \eta \rangle$. Moreover, $\langle \eta \rangle(e) = \sup \eta$.

3. Distributivity of the lattice of L-subgroups

The set $L(G)$ of L-subgroups of a group G forms a lattice under the L-set inclusion where the join and the meet operations for a family $\{\eta_i\}_{i \in I}$ of L-subgroups of G are defined as follows:

$$\bigvee_{i \in I} \eta_i = \left\langle \bigcup_{i \in I} \eta_i \right\rangle \text{ and } \bigwedge_{i \in I} \eta_i = \bigcap_{i \in I} \eta_i \,.$$

Here $\left\langle \bigcup_{i \in I} \eta_i \right\rangle$ is the L-subgroup of G generated by the union of the family $\{\eta_i : i \in I\}$. Under these operations the lattice $L(G)$ forms a complete lattice and its subset $NL(G)$ of normal L-subgroups constitutes a complete modular sublattice.

In classical algebra, a generalized cyclic group can be characterized by the distributivity of its lattice of subgroups. Here we demonstrate that such groups can also be characterized in terms of the distributivity of its lattice of L-subgroups.

Definition 3.1 ([15])**.** A group G is said to be generalized cyclic if for any two elements $a, b \in G$, there exists an element $c \in G$ such that

$$a = c^m \text{ and } b = c^n \text{ for some integers } m \text{ and } n.$$

The group of rational numbers under addition is an example of a generalized cyclic group which is not cyclic. It is worthwile to mention here that a generalized cyclic group G is an Abelian group. Moreover for its subgroup lattice $\mathcal{L}(G)$, the join and the meet of $H, K \in \mathcal{L}(G)$ are defined by :

$$H \vee K = HK \quad \text{and} \quad H \wedge K = H \cap K.$$

The following result is well known and can be found in Birkhoff [15].

Theorem 3.2. *A group G is generalized cyclic if and only if its subgroup lattice $\mathcal{L}(G)$ is distributive.*

Let us denote by $L(G)$ the lattice of L-subgroups of G. Then we have the following:

Theorem 3.3. *If a group G is generalized cyclic, then the lattice $L(G)$ is distributive.*

Proof. Suppose that G is a generalized cyclic. Since the distributive inequality holds in every lattice, it is sufficient to establish that if θ, ϕ and $\chi \in L(G)$, then

$$\theta \wedge (\phi \vee \chi) \subseteq (\theta \wedge \phi) \vee (\theta \wedge \chi).$$

In view of Theorem 2.8, we have

$$\phi \vee \chi = \phi^{t_\chi} \circ \chi^{t_\phi}$$

and

$$(\theta \wedge \phi) \vee (\theta \wedge \chi) = (\theta \wedge \phi)^{t_{\theta \wedge \chi}} \circ (\theta \wedge \chi)^{t_{\theta \wedge \phi}}.$$

Thus we shall prove

$$\theta \wedge (\phi^{t_\chi} \circ \chi^{t_\phi}) \subseteq (\theta \wedge \phi)^{t_{\theta \wedge \chi}} \circ (\theta \wedge \chi)^{t_{\theta \wedge \phi}}.$$

Let $x \in G$ and define the following subset of $G \times G$:

$$D(x) = \{(u, v) \in G \times G : x = uv\}.$$

Now, consider

$$\begin{aligned}(\theta \wedge (\phi^{t_\chi} \circ \chi^{t_\phi}))(x) &= \theta(x) \wedge (\phi^{t_\chi} \circ \chi^{t_\phi})(x) \\ &= \theta(x) \wedge \left\{ \bigvee_{(u,v)\in D(x)} (\phi^{t_\chi}(u) \wedge \chi^{t_\phi}(v)) \right\} \\ &= \bigvee_{(u,v)\in D(x)} (\theta(x) \wedge \phi^{t_\chi}(u) \wedge \chi^{t_\phi}(v)) \\ &\qquad \text{(since } L \text{ is a complete Heyting algebra)} \\ &= \bigvee_{(u,v)\in D(x)} a_{u,v},\end{aligned}$$

where $a_{u,v} = \theta(x) \wedge \phi^{t_\chi}(u) \wedge \chi^{t_\phi}(v)$ for $(u, v) \in D(x)$. Note that $x \in \theta_{a_{u,v}}$ and $x = uv \in \phi^{t_\chi}_{a_{u,v}} \chi^{t_\phi}_{a_{u,v}}$, which implies

$$x \in \theta_{a_{u,v}} \cap \phi^{t_\chi}_{a_{u,v}} \chi^{t_\phi}_{a_{u,v}}.$$

Since θ, ϕ and $\chi \in L(G)$, the tip extended pair ϕ^{t_χ}, $\chi^{t_\phi} \in L(G)$. By Theorem 2.3, the level subsets $\theta_{a_{u,v}}$, $\phi^{t_\chi}_{a_{u,v}}$ and χ^{t_ϕ} are subgroups of G. Since G is a generalized cyclic group, by Theorem 3.2, its subgroup lattice is distributive. Consequently,

$$\theta_{a_{u,v}} \cap \phi^{t_\chi}_{a_{u,v}} \chi^{t_\phi}_{a_{u,v}} = (\theta_{a_{u,v}} \cap \phi^{t_\chi}_{a_{u,v}})(\theta_{a_{u,v}} \cap \chi^{t_\phi}_{a_{u,v}}).$$

Then

$$x \in (\theta_{a_{u,v}} \cap \phi^{t_\chi}_{a_{u,v}})(\theta_{a_{u,v}} \cap \chi^{t_\phi}_{a_{u,v}}).$$

Thus $x = yz$, where $y \in \theta_{a_{u,v}} \cap \phi^{t_\chi}_{a_{u,v}}$ and $z \in \theta_{a_{u,v}} \cap \chi^{t_\phi}_{a_{u,v}}$. This implies

$$(\theta \wedge \phi^{t_\chi})(y) \geq a_{u,v} \text{ and } (\theta \wedge \chi^{t_\phi})(\mathrm{z}) \geq a_{u,v}.$$

So

$$a_{u,v} \leq \left(\theta \wedge \phi^{t_\chi}\right)(\mathrm{y}) \wedge \left(\theta \wedge \chi^{t_\phi}\right)(\mathrm{z}).$$

Consequently,

$$\begin{aligned}(\theta \wedge (\phi^{t_\chi} \circ \chi^{t_\phi}))(x) &= \bigvee_{(u,v)\in D(x)} a_{u,v} \\ &\leq \bigvee_{(r,s)\in D(x)} \left\{ \left(\theta \wedge \phi^{t_\chi}\right)(\mathrm{r}) \wedge \left(\theta \wedge \chi^{t_\phi}\right)(\mathrm{s}) \right\} \\ &= (\theta \wedge \phi^{t_\chi}) \circ (\theta \wedge \chi^{t_\phi})(x).\end{aligned}$$

Hence

$$(\theta \wedge (\phi^{t_\chi} \circ \chi^{t_\phi})) \subseteq (\theta \wedge \phi^{t_\chi}) \circ (\theta \wedge \chi^{t_\phi}).$$

In view of the definition of a tip extended pair of L-subgroup, the following is easy to verify:

$$\theta \wedge \phi^{t_\chi} = (\theta \wedge \phi)^{t_{\theta\wedge\chi}} \quad \text{and} \quad \theta \wedge \chi^{t_\phi} = (\theta \wedge \chi)^{t_{\theta\wedge\phi}}.$$

Therefore

$$\theta \wedge (\phi^{t_\chi} \circ \chi^{t_\phi}) \subseteq (\theta \wedge \phi)^{t_{\theta\wedge\chi}} \circ (\theta \wedge \chi)^{t_{\theta\wedge\phi}}.$$

This completes the proof.

Below, an alternative proof of the above theorem is provided without using the notion of tip extended pair of L-subgroups. However, in this proof the lattice L is a chain and hence the theorem looses its generality.

Alternative proof of the above theorem.

Let θ, ϕ and $\chi \in L(G)$. Then we show that

$$\theta \wedge (\phi \vee \chi) \subseteq (\theta \wedge \phi) \vee (\theta \wedge \chi).$$

Note that the tip of the L-subset $\phi \cup \chi$ is $\phi(e) \vee \chi(e)$. By Theorem 2.10,

$$\phi \vee \chi(x) = \bigvee_{a \leq \phi(e)\vee\chi(e)} \{a : x \in \langle(\phi \cup \chi)_a\rangle\}.$$

Now, consider

$$\begin{aligned}
\theta \wedge (\phi \vee \chi)(x) &= \theta(x) \wedge \left\{ \bigvee_{a \leq \phi(e)\vee\chi(e)} \{a : x \in \langle(\phi \cup \chi)_a\rangle\} \right\} \\
&= \bigvee_{a \leq \phi(e)\vee\chi(e)} \{\theta(x) \wedge a : x \in \langle \phi_a \cup \chi_a \rangle\} \\
&\qquad \text{(using Proposition 2.1, since } L \text{ is a chain)} \\
&= \bigvee_{c \in S} c,
\end{aligned}$$

where $S = \{\theta(x) \wedge a : a \leq \phi(e) \vee \chi(e) \text{ and } x \in \langle \phi_a \cup \chi_a \rangle\}$. Note that the tip of the L-subset $(\theta \cap \phi) \cup (\theta \cap \chi)$ is $\theta(e) \wedge \{\phi(e) \vee \chi(e)\}$. Further, define the following subset of L:

$$T = \{b : b \leq \theta(e) \wedge (\phi(e) \vee \chi(e)) \text{ and } x \in \langle \theta_b \cap (\phi_b \cup \chi_b)\rangle\}.$$

We claim that $S \subseteq T$. For this, let $c \in S$. Then

$$c = \theta(x) \wedge a \quad \text{where} \quad a \leq \phi(e) \vee \chi(e) \text{ and } x \in \langle \phi_a \cup \chi_a \rangle. \tag{3.1}$$

Note that as $c \leq a \leq \phi(e) \vee \chi(e)$, the level subset $(\phi \cup \chi)_c = \phi_c \cup \chi_c$ is non empty. If either ϕ_c or χ_c is empty, then cleary $S \subseteq T$. So we assume that $\phi_c \neq \phi \neq \chi_c$. Hence by Theorem 2.3, ϕ_c and χ_c are subgroups of G. As G is a generalized cyclic group, $\langle \phi_c \cup \chi_c \rangle = \phi_c \chi_c$. Also, the subgroup lattice of a generalized cyclic group is distributive, we have

$$\theta_c \cap (\phi_c \chi_c) = (\theta_c \cap \phi_c)(\theta_c \cap \chi_c).$$

At this point, by (3.1), note that $x \in \theta_c$ and $x \in \langle \phi_c \cup \chi_c \rangle = \phi_c\chi_c$ so that $x \in \theta_c \cap (\phi_c\chi_c)$. This implies $x \in (\theta_c \cap \phi_c)(\theta_c \cap \chi_c)$. But

$$(\theta_c \cap \phi_c)(\theta_c \cap \chi_c) = \langle (\theta_c \cap \phi_c) \cup (\theta_c \cap \chi_c) \rangle = \langle \theta_c \cap (\phi_c \cup \chi_c) \rangle.$$

Again by (3.1) and the above equation, it follows that $S \subseteq T$. Now, by Proposition 2.1, $\theta_c \cap (\phi_c \cup \chi_c) = (\theta \cap (\phi \cup \chi))_c$ so that we have

$$T = \{b : b \leq \theta(e) \wedge (\phi(e) \vee \chi(e)) \text{ and } x \in \langle (\theta \cap (\phi \cup \chi))_b \rangle \}.$$

Consequently,

$$\theta \wedge (\phi \vee \chi)(x) = \bigvee_{c \in S} c \leq \bigvee_{b \in T} b = (\theta \cap \phi) \vee (\theta \cap \chi)(x). \text{ (By Theorem 2.10)}$$

This proves the lattice of L-subgroup of a generalized cyclic group is distributive. □

In the reverse direction, we have

Theorem 3.4. *Let G be a group. Then the subgroup lattice $\mathcal{L}(G)$ is distributive if the lattice $L(G)$ is distributive.*

Proof. Let $L_c(G)$ be the set of characteristics functions of the members of $\mathcal{L}(G)$. Then it is easy to verify that $L_c(G)$ is a sublattice of $L(G)$. Moreover, the mapping Chi from $\mathcal{L}(G)$ into $L(G)$ defined by

$$Chi : A \to 1_A, \quad A \in \mathcal{L}(G),$$

where 1_A is the characteristic function of A, is a lattice monomorphism such that the image of $\mathcal{L}(G)$ under Chi is $L_c(G)$. Now $L_c(G)$ being a sublattice of a distributive lattice $L(G)$ is a distributive lattice. Consequently the lattice $\mathcal{L}(G)$ being isomorphic to $L_c(G)$ is also distributive. □

Thus in view of Theorems 3.3 and 3.4, we obtain the following:

Theorem 3.5. *A group G is generalized cyclic if and only if the lattice $L(G)$ is distributive.*

If $L = [0, 1]$, then as a corollary we obtain the following:

Corollary 3.6. *A group G is generalized cyclic if and only if the lattice of fuzzy subgroups of G is distributive.*

4. Modularity

This section deals with the modularity of the lattice of normal L-subgroups. Notice that a pair of normal L-subgroups θ and ϕ of G, $\theta \circ \phi$ is a normal L-subgroup of G generated by the union $\theta \cup \phi$ if and only if θ and ϕ have the same tips (see [1]) whereas $\theta^{t_\phi} \circ \phi^{t_\theta}$ is a normal L-subgroup of G generated by the union $\theta \cup \phi$ (see Theorem 2.8).

Theorem 4.1. *The set $NL(G)$ of normal L-subgroups of G is a lattice under the ordering of L-set inclusion, where the join '$\vee$' and the meet '$\wedge$' in $NL(G)$ are defined as follows:*

$$\theta \vee \phi = \theta^{t_\phi} \circ \phi^{t_\theta}, \text{ and } \theta \wedge \phi = \theta \cap \phi.$$

Theorem 4.2. *The lattice $NL(G)$ of normal L-subgroups of G is modular.*

Proof. Since the modular inequality holds in every lattice, it is sufficient to establish that if θ, ϕ and $\chi \in NL(G)$ and $\theta \supseteq \phi$, then

$$\theta \wedge (\phi \vee \chi) \subseteq \phi \vee (\theta \wedge \chi). \tag{4.1}$$

In view of Theorem 2.8, we have

$$\phi \vee \chi = \phi^{t_\chi} \circ \chi^{t_\phi} \quad \text{and} \quad \phi \vee (\theta \wedge \chi) = \phi^{t_{\theta\wedge\chi}} \circ (\theta \wedge \chi)^{t_\phi}.$$

Thus we shall prove

$$\theta \wedge (\phi^{t_\chi} \circ \chi^{t_\phi}) \subseteq \phi^{t_{\theta\wedge\chi}} \circ (\theta \wedge \chi)^{t_\phi}.$$

Let $x \in G$. suppose $x = e$. Then by using the definition of a tip extended pair of L-subgroups, we get

$$\begin{aligned} \theta \wedge (\phi^{t_\chi} \circ \chi^{t_\phi})(e) &= \theta(e) \wedge \{(\phi(e) \vee \chi(e)) \wedge (\phi(e) \vee \chi(e))\} \\ &= \theta(e) \wedge (\phi(e) \vee \chi(e)) \\ &= \phi(e) \vee (\theta(e) \wedge \chi(e)) \quad (\text{as } L \text{ is distributive and } \theta \supseteq \phi) \\ &= \phi(e) \vee (\theta \wedge \chi)(e) \\ &= (\phi^{t_{\theta\wedge\chi}} \circ (\theta \wedge \chi)^{t_\phi})(e). \end{aligned}$$

Thus (4.1) holds.

Suppose $x \neq e$. Further, observe the following:

As $\chi^{t_\phi}(e) = \phi^{t_\chi}(e) = \phi(e) \vee \chi(e)$, we have

$$\phi^{t_\chi}(x) \wedge \chi^{t_\phi}(e) = \phi^{t_\chi}(x) = \phi(x) \tag{4.2}$$

and

$$\phi^{t_\chi}(e) \wedge \chi^{t_\phi}(x) = \chi^{t_\phi}(x) = \chi(x). \tag{4.3}$$

Also $\theta \supseteq \phi$. This implies that

$$\theta(x) \wedge \phi(x) = \phi(x). \tag{4.4}$$

Next if $x = uv$, then as θ is an L-subgroup, we get

$$\theta(x) \wedge \theta(u) \leq \theta(u^{-1}x) = \theta(v). \tag{4.5}$$

Since the tip of an L-subgroup is attained at the identity element $'e'$, it follows that

$$(\theta \wedge \chi)(x) \vee \phi(x) \leq (\theta \wedge \chi)(e) \vee \phi(e).$$

By the definition of a tip-extended pair $\phi^{t_{\theta\wedge\chi}}$ and $(\theta \wedge \chi)^{t_\phi}$, it follows that

$$\phi^{t_{\theta\wedge\chi}}(e) = (\theta \wedge \chi)(e) \vee \phi(e) = (\theta \wedge \chi)^{t_\phi}(e). \tag{4.6}$$

Consequently,

$$\phi(x) \vee (\theta \wedge \chi)(x) \leq \phi^{t_{\theta\wedge\chi}}(e) = (\theta \wedge \chi)^{t_\phi}(e),$$

which implies

(4.7)

$$\phi(x) \vee (\theta \wedge \chi)(x) = \phi^{t_{\theta\wedge\chi}}(e) \wedge (\phi(x) \vee (\theta \wedge \chi)(x)) = \phi^{t_{\theta\wedge\chi}}(e) \wedge (\phi^{t_{\theta\wedge\chi}}(x) \vee (\theta \wedge \chi)^{t_\phi}(x)).$$

Define the following subset of $G \times G$:

$$D(x) = \{(u, v) \in G \times G : x = uv\}.$$

Then consider the left hand side of the inequality (4.1)

$$
\begin{aligned}
(\theta \wedge (\phi^{t_\chi} \circ \chi^{t_\phi}))(x) &= \theta(x) \wedge (\phi^{t_\chi} \circ \chi^{t_\phi})(x) \\
&= \theta(x) \wedge \left(\bigvee_{(u,v)\in D(x)} (\phi^{t_\chi}(u) \wedge \chi^{t_\phi}(v)) \right) \\
&= \bigvee_{(u,v)\in D(x)} (\theta(x) \wedge \phi^{t_\chi}(u) \wedge \chi^{t_\phi}(v)) \\
&\qquad\qquad \text{(since } L \text{ is a complete Heyting algebra)} \\
&= \left(\bigvee_{\substack{(u,v)\in D(x)\\ u\neq e, v\neq e}} (\theta(x) \wedge \phi^{t_\chi}(u) \wedge \chi^{t_\phi}(v)) \right) \\
&\quad \bigvee \left(\theta(x) \wedge \phi^{t_\chi}(x) \wedge \chi^{t_\phi}(e)\right) \bigvee \left(\theta(x) \wedge \phi^{t_\chi}(e) \wedge \chi^{t_\phi}(x)\right) \\
&= \left(\bigvee_{\substack{(u,v)\in D(x)\\ u\neq e, v\neq e}} (\theta(x) \wedge \phi(u) \wedge \chi(v)) \right) \\
&\quad \bigvee ((\theta(x) \wedge \phi(x)) \vee (\theta(x) \wedge \chi(x))) \quad \text{(by (4.2) and (4.3))} \\
&= \left(\bigvee_{\substack{(u,v)\in D(x)\\ u\neq e, v\neq e}} (\theta(x) \wedge \theta(u) \wedge \phi(u) \wedge \chi(v)) \right) \\
&\quad \bigvee (\phi(x) \vee (\theta(x) \wedge \chi(x))) \quad \text{(by (4))} \\
&\leq \left(\bigvee_{\substack{(u,v)\in D(x)\\ u\neq e, v\neq e}} (\theta(v) \wedge \phi(u) \wedge \chi(v)) \right) \bigvee (\phi(x) \vee (\theta \wedge \chi(\text{x}))) \text{ (by (4.5))} \\
&= \left(\bigvee_{\substack{(u,v)\in D(x)\\ u\neq e, v\neq e}} \{\phi(u) \wedge (\theta \wedge \chi)(v)\} \right) \\
&\quad \bigvee (\phi^{t_{\theta\wedge\chi}}(e) \wedge (\phi^{t_{\theta\wedge\chi}}(x) \vee (\theta \wedge \chi)^{t_\phi}(x))) \quad \text{(by (4.7))} \\
&= \left(\bigvee_{\substack{(u,v)\in D(x)\\ u\neq e, v\neq e}} (\phi^{t_{\theta\wedge\chi}}(u) \wedge (\theta \wedge \chi)^{t_\phi}(v)) \right) \\
&\quad \bigvee ((\phi^{t_{\theta\wedge\chi}}(e) \wedge \phi^{t_{\theta\wedge\chi}}(x)) \vee (\phi^{t_{\theta\wedge\chi}}(e) \wedge (\theta \wedge \chi)^{t_\phi}(x))) \\
&\qquad \text{(by the definition of the tip-extended pair } \phi^{t_{\theta\wedge\chi}}, \\
&\qquad (\theta \wedge \chi)^{t_\phi} \text{ and the distributivity of } L) \\
&= \left(\bigvee_{\substack{(u,v)\in D(x)\\ u\neq e, v\neq e}} \left(\phi^{t_{\theta\wedge\chi}}(u) \wedge (\theta \wedge \chi)^{t_\phi}(v)\right) \right) \\
&\quad \bigvee ((\phi^{t_{\theta\wedge\chi}}(e) \wedge (\theta \wedge \chi)^{t_\phi}(x)) \vee \{\phi^{t_{\theta\wedge\chi}}(x) \wedge (\theta \wedge \chi)^{t_\phi}(e)\}) \quad \text{(by (4.6))} \\
&= \bigvee_{(u,v)\in D(x)} (\phi^{t_{\theta\wedge\chi}}(u) \wedge (\theta \wedge \chi)^{t_\phi}(v)) \\
&= \phi^{t_{\theta\wedge\chi}} \circ (\theta \wedge \chi)^{t_\phi}(x).
\end{aligned}
$$

Thus

$$\theta \wedge (\phi \vee \chi) \subseteq \phi \vee (\theta \wedge \chi).$$

This establishes the modularity of the lattice $NL(G)$. □

5. An application of subdirect product theorem

In this section, we establish the main results of this paper by an application of subdirect product theorem proved by T. Head [16] and by a subdirect product theorm proved in [2].

For any algebra X, T. Head by using convolutional method very conveniently and elegantly mirrored any n-ary operation of X to its fuzzy power set $F(X)$, power set $P(X)$, crisp set $C(X)$ and finally to the cartesian product $C(X)^J$ where $J = [0, 1)$. The intricate relationship and interplay of these subsets produced some amazing results which are formulated in the form of metatheorem and subdirect product theorems. The purpose of the formulation of these results was to obtain the fuzzy versions of the corresponding crisp results of algebra. T. Head successfully accomplished this task and demonstrated how to extend the results of semigroup to fuzzy setting by an application of metatheorem. The formulation of metatheorem and subdirect product theorem are based on the notions of Rep function and convolutional extension. T. Head has presented the fuzzy power algebra as a subdirect product of copies of its associated crisp power algebra. Recall that an algebra A is said to be a subdirect product of a family of algebras $\{A_b : b \text{ in } B\}$, where B is an arbitrary index set, if A is isomorphic to a subalgebra of the product algebra $\{A_b : b \text{ in } B\}$ with the property that its projection into each co-ordinate space A_b is a surjection. Recall that if X is an algebra having the n-ary operations $*_1, \ldots, *_k$, $n \geq 1$, then these operations extends to operations on $P(X)$, $C(X)$ and L^X by convolutional extension method. Moreover, as $P(X)$, $C(X)$ and L^X have two additional operations sup and inf, these sets become $(*_1, \ldots, *_k, \inf, \sup)$-algebras. Since the function $Rep : L^X \to C(X)^J$ commutes with $*_1, \ldots, *_k$, finite inf and arbitrary sup ; the function Rep turns out to be an injective homomorphism of the algebra L^X into the product algebra $C(X)^J$. Therefore Rep is an algebraic as well as order theoretic isomorphism of L^X with its image $I(X)$. Here we restate the subdirect product theorem due to T. Head for the algebra of L^X where the lattice L is a complete chain.

Firstly, we recall the definition of Rep function as given in [16]. Here, the set $F(X)$ of fuzzy subsets of X has been replaced by the set of L-subset L^X of X, where L is a complete chain.

Definition 5.1. For a non empty set X, let $Rep : L^X \to C(X)^J$ where $J = L \sim \{1\}$, be defined for each $\mu \in L^X$, for each $a \in J$ and $x \in X$, by

$$Rep(\mu)(a)(x) = \begin{cases} 0 & \text{if } \mu(x) \leq a, \\ 1 & \text{otherwise.} \end{cases}$$

The subdirect product theorem is as follows :

Theorem 5.2. *Let X be an algebra having n-ary operations $*_1, \ldots, *_k$, for various values of $n \geq 1$. Then Rep:$L^X \to C(X)^J$ is a representation of the $(\wedge, \vee, *_1, \ldots, *_k)$-algebra L^X as subdirect product of copies of the $(\wedge, \vee, *_1, \ldots, *_k)$-algebra $C(X)$.*

T. Head [16] has also established a subdirect product theorem for a restricted class of fuzzy subgroups namely fuzzy normal subgroups. For this purpose, he had to demonstrate that the function Rep commutes with the inf of the lattice of fuzzy normal subgroups of G and also with its sup. The first requirement is satisfied since the inf of lattice of fuzzy normal subgroups of G coincides with inf of lattice of fuzzy subsets of G. To show the other commutation, for a pair of fuzzy subgroups η and θ, T. Head has defined what he calls a tip extended pair of fuzzy subgroups η^{t_θ} and θ^{t_η} (see Erratum [17]). Then he asserts that $\sup\{\eta, \theta\} = \eta^{t_\theta} * \theta^{t_\eta}$, where the operation $*$ in the lattice of fuzzy normal subgroups of G is the convolutional extension of the binary operation $*$ of the group G [16] which coincides with the set product $\circ$ defined by Liu [19]. The function Rep commutes with the sup of lattice of fuzzy normal subgroups of G follows in view of the fact that the Rep function commutes with the convolutional extension $*$.

Let $L_c(G)$ denotes the set of characteristic functions of all subgroups of G and $NL_c(G)$ denotes the set of characteristic functions of all normal subgroups of G. Then it is well known $L_c(G)$ is a sublattice of the lattice $L(G)$ of L-subgroups of G and $NL_c(G)$ is a sublattice of $NL(G)$ the lattice of normal L-subgroups of G. T. Head has also established a version of subdirect product theorem for the lattice of fuzzy normal subgroups. We restate this theorem for normal L-subgroups of G when L is a complete chain. Let $NLRep$ denotes the restriction of the function $Rep : L^G \to C(G)^J$ to the set $NL(G)$ of normal L-subgroups of G and $NLI(G)$ denotes the image of $NL(G)$ under the mapping $NLRep$. Then

Theorem 5.3. *Let $\langle G, ^{-1}, *\rangle$ be a group. Then $NLRep : NL(G) \to NL_c(G)^J$ is an isomorphism of the lattice $NL(G)$ onto $NLI(G)$ that represents $NL(G)$ as a subdirect product of the copies of $NL_c(G)$ in the lattice $NL_c(G)^J$.*

Next, we exhibit that Theorem 4.2 can be obtained as a simple corollary to the above subdirect product theorem. In this result, let us denote the subgroup lattice of G by $\mathcal{L}(G)$ and by $\mathcal{L}_n(G)$ the subgroup lattice of normal subgroups of G.

Corollary 5.4. *The lattice $NL(G)$ is modular.*

Proof. Since the function $chi : \mathcal{L}_n(G) \to NL_c(G)$ is one to one and onto order preserving mapping which commutes with the convolutional extension of the binary operation of G to $NL(G)$, chi provides an algebraic as well as order theoretic isomorphism between the lattices $\mathcal{L}_n(G)$ and $NL_c(G)$. Now, since $\mathcal{L}_n(G)$ is modular, $NL_c(G)$ is also modular. Then the cartisian product $NL_c(G)^J$ is also modular. Now, any sublattice, in particulat $NLI(G)$ of $NL_c(G)^J$ is also modular. Thus by subdirect product theorem, the lattice $NL(G)$ is a modular lattice. □

In [2], a new subdirect product theorem for L-subgroups of G has been established provided L is a complete chain. If $LRep$ denotes the restriction of the function $Rep : L^G \to C(G)^J$ to the set $L(G)$ of L-subgroups of G and $LI(G)$ denotes the image of $L(G)$ under the mapping $LRep$, then we have the following.

Theorem 5.5. *Let $\langle G, ^{-1}, *\rangle$ be a group. Then $LRep : L(G) \to L_c(G)^J$ is an isomorphism of the lattice $L(G)$ onto $LI(G)$ that represents $L(G)$ as a subdirect product of the copies of $L_c(G)$ in the lattice $L_c(G)^J$.*

Below we demonstrate that Theorem 3.3 can be obtained as a simple corollary to the above subdirect product theorem.

Corollary 5.6. *A group G is generalized cyclic if and only if the lattice $L(G)$ is distributive.*

Proof. ($\Rightarrow$): As G is a generalized cyclic group, by Theorem 3.2, the subgroup lattice $\mathcal{L}(G)$ is distributive. Since the function $chi : \mathcal{L}(G) \to L_c(G)$ provides an algebraic as well as order theoretic isomorphism between the subgroup lattice $\mathcal{L}_(G)$ and crisp L-subgroup lattice $L_c(G)$, it follows that the lattice $L_c(G)$ is a distributive lattice. Hence the cartisian product $L_c(G)^J$ is also distributive. Now, any sublattice, in particulat $LI(G)$, of $L_c(G)^J$ is also distributive. Then by subdirect product theorem, the lattice $L(G)$ is a distributive lattice.

($\Leftarrow$): Since the L-subgroup lattice is distributive, its sublattice $L_c(G)$ of crisp L-subgroups of G is also distributive. As the function $chi : \mathcal{L}(G) \to L_c(G)$ provides an algebraic as well as order theoretic isomorphism, it follows that the subgroup lattice $\mathcal{L}(G)$ is a distributive lattice. Consequently, by Theorem 3.2, G is a generalized cyclic group. □

We conclude this note with the following conjecture in a completely distributive lattice.

Conjecture. A lattice polynomial identity P is valid in the subgroup latice $\mathcal{L}(G)$ of a group G if and only if it is valid in the L-subgroup lattice $L(G)$ of G.

The same conjecture can be made for the lattice of ideals of a ring and the lattice of L-ideals for a completely distributive lattice. These conjectures are already established and proved in fuzzy setting by T. Head in his important papers [16, 17] by using metatheorem approach.

6. Conclusion

N. Ajmal and K. V. Thomas studied the modularity of the lattice of fuzzy normal subgroups in a systematic and stepwise manner. In 1994, they established the lattice of fuzzy normal subgroups is modular. The notion of strong level subsets was used very effectively in the development their paper and suggested further application of this notion in the growth of fuzzy algebraic substructures. In 1995, T. Head defined Rep function by using this notion and then formulated metatheorem and subdirect product theorem. Using this subdirect product theorem, T. Head very easily and conveniently extended the modularity of the lattice of normal subgroups to fuzzy setting. In fact, results of fuzzy algebra which are extensions of results from classical algebra become just simple instances of these indigenous results. However, for lattice valued fuzzy subsets metatheorem and subdirect theorem are not applicable. Therefore we suggest the researchers pursuing studies in these areas to switch over to L-setting by investigating properties of L-subalgebras.

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