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Continuities and neighborhood structures in intuitionistic smooth topological spaces

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Abstract. First, we introduce the concepts of an intuitionistic smooth [resp. weak, strong] continuity and an intuitionistic smooth homeomorphism, and we obtained some their properties. Next, we define an intuitionistic smooth subspace and study its some properties. Finally, we define an intuitionistic smooth neighborhood system and an intuitionistic smooth Q- neighborhood system and we obtain the the characterization (See Theorem 5.5).

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1. Introduction

In 1965, Zadeh [37] introduced the concept of a fuzzy set as the generalization of a set. In 1986, Chang [8] was the first to introduce the notion of a fuzzy topology on a set X. After that, many researchers [16, 17, 18, 19, 20, 23, 24, 25, 28, 29, 36] have investigated several properties, e.g., fuzzy closure operator, fuzzy compactness, fuzzy connectedness, separation axioms, regularity axioms, normality axioms, neighborhood structures and product, etc. in fuzzy topological spaces.

However, in their definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy set was absent. In 1985, Kubiak [21] and Šostak [35] introduced the concept of gradation of openness (closedness) of fuzzy sets in a set X and gave the definition of a fuzzy topology on X as an extension of Chang's fuzzy topology [8]. After then, many researchers [9, 10, 14, 15, 26, 27, 30] studied fuzzy topology in the above sense.

In 1975, Zadeh [38] introduced the idea of interval-valued fuzzy sets. In 1986, Atanassov [2] introduced the idea of intuitionistic fuzzy set. After then many researchers [2, 3, 4, 5, 6, 7] have worked mainly on operators and relations on intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. In particular, In 2010, Cheong and Hur [11] introduced the concept of an intuitionistic interval-valued fuzzy set and studied its basic properties. In 1997, Çoker [12] introduced the idea of the topology of intuitionistic fuzzy sets. Moreover, Samanta and Mondal [32, 33] introduced the definitions of the topology of interval-valued fuzzy sets and the topology of interval-valued intuitionistic fuzzy sets, respectively.

On the other hand, Çoker and Demirci [13], and Samanta and Mondal [31, 34] defined intuitionistic gradation of openness (in short IGO) of fuzzy sets in Šostak′s sense thereby gave the definition of an intuitionsitic fuzzy topology (in short, IFT). In particular, in 2005, Abbas [1] introduced the notions of some intuitionistic fuzzy compactness in intuitionistic topological spaces and investigated some their properties. They mainly dealt with intuitionistic gradation of openness of fuzzy sets in the sense of Chang. But in 2010, Lim et al. [22] dealt with it in Lowen′s sense.

In this paper, we introduce the concepts of an intuitionistic smooth [resp. weak, strong] continuity and an intuitionistic smooth homeomorphism, and we obtained some of their properties. Next, we define an intuitionistic smooth subspace and study its some properties. Finally, we define an intuitionistic smooth neighborhood system and an intuitionistic smooth Q-neighborhood system and we obtain the its characterization, respectively (See Theorem 5.5).

2. Preliminaries

In this section, we will list some concepts and results which are needed in the next sections. Throughout this paper, X, Y, Z, etc. always denote nonempty (ordinary) sets. We will write $I = [0,1], I_0 = (0,1]$ and $I_1 = [0,1)$.

Definition 2.1 ([37])**.** A mapping $A : X \to I$ is called a fuzzy set in X. $\mathbf{0}$ and $\mathbf{1}$ are called the empty fuzzy set and the whole fuzzy set in X defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for each $x \in X$, respectively. The set $\{x \in X : A(x) > 0\}$ is called a support of A and is denoted by $S(A)$ or A_0.

We will denote the set of all fuzzy sets as I^X.

From [8], we can see that $(I^X, \cup, \cap, \mathbf{0}, \mathbf{1})$ is a complete distributive lattice satisfying the DeMorgan′s Laws with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$.

Definition 2.2 ([8, 29])**.** Let $f : X \to Y$ be a mapping, let $A \in I^X$ and let $B \in I^Y$.

(i) The image of A under f, denoted by $f(A)$, is a fuzzy set in Y defined as follows: for each $y \in Y$,

$$f(A)(y) = \begin{cases} \bigvee_{f(x)=y} A(x) & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{if } f^{-1}(y) = \varnothing, \end{cases}$$

(ii) The preimage of B under f, denoted by $f^{-1}(B)$, is a fuzzy set in X as follows: for each $x \in X$,

$$f^{-1}(B)(x) = B(f(x)).$$

Let $I \oplus I = \{(a, b) \in I \times I : a + b \leq 1\}$, let $(a, b), (c, d) \in I \oplus I$ and let $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Gamma} \subset I \oplus I$. We define the following(See [11]):

(i) $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \geq d$,

(ii) $(a, b) = (c, d)$ iff $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$,

(iii) $(a, b)^c = (b, a)$, where $(a, b)^c$ denotes the complement of (a, b),

(iv) $\bigvee_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigvee_{\alpha \in \Gamma} a_\alpha, \bigwedge_{\alpha \in \Gamma} b_\alpha)$,

(v) $\bigwedge_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigwedge_{\alpha \in \Gamma} a_\alpha, \bigvee_{\alpha \in \Gamma} b_\alpha)$.

Each member (a, b) of $I \oplus I$ is called an intuitionistic point. When the elements of $I \oplus I$ are denoted be capital letters $M, N, \cdots$, we write $M = (\mu_M, \nu_M), N = (\mu_N, \nu_N), \cdots$, where μ_M and ν_M are the membership and the nonmembership points, respectively.

From Theorem 2.1 in [11], we can see that $(I \oplus I, \leq)$ is a complete distributive lattice with the greatest element $(1, 0)$ and the least element $(0, 1)$ satisfying DeMorgan's laws.

The following is the modification of the concept of the concept of intuitionistic fuzzy sets introduced by Atanassov (See [2]).

Definition 2.3 ([11]). A mapping $A : X \to I \oplus I$ is called an intuitionistic fuzzy set in X and we write $A(x) = (\mu_A(x), \nu_A(x))$ for each $x \in X$. $\widetilde{0}$ and $\widetilde{1}$ are the empty intuitionistic fuzzy set and the whole intuitionistic fuzzy set in X given by $\widetilde{0}(x) = (0, 1)$ and $\widetilde{1}(x) = (1, 0)$, respectively.

We denote the set of all intuitionistic fuzzy sets in X as $(I \oplus I)^X$.

Definition 2.4 ([2]). Let $A, B \in (I \oplus I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset (I \oplus I)^X$. Then the union $\bigcup_{\alpha \in \Gamma} A_\alpha$, the intersection $\bigcap_{\alpha \in \Gamma} A_\alpha$, the complement A^c of A and the inclusion $A \subset B$ are defined as follows: for each $x \in X$,

(i) $(\bigcup_{\alpha \in \Gamma} A_\alpha)(x) = (\bigvee_{\alpha \in \Gamma} \mu_{A_\alpha}(x), \bigwedge_{\alpha \in \Gamma} \nu_{A_\alpha}(x))$,

(ii) $(\bigcap_{\alpha \in \Gamma} A_\alpha)(x) = (\bigwedge_{\alpha \in \Gamma} \mu_{A_\alpha}(x), \bigvee_{\alpha \in \Gamma} \nu_{A_\alpha}(x))$,

(iii) $A^c(x) = (\nu_A(x), \mu_A(x))$,

(iv) $A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

We can easily see that $((I \oplus I)^X, \cup, \cap)$ is a complete distributive lattice with the least element $\widetilde{0}$ and the greatest element $\widetilde{1}$ satisfying DeMorgan$'$s laws.

A fuzzy set $A \in I^X$ is called a fuzzy point in a nonempty set X with the value $\lambda \in I_0$ and the support $x \in X$, denoted by x_λ, if $A(x) = \lambda$ and $A(y) = 0$, for each $x \neq y \in X$. We will denote the set of all fuzzy points in X as $F_P(X)$. A fuzzy point x_λ in X is said to belong to a fuzzy set A denoted by $x_\lambda \in A$, if $A(x) \geq \lambda$. Let $x_\lambda \in \mathrm{F_P(X)}$ and let $A \in I^X$. Then x_λ is said to be quasi-coincident with A, denoted by $x_\lambda q A$, if $A(x) + \lambda > 1$. The negation of the statements $x_\lambda \in A$ and $x_\lambda q A$ will be symbolized by the notations $x_\lambda \notin A$ and $x_\lambda \overline{q} A$, respectively (See [28]).

Result 2.5 ([28], Proposition 2.3). *Let* $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$. *Then* $x_\lambda q(\bigcup_{\alpha\in\Gamma} A_\alpha)$ *if and only if* $\exists\ \alpha \in \Gamma$ *such that* $x_\lambda q A_\alpha$.

For each $\alpha \in I$, a mapping $\boldsymbol{\alpha} : X \to I$ is called the $\alpha-$constant fuzzy set in X, if $\boldsymbol{\alpha}(x) = \alpha$, for each $x \in X$.

Definition 2.6 ([23]). A fuzzy topology or Lowen′s fuzzy topology on a set X is a collection $\delta \subset I^X$ satisfying the following three axioms:
(FT1) $\boldsymbol{\alpha} \in \delta,\ \forall\ \alpha \in I$,
(FT2) $A \cap B \in \delta,\ \forall\ A, B \in \delta$,
(FT3) $\bigcup_{\alpha\in\Gamma} A_\alpha \in \delta,\ \forall\ \{A_\alpha\}_{\alpha\in\Gamma} \subset \delta$.
The pair (X, δ) is called a fuzzy topological space(in short, fts) and we denote the set of all fuzzy topologies on X as $FT(X)$. A fuzzy set A is said to the closed in X, if $A^c \in \delta$.
If we replace the condition on (FT1) by the weaker condition
(FT1)′ $\mathbf{0}, \mathbf{1} \in \delta$
then δ is called a quasi-fuzzy topology or Chang ′s fuzzy topology on X (See [8]). The pair (X, δ) is called a Chang ′s fuzzy topological space. And we will denote the set of all Chang's fuzzy topologies on X as $QFT(X)$.

Definition 2.7 ([28]). Let (X, δ) be fts.
(i) A subfamily $\mathfrak{B}$ of δ is called a base for δ, if for each $A \in \delta$, $\exists\ \mathfrak{B}_A \subset \mathfrak{B}$ s.t. $A = \bigcup \mathfrak{B}_A$.
(ii) A subfamily $\mathcal{S}$ of δ is called a subbase for δ, if the family $\mathfrak{B} = \{\bigcap \mathcal{F} : \mathcal{F}$ is a finite subset of $\mathcal{S}\}$ is a base for δ.

Definition 2.8 ([28]). Let (X, δ) be a fts, let $A \in I^X$ and let $x_\lambda \in F_P(X)$.
(i) A is called a neighborhood (in short, nbd) of x_λ, if

$$\exists\ B \in \delta \text{ s.t. } x_\lambda \in B \subset A.$$

(ii) A is called a Q-neighborhood (in short, Q-nbd) of x_λ, if

$$\exists\ B \in \delta \text{ s.t. } x_\lambda q B \subset A.$$

The family consisting of all the nbds [resp. Q-nbds] of x_λ is called the system of nbds [resp. Q-nbds] of x_λ and is denoted by $\mathcal{N}(x_\lambda)$ [resp.$\mathcal{N}_Q(x_\lambda)$]. A nbd [resp. Q-nbd] A is said to be open, if $A \in \delta$.

Result 2.9 ([28], Proposition 2.4). *Let* (X, δ) *be an* fts *and let* $\mathfrak{B} \subset \delta$. *Then* $\mathfrak{B}$ *is a base for* δ *if and only if for each* $x_\lambda \in \mathrm{F_P(X)}$ *and for each* $U \in \mathcal{N}_Q(x_\lambda)$, $\exists\ B \in \mathfrak{B}$ *such that* $x_\lambda q B \subset U$.

The following is the modification of the notion of an intuitionistic gradation of openness introduced by Mondal and Samanta [34].

Definition 2.10 ([22]). Let X be a nonempty set. Then a mapping $\tau = (\mu_\tau, \nu_\tau) : I^X \to I \oplus I$ is called an intuitionistic smooth topology(in short, *ist*) or a family of intuitionistic fuzzy open sets on X, if it satisfies the following conditions :
(IST1) $\tau(\boldsymbol{\alpha}) = (1, 0),\ \forall\ \alpha \in I$,
(IST2) $\mu_\tau(A \cap B) \geq \mu_\tau(A) \wedge \mu_\tau(B)$ and $\nu_\tau(A \cap B) \leq \nu_\tau(A) \vee \nu_\tau(B),\ \forall\ A, B \in I^X$,

(IST3) $\mu_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mu_\tau(A_\alpha)$ and $\nu_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \leq \bigvee_{\alpha\in\Gamma} \nu_\tau(A_\alpha),\ \forall\ \{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.

The pair (X, τ) is called an intuitionistic smooth topological space (in short, *ists*).

In the above Definition, if we replace the condition(IST1) by the following condition

(IST1)$'$ $\tau(\mathbf{0}) = \tau(\mathbf{1}) = (1, 0)$,

then τ [resp. (X, τ)] is called an intuitionistic gradation of openness [resp. intuitionistic fuzzy topological space] (See [34]). In this case, we will call τ [resp. (X, τ)] as an intuitionistic quasi-smooth topology [resp. intuitionistic quasi-smooth topological space (in short, iqsts)]. We will denote the set of all intuitionistic quasi-smooth [resp. smooth] topologies on X as $IQST(X)$ [resp. $IST(X)$]. It is clear that if (X, τ) is an *ists*, then it is an *iqsts*, i.e., $IST(X) \subset IQST(X)$.

Let $2 = \{0, 1\}$. Since we can regard $(0, 1)$ and $(1, 0) \in 2 \oplus 2$ as $(0, 1) = \mathbf{0}$ and $(1, 0) = \mathbf{1}$, respectively, restricting the definition to $\tau : I^X \to 2 \oplus 2$ gives us the definition of a fuzzy topology in the sense of Lowen (See [23]). Putting $\tau : 2^X \to 2 \oplus 2$, we even obtain a (crisp) topology on X again. Of course, if τ is an *iqsts*, the alternation of the definition to $\tau : I^X \to 2 \oplus 2$ gives us the definition of the fuzzy topology in the sense of Chang (See [8]).

Definition 2.11 ([22])**.** Let X be a nonempty set. Then a mapping $\mathcal{C} = (\mu_\mathcal{C}, \nu_\mathcal{C}) : I^X \to I \oplus I$ is called an intuitionistic smooth cotopology (in short, *isct*) on X if it is satisfies the following conditions:

(ISCT1) $\mathcal{C}(\boldsymbol{\alpha}) = (1, 0),\ \forall\ \alpha \in I$,

(ISCT2) $\mu_\mathcal{C}(A \cup B) \geq \mu_\mathcal{C}(A) \wedge \mu_\mathcal{C}(B)$
and
$\nu_\mathcal{C}(A \cup B) \leq \nu_\mathcal{C}(A) \vee \nu_\mathcal{C}(B),\ \forall\ A, B \in I^X$,

(ISCT3) $\mu_\mathcal{C}(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mu_\mathcal{C}(A_\alpha)$
and
$\nu_\mathcal{C}(\bigcap_{\alpha\in\Gamma} A_\alpha) \leq \bigvee_{\alpha\in\Gamma} \nu_\mathcal{C}(A_\alpha),\ \forall\ \{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.

The pair $(X, \mathcal{C})$ is called an intuitionistic smooth cotopological space(in short, *iscts*).

In the above Definitions, if we replace the condition (ISCT1) by the condition

(ISCT1)$'$ $\tau(\mathbf{0}) = \tau(\mathbf{1}) = (1, 0)$,

then τ [resp. (X, τ)] is called an intuitionistic gradation of closedness [resp. intuitionistic fuzzy cotopological space] (See [34]). In this case, we will call $\mathcal{C}$ [resp. $(X, \mathcal{C})$] as an intuitionistic quasi-smooth cotopology [resp. intuitionistic quasi-smooth cotopological space (in short, *iqscts*)]. We will denote the set of all intuitionistic quasi-smooth [resp. smooth] cotopologies on X as $IQSCT(X)$ [resp. $ISCT(X)$]. It is clear that if $(X, \mathcal{C})$ is an iscts, then it is an *isqcts*, i.e., $ISCT(X) \subset IQSCT(X)$.

Definition 2.12 ([22])**.** Let (X, τ) be an *ists* and let $(\lambda, \mu) \in I \oplus I$. We define $[\tau]_{(\lambda,\mu)}$ and $[\tau]^*_{(\lambda,\mu)}$ as follows, respectively:

(i) $[\tau]_{(\lambda,\mu)} = \{A \in I^X : \mu_\tau(A) \geq \lambda,\ \nu_\tau(A) \leq \mu\}$,

(ii) $[\tau]^*_{(\lambda,\mu)} = \{A \in I^X : \mu_\tau(A) > \lambda,\ \nu_\tau(A) < \mu\}$.

We call $[\tau]_{(\lambda,\mu)}$ [resp. $[\tau]^*_{(\lambda,\mu)}$] the $(\lambda, \mu)-$level [resp. strong $(\lambda, \mu)-$level] of τ. If $(\lambda, \mu) = (0, 1)$, then $[\tau]_{(0,1)} = I^X$, i.e., $[\tau]_{(0,1)}$ is the discrete fuzzy topology on X and if $(\lambda, \mu) = (1, 0)$, then $[\tau]^*_{(1,0)} = \varnothing$. Moreover, we can easily see that for any $(\lambda, \mu) \in I \oplus I$, $[\tau]^*_{(\lambda,\mu)} \subset \tau_{(\lambda,\mu)}$.

Result 2.13 ([22], Proposition 3.8). (1) *Let* $\{\delta_{(\lambda,\mu)}\}_{(\lambda,\mu)\in I\oplus I}$ *be a descending family of fuzzy topologies on* X *such that* $\delta_{(0,1)}$ *is the discrete fuzzy topology on* X*. We define the mapping* $\tau : I^X \to I\oplus I$ *as follows: for each* $A\in I^X$,

$$\tau(A) = (\bigvee_{A\in\delta_{(\lambda,\mu)}} \lambda, \bigwedge_{A\in\delta_{(\lambda,\mu)}} \mu).$$

Then $\tau \in$*IST(X).*

(2) *If* $\delta_{(\lambda,\mu)} = \bigcap_{(\lambda',\mu')<(\lambda,\mu)} \delta_{(\lambda',\mu')}$*, for each* $(\lambda,\mu)\in I_0\oplus I_1$*, then* $[\tau]_{(\lambda,\mu)} = \delta_{(\lambda,\mu)}$.

(3) *Alternatively, if* $\delta_{(\lambda,\mu)} = \bigcup_{(\lambda',\mu')>(\lambda,\mu)} \delta_{(\lambda',\mu')}$*, for each* $(\lambda,\mu)\in I_1\oplus I_0$*, then* $[\tau]^*_{(\lambda,\mu)} = \delta_{(\lambda,\mu)}$.

Result 2.14 ([22], Proposition 3.11). *Let* (X,δ) *be a fuzzy topological space and for each* $(\lambda,\mu)\in I_0\oplus I_1$*, we define the mapping* $\delta^{(\lambda,\mu)} : I^X \to I\oplus I$ *as follows: for each* $A\in I^X$,

$$\delta^{(\lambda,\mu)}(A) = \begin{cases} (1,0) & if\ A\ is\ a\ constant\ fuzzy\ set, \\ (\lambda,\mu) & if\ A\in\delta\ but\ not\ a\ constant\ fuzzy\ set, \\ (0,1) & otherwise. \end{cases}$$

Then $\delta^{(\lambda,\mu)} \in$*IST(X) such that* $[\delta^{(\lambda,\mu)}]_{(\lambda,\mu)} = \delta$.

In this case, $\delta^{(\lambda,\mu)}$ is called $(\lambda,\mu)-$th intuitionistic gradation on X and $(X,\delta^{(\lambda,\mu)})$ is called a $(\lambda,\mu)-$th graded intuitionistic fuzzy topological space.

3. Intuitionistic smooth continuous mapping

If we want to consider a category of intuitionistic smooth topological spaces, we need, besides the objects, in the category, also a description of the morphism between them. For crisp topological spaces, these morphisms were continuous mappings. For fuzzy topological spaces, as can be seen in, e.g. [23], a mapping $f : (X,\delta) \to (Y,\delta')$ is said to be fuzzy continuous, if $f^{-1}(A)\in\delta$, for each $A\in\delta'$.

Now we define an intuitionistic smooth continuous mapping.

Definition 3.1. Let (X,τ_1) and (Y,τ_2) be two ists. Then a mapping $f : X\to Y$ is said to be:

(i) intuitionistic smooth continuous [34], if it satisfies the following conditions: for each $A\in I^Y$,

$$\mu_{\tau_2}(A) \le \mu_{\tau_1}(f^{-1}(A))$$

and

$$\nu_{\tau_2}(A) \ge \nu_{\tau_1}(f^{-1}(A)),$$

(ii) intuitionistic smooth weakly continuous, if it satisfies the following conditions: for each $A\in I^Y$, $\mu_{\tau_2}(A) > 0$ and $\nu_{\tau_2}(A) < 1$ imply $\mu_{\tau_1}(f^{-1}(A)) > 0$ and $\nu_{\tau_1}(f^{-1}(A)) < 1$,

(iii) intuitionistic smooth strongly continuous, if $\tau_2(A) = \tau_1(f^{-1}(A))$, $\forall\ A\in I^Y$.

It is clear that intuitionistic smooth strong continuity ⇒ intuitionistic smooth continuity ⇒ intuitionistic smooth weak continuity. However the converse may not be true, in general.

Example 3.2. (1) Let $X = \{a,b,c,d\}$, and let A and B be two fuzzy sets in X defined as follows:

$$A(x) = \begin{cases} 1 & \text{if } x = b,d \\ 0 & \text{if } x = a,c \end{cases}$$

and

$$B(x) = \begin{cases} 1 & \text{if } x = a, c, \\ 0 & \text{if } x = b, d. \end{cases}$$

For each $i = 1, 2$, we define the mapping $\tau_i : I^X \to I \oplus I$ as follows:

$$\tau_i(\boldsymbol{\alpha}) = (1, 0),\ \tau_1(A) = \tau_1(B) = (1, 0),\ \tau_2(A) = \tau_2(B) = (\frac{1}{2}, \frac{1}{2}).$$

Then it is clear that (X, τ_1) and (X, τ_2) are two *ists*. Consider the identity mapping $1_X : (X, \tau_2) \to (X, \tau_1)$. Then we can easily see that 1_X is intuitionistic smooth weakly continuous, but not intuitionistic smooth continuous.

(2) Let O be the set of all odd numbers of $\mathbb{N}$ and let $A_n = \{1, 3, \dots, 2n-1\}$. For each $i = 1, 2$, we define the mapping $\tau_i : I^{\mathbb{N}} \to I \oplus I$ as follows: for each $A \in I^X$,

$$\tau_i(A) = \begin{cases} (\frac{1}{i}, 1 - \frac{1}{i}) & \text{if } A = \chi_O, \\ (max\{\frac{1}{i}, \frac{1}{2n-1}\},\ min\{\frac{1}{i}, \frac{1}{2n-1}\}) & \text{if } A = \chi_{A_n}, \\ (1, 0) & \text{if } A \in I^X \setminus \{\chi_O, \chi_{A_n}\}. \end{cases}$$

Then clearly (X, τ_i) is an *ists*. Consider two identity mappings $1_X : (X, \tau_2) \to (X, \tau_1)$ and $1^*_X : (X, \tau_1) \to (X, \tau_2)$. Then we can easily see that 1_X in intuitionistic smooth weakly continuous but not intuitionistic smooth continuous, and 1^*_X is intuitionistic smooth continuous but not intuitionistic smooth strongly continuous.

Remark 3.3. Let (X, τ) and (Y, τ_2) be *ists* and let $f : X \to Y$ be a mapping. Then it is obvious that if $f : (X, \tau_1) \to (Y, \tau_2)$ is intuitionistic smooth [resp. weakly and strongly] continuous, then $f : (X, \mu_{\tau_1}) \to (Y, \nu_{\tau_2})$ and $f : (X, \nu_{\tau_1}{}^c) \to (Y, \nu_{\tau_2}{}^c)$ are smooth [resp. weakly and strongly] continuous.

Proposition 3.4. *Constant mappings are always intuitionistic smooth continuous.*

Proof. Let (X, τ_1) and (Y, τ_2) be two *ists* and let $f : X \to Y$ be any constant mapping, say $f(x) = y_0 \in Y$, for each $x \in X$, where $y_0 \in Y$ is fixed. Let $A \in I^Y$ and let $x \in X$. Then $f^{-1}(A)(x) = A(f(x)) = A(y_0)$. Thus $f^{-1}(A)$ is a constant fuzzy set in X. So $\tau_1(f^{-1}(A)) = (1, 0)$. Hence $\mu_{\tau_1}(f^{-1}(A)) = 1 \geq \mu_{\tau_2}(A)$ and $\nu_{\tau_1}(f^{-1}(A)) = 0 \leq \nu_{\tau_2}(A)$. Therefore f is intuitionistic smooth continuous. □

The following is the immediate result of Definition 3.1.

Proposition 3.5. *The identity mapping* $1_X : (X, \tau_1) \to (X, \tau_1)$ *is intuitionistic smooth continuous.*

The following is the immediate result of Definitions 2.12 and 3.1.

Theorem 3.6. *Let* $f : (X, \tau_1) \to (Y, \tau_2)$ *be a mapping. Then*

(1) *f is intuitionistic smooth continuous if and only if for each* $A \in I^Y$,

$$\mu_{c_{\tau_2}}(A) \leq \mu_{c_{\tau_1}}(f^{-1}(A)) \text{ and } \nu_{c_{\tau_2}}(A) \geq \nu_{c_{\tau_1}}(f^{-1}(A)),$$

(2) *f is intuitionistic weakly smooth continuous if and only if* $\mu_{c_{\tau_2}}(A) > 0$ *and* $\mu_{c_{\tau_2}}(A) < 1$ *imply* $\mu_{c_{\tau_1}}(f^{-1}(A)) > 0$ *and* $\mu_{c_{\tau_1}}(f^{-1}(A)) < 1$, *for each* $A \in I^Y$.

Proposition 3.7. *Let* $(X, \tau_1), (Y, \tau_2)$ *and* (Z, τ_3) *are ists. If* $f : X \to Y$ *and* $g : Y \to Z$ *are intuitionistic smooth continuous, then so is* $g \circ f$.

Proof. Let $A \in I^Z$. Then

$$\begin{aligned}\mu_{\tau_3}(A) &\le \mu_{\tau_2}(g^{-1}(A)) \text{ [Since } g \text{ is intuitionistic smooth continuous]}\\ &\le \mu_{\tau_1}(f^{-1}(g^{-1}(A))) \text{ [Since } f \text{ is intuitionistic smooth continuous]}\\ &= \mu_{\tau_1}((g\circ f)^{-1}(A))\end{aligned}$$

and

$$\begin{aligned}\nu_{\tau_3}(A) &\ge \nu_{\tau_2}(g^{-1}(A))\\ &\ge \nu_{\tau_1}(f^{-1}(g^{-1}(A)))\\ &= \nu_{\tau_1}((g\circ f)^{-1}(A)).\end{aligned}$$

Thus $g \circ f$ is intuitionistic smooth continuous. □

By Propositions 3.5 and 3.7, we can see that IST(X) forms a concrete category.

Theorem 3.8. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping between intuitionistic smooth topological spaces. Then f is intuitionistic smooth continuous if and only if for each $(\lambda, \mu) \in I \oplus I$, $f : (X, [\tau_1]_{(\lambda,\mu)}) \to (Y, [\tau_2]_{(\lambda,\mu)})$ is fuzzy continuous.*

Proof. (⇒): Suppose f is intuitionistic smooth continuous. Then clearly $\mu_{\tau_2}(A) \le \mu_{\tau_1}(f^{-1}(A))$ and $\nu_{\tau_2}(A) \ge \nu_{\tau_1}(f^{-1}(A))$, $\forall\ A \in I^Y$. Let $B \in [\tau_2]_{(\lambda,\mu)}$, for each $(\lambda, \mu) \in I \oplus I$. Then $\lambda \le \mu_{\tau_2}(B) \le \mu_{\tau_1}(f^{-1}(B))$ and $\mu \ge \nu_{\tau_2}(B) \ge \nu_{\tau_1}(f^{-1}(B))$. Thus $f^{-1}(B) \in [\tau_1]_{(\lambda,\mu)}$. So $f : (X, [\tau_1]_{(\lambda,\mu)}) \to (Y, [\tau_2]_{(\lambda,\mu)})$ is fuzzy continuous.

(⇐): Suppose the necessary condition holds and let $A \in I^Y$.

If $\tau_2(A) = (0, 1)$, then clearly, $\mu_{\tau_2}(A) = 0 \le \mu_{\tau_1}(f^{-1}(A))$ and $\nu_{\tau_2}(A) = 1 \ge \nu_{\tau_1}(f^{-1}(A))$.

If $\tau_2(A) = (\lambda, \mu) \in I_0 \oplus I_1$, then $A \in [\tau_2]_{(\lambda,\mu)}$. By the hypothesis, $f^{-1}(A) \in [\tau_1]_{(\lambda,\mu)}$. Thus $\mu_{\tau_2}(A) = \lambda \le \mu_{\tau_1}(f^{-1}(A))$ and $\nu_{\tau_2}(A) = \mu \ge \nu_{\tau_1}(f^{-1}(A))$. So, in all cases, $\mu_{\tau_2}(A) \le \mu_{\tau_1}(f^{-1}(A))$ and $\nu_{\tau_2}(A) \ge \nu_{\tau_1}(f^{-1}(A))$. Hence $f : (X, \tau_1) \to (Y, \tau_2)$ is intuitionistic smooth continuous. □

Theorem 3.9. *Let (X, δ_1) and (Y, δ_2) be fts and let $f : X \to Y$ be a mapping. Then $f : (X, \delta_1) \to (Y, \delta_2)$ is fuzzy continuous if and only if $f : (X, \delta_1^{(\lambda,\mu)}) \to (Y, \delta_2^{(\lambda,\mu)})$ is intuitionistic smooth continuous, for each $(\lambda, \mu) \in I_0 \oplus I_1$.*

Proof. (⇒): Suppose $f : (X, \delta_1) \to (Y, \delta_2)$ is fuzzy continuous and let $A \in I^Y$. Then we have the following possibilities:

(i) A is a constant fuzzy set,

(ii) $A \in \delta_2 \setminus \{\boldsymbol{\alpha} : \alpha \in I\}$,

(iii) $A \notin \delta_2$.

Suppose A is a constant fuzzy set. Then $\delta_2^{(\lambda,\mu)}(A) = (1, 0)$, and $f^{-1}(A) = \mathbf{0}$ or $\mathbf{1}$. Thus $\delta_1^{(\lambda,\mu)}(f^{-1}(A)) = (1, 0)$. So

$$\mu_{\delta_2^{(\lambda,\mu)}}(A) = 1 = \mu_{\delta_1^{(\lambda,\mu)}}(f^{-1}(A))$$

and

$$\nu_{\delta_2^{(\lambda,\mu)}}(A) = 0 = \nu_{\delta_1^{(\lambda,\mu)}}(f^{-1}(A)).$$

Suppose $A \in \delta_2 \setminus \{\boldsymbol{\alpha} : \alpha \in I\}$. Then $\delta_2^{(\lambda,\mu)}(A) = (\lambda, \mu)$. Since $f : (X, \delta_1) \to (Y, \delta_2)$ is fuzzy continuous, $f^{-1}(A) \in \delta_1$. Moreover, $f^{-1}(A)$ is not a constant fuzzy set in X. Thus $\delta_1^{(\lambda,\mu)}(f^{-1}(A)) = (\lambda, \mu)$. So

$$\mu_{\delta_2^{(\lambda,\mu)}}(A) = \lambda = \mu_{\delta_1^{(\lambda,\mu)}}(f^{-1}(A))$$

and

$\nu_{\delta_2^{(\lambda,\mu)}}(A) = \mu = \nu_{\delta_1^{(\lambda,\mu)}}(f^{-1}(A)).$

Suppose $A \notin \delta_2$. Then $\delta_2^{(\lambda,\mu)}(A) = (0, 1)$. Thus

$\mu_{\delta_2^{(\lambda,\mu)}}(A) = 0 \leq \mu_{\delta_1^{(\lambda,\mu)}}(f^{-1}(A))$

and

$\nu_{\delta_2^{(\lambda,\mu)}}(A) = 1 \geq \nu_{\delta_1^{(\lambda,\mu)}}(f^{-1}(A)).$

Hence, in all cases, $\mu_{\delta_2^{(\lambda,\mu)}}(A) \leq \mu_{\delta_1^{(\lambda,\mu)}}(f^{-1}(A))$ and $\nu_{\delta_2^{(\lambda,\mu)}}(A) \geq \nu_{\delta_1^{(\lambda,\mu)}}(f^{-1}(A))$. Therefore $f : (X, \delta_1^{(\lambda,\mu)}) \to (Y, \delta_2^{(\lambda,\mu)})$ is intuitionistic smooth continuous, for each $(\lambda, \mu) \in I_0 \oplus I_1$.

($\Leftarrow$): Suppose the necessary condition holds. From Result 2.14 and Theorem 3.8, we can see that $f : (X, \delta_1) \to (Y, \delta_2)$ is fuzzy continuous. This completes the proof. □

Theorem 3.10. *Let (X, τ) be an ists and let $f : X \to Y$ be a mapping. Let $\{\delta_{(\lambda,\mu)}\}_{(\lambda,\mu)\in I_0\oplus I_1}$ be a descending family of fuzzy topologies on Y and let τ' be the ist on Y generated be this family. For each $(\lambda, \mu) \in I_0 \oplus I_1$, let $\mathcal{B}_{(\lambda,\mu)}$ be a base and let $\mathcal{S}_{(\lambda,\mu)}$ be a subbase for $\delta_{(\lambda,\mu)}$. Then*

(1) *$f : (X, \tau) \to (Y, \tau')$ is intuitionistic smooth continuous if and only if $\lambda \leq \mu_\tau(f^{-1}(A))$ and $\mu \geq \nu_\tau(f^{-1}(A))$, $\forall\ A \in \delta_{(\lambda,\mu)}$, $\forall\ (\lambda, \mu) \in I_0 \oplus I_1$,*

(2) *$f : (X, \tau) \to (Y, \tau')$ is intuitionistic smooth continuous if and only if $\lambda \leq \mu_\tau(f^{-1}(A))$ and $\mu \geq \nu_\tau(f^{-1}(A))$, $\forall\ A \in \mathcal{B}_{(\lambda,\mu)}$, $\forall\ (\lambda, \mu) \in I_0 \oplus I_1$,*

(3) *$f : (X, \tau) \to (Y, \tau')$ is intuitionistic smooth continuous if and only if $\lambda \leq \mu_\tau(f^{-1}(A))$ and $\mu \geq \nu_\tau(f^{-1}(A))$, $\forall\ A \in \mathcal{S}_{(\lambda,\mu)}$, $\forall\ (\lambda, \mu) \in I_0 \oplus I_1$.*

Proof. (1) ($\Rightarrow$): Suppose $f : (X, \tau) \to (Y, \tau')$ is intuitionistic smooth continuous. For each $(\lambda, \mu) \in I_0 \oplus I_1$, let $A \in \delta_{(\lambda,\mu)}$. Then

$\mu_\tau(f^{-1}(A)) \geq \mu_{\tau'}(A)$ [By the hypothesis]

$\geq \lambda$ [By Result 2.13 (1)]

and

$\nu_\tau(f^{-1}(A)) \leq \nu_{\tau'}(A) \leq \mu.$

($\Leftarrow$): Suppose the necessary conditions holds. Let $A \in I^Y$ and let $\tau'(A) = (\lambda, \mu) \in I_0 \oplus I_1$. Then $A \in \delta_{(\lambda,\mu)}$. Thus, by hypothesis,

$\mu_\tau(f^{-1}(A)) \geq \lambda = \mu_{\tau'}(A)$

and

$\nu_\tau(f^{-1}(A)) \leq \mu = \nu_{\tau'}(A) \leq \mu.$

So $f : (X, \tau) \to (Y, \tau')$ is intuitionistic smooth continuous.

Arguing as above and using Definition 2.8, we get (2) and (3). □

Definition 3.11. Let τ_1 and τ_2[resp. $\mathcal{C}_1$ and $\mathcal{C}_2$] be intuitionistic smooth topologies [resp. intuitionistic smooth cotopologies] on X and Y, respectively. Then a mapping $f : X \to Y$ is said to be intuitionistic smooth open [resp. closed], if $\mu_{\tau_1}(A) \leq \mu_{\tau_2}(f(A))$ and $\nu_{\tau_1}(A) \geq \nu_{\tau_2}(f(A))$ [resp. $\mu_{\mathcal{C}_1}(A) \leq \mu_{\mathcal{C}_2}(f(A))$ and $\nu_{\mathcal{C}_1}(A) \geq \nu_{\mathcal{C}_2}(f(A))$], $\forall\ A \in I^X$.

Definition 3.12. Let (X, τ_1) and (Y, τ_2) be two *ists*. Then a mapping $f : X \to Y$ is called an intuitionistic smooth homeomorphism, if f is bijective, and f and f^{-1} are intuitionistic smooth continuous.

The following is the immediate result of Definitions 3.1, 3.11 and Theorem 3.6 (1).

Proposition 3.13. *Let X and Y be two sets and let $\{\delta_{(\lambda,\mu)}\}_{(\lambda,\mu)\in I_0\oplus I_1}$ and $\{\delta'_{(\lambda,\mu)}\}_{(\lambda,\mu)\in I_0\oplus I_1}$ be two descending families of fuzzy topologies on X and Y, respectively, and let $f : X \to Y$ be a mapping. If $f : (X, \delta_{(\lambda,\mu)}) \to (Y, \delta'_{(\lambda,\mu)})$ is fuzzy continuous [resp. open and closed] for each $(\lambda, \mu) \in I_0 \oplus I_1$, then $f : (X, \tau) \to (Y, \tau')$ is intuitionistic smooth continuous[resp. open and closed] w.r.t. the intuitionistic smooth topologies τ and τ' generated from the families $\{\delta_{(\lambda,\mu)}\}$ and $\{\delta'_{(\lambda,\mu)}\}$.*

4. Intuitionistic smooth subspace

Let $Y \subset X$ and let $A \in I^X$. Then the restriction of A on Y is denoted by $A\,|_Y$. For each $A \in I^Y$, the extension of A on X, denoted by A_X, is defined as follows: for each $x \in X$,

$$A_X(x) = \begin{cases} A(x) & \text{if } x \in Y, \\ 0 & \text{if } x \in X \setminus Y. \end{cases}$$

Proposition 4.1. *Let (X, τ) be an ists and let $Y \subset X$. We define the mapping $\tau_Y : I^Y \to I \oplus I$ as follows: for each $A \in I^Y$,*

$$\tau_Y(A) = (\bigvee_{B\in I^X,\ A=B|_Y} \mu_\tau(B), \bigwedge_{B\in I^X,\ A=B|_Y} \nu_\tau(B)).$$

Then τ_Y is an intuitionistic smooth topology on Y, $\mu_{\tau_Y}(A) \geq \mu_\tau(A_X)$ and $\nu_{\tau_Y}(A) \leq \nu_\tau(A_X)$, for each $A \in I^Y$.

In this case, the intuitionistic smooth topological space (Y, τ_Y) is called a subspace of (X, τ) and τ_Y is called the intuitionistic smooth topology on Y induced by τ.

Proof. For each $A \in I^Y$, let $A = B\,|_Y$ and $B \in I^X$. Since $\tau \in$IST(X), $\mu_\tau(B) \leq 1 - \nu_\tau(B)$. Thus

$$\bigvee_{B\in I^X,\ A=B|_Y} \mu_\tau(B) \leq \bigvee_{B\in I^X,\ A=B|_Y} (1 - \nu_\tau(B)) = 1 - \bigwedge_{B\in I^X,\ A=B|_Y} \nu_\tau(B).$$

So $\mu_{\tau_Y}(A) \leq 1 - \nu_{\tau_Y}(A)$. Hence $\tau_Y : I^Y \to I \oplus I$ is a mapping.

It is obvious that the condition (IST1) holds.

Let $A, B \in I^Y$. Then

$$\mu_{\tau_Y}(A) = \bigvee_{C\in I^X,\ A=C|_Y} \mu_\tau(C),\ \mu_{\tau_Y}(B) = \bigvee_{D\in I^X,\ B=D|_Y} \mu_\tau(D)$$

and

$$\nu_{\tau_Y}(A) = \bigwedge_{C\in I^X,\ A=C|_Y} \nu_\tau(C),\ \nu_{\tau_Y}(B) = \bigwedge_{D\in I^X,\ B=D|_Y} \nu_\tau(D).$$

Thus

$$\begin{aligned} \mu_{\tau_Y}(A \cap B) &= \bigvee_{C\cap D\in I^X,\ A\cap B=(C\cap D)|_Y} \mu_\tau(C \cap D) \\ &\geq \bigvee_{C\cap D\in I^X,\ A\cap B=(C\cap D)|_Y} [\mu_\tau(C) \wedge \mu_\tau(D)] \text{ [Since } \tau \in \text{IST}(X)] \\ &= (\bigvee_{C\in I^X,\ A=C|_Y} \mu_\tau(C)) \wedge (\bigvee_{D\in I^X,\ B=D|_Y} \mu_\tau(D))] \\ &= \mu_{\tau_Y}(A) \wedge \mu_{\tau_Y}(B). \end{aligned}$$

Similarly, we have $\nu_{\tau_Y}(A \cap B) \leq \nu_{\tau_Y}(A) \vee \nu_{\tau_Y}(B)$. So the condition (IST2) holds.

Now let $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^Y$. Then, for each $\alpha \in \Gamma$,

$$\mu_{\tau_Y}(A_\alpha) = \bigvee_{B_\alpha\in I^X,\ A_\alpha=B_\alpha|_Y} \mu_\tau(B_\alpha)$$

and

$$\nu_{\tau_Y}(A_\alpha) = \bigwedge_{B_\alpha\in I^X,\ A_\alpha=B_\alpha|_Y} \nu_\tau(B_\alpha).$$

Thus

$$\begin{aligned}\mu_{\tau_Y}(\bigcup_{\alpha\in\Gamma}A_\alpha) &= \bigvee_{\bigcup_{\alpha\in\Gamma}A_\alpha\in I^X,\ \bigcup_{\alpha\in\Gamma}B_\alpha=(\bigcup_{\alpha\in\Gamma}B_\alpha)|_Y}\mu_\tau(\bigcup_{\alpha\in\Gamma}B_\alpha)\\ &\geq \bigvee_{B_\alpha\in I^X,\ A_\alpha=B_\alpha|_Y}[\bigwedge_{\alpha\in\Gamma}\mu_\tau(B_\alpha)]\\ &\geq \bigwedge_{\alpha\in\Gamma}[\bigvee_{B_\alpha\in I^X,\ A_\alpha=B_\alpha|_Y}\mu_\tau(B_\alpha)]\\ &= \bigwedge_{\alpha\in\Gamma}\mu_{\tau_Y}(A_\alpha).\end{aligned}$$

Similarly, we have $\nu_{\tau_Y}(\bigcup_{\alpha\in\Gamma}A_\alpha)\leq\bigvee_{\alpha\in\Gamma}\nu_{\tau_Y}(A_\alpha)$. So the condition (IST3) holds. Hence $\tau_Y\in$IST(Y).

It is clear that $\mu_{\tau_Y}(A)\geq\mu_\tau(A_X)$ and $\nu_{\tau_Y}(A)\leq\nu_\tau(A_X)$, for each $A\in I^Y$ from the definition of τ_Y. □

Proposition 4.2. *Let (Y,τ_Y) be an intuitionistic smooth subspace of (X,τ) and let $A\in I^Y$. Then*

(1) $\mathcal{C}_{\tau_Y}(A)=(\bigvee_{B\in I^X,A=B|_Y}\mu_{\mathcal{C}_\tau}(B),\bigwedge_{B\in I^X,A=B|_Y}\nu_{\mathcal{C}_\tau}(B))$, *where* $\mathcal{C}_{\tau_Y}(A)=\tau_Y(A^c)$,

(2) *if* $Z\subset Y\subset X$, *then* $\tau_Z=(\tau_Y)_Z$.

Proof. (1) Let $A\in I^Y$. Then

$$\begin{aligned}\mu_{\mathcal{C}_{\tau_Y}}(A)&=\mu_{\tau_Y}(A^c)\\ &=\bigvee_{B\in I^X,\ A^c=B|_Y}\mu_\tau(B)\\ &=\bigvee_{B^c\in I^X,\ A=B^c|_Y}\mu_\tau(B)\\ &=\bigvee_{B^c\in I^X,\ A=B^c|_Y}\mu_{\mathcal{C}_\tau}(B^c)\\ &=\bigvee_{C\in I^X,\ A=C|_Y}\mu_{\tau_Y}(A).\end{aligned}$$

On the other hand,

$$\begin{aligned}\nu_{\mathcal{C}_{\tau_Y}}(A)&=\nu_{\tau_Y}(A^c)\\ &=\bigwedge_{B\in I^X,\ A^c=B|_Y}\nu_\tau(B)\\ &=\bigwedge_{B^c\in I^X,\ A=B^c|_Y}\nu_\tau(B)\\ &=\bigwedge_{B^c\in I^X,\ A=B^c|_Y}\nu_{\mathcal{C}_\tau}(B^c)\\ &=\bigwedge_{C\in I^X,\ A=C|_Y}\nu_{\mathcal{C}_{\tau_Y}}(C).\end{aligned}$$

Thus the result holds.

(2) Let $A\in I^Z$. Then

$$\begin{aligned}\mu_{(\tau_Y)_Z}(A)&=\bigvee_{B\in I^Y,\ A=B|_Z}\mu_{\tau_Y}(B)\\ &=\bigvee_{B\in I^Y,A=B|_Z}[\bigvee_{C\in I^X,\ B=C|_Y}\mu_\tau(C)]\\ &=\bigvee_{C\in I^X,\ A=C|_Z}\mu_\tau(C)\\ &=\mu_{\tau_Z}(A).\end{aligned}$$

On the other hand,

$$\begin{aligned}\nu_{(\tau_Y)_Z}(A) &= \bigwedge_{B\in I^Y,\ A=B|_Z} \nu_{\tau_Y}(B)\\ &= \bigwedge_{B\in I^Y, A=B|_Z} [\bigwedge_{C\in I^X, B=C|_Y} \nu_\tau(C)]\\ &= \bigvee_{C\in I^X,\ A=C|_Z} \nu_\tau(C)\\ &= \nu_{\tau_Z}(A).\end{aligned}$$

Thus $\tau_Z = (\tau_Y)_Z$. □

Proposition 4.3. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an intuitionistic smooth continuous mapping and let $Z \subset X$. Then the restriction mapping $f\ |_Z: (Z, (\tau_1)_Z) \to (Y, \tau_2)$ is also intuitionistic smooth continuous.*

Proof. Let $A \in I^Y$. Then

$$\begin{aligned}\mu_{(\tau_1)_Z}((f\ |_Z)^{-1}(A)) &= \bigvee\{\mu_{\tau_1}(B) : B \in I^X,\ (f\ |_Z)^{-1}(A) = B\ |_Z\}\\ &\geq \mu_{\tau_1}(f^{-1}(A))\\ &\geq \mu_{\tau_2}(A)\end{aligned}$$

and

$$\begin{aligned}\nu_{(\tau_1)_Z}((f\ |_Z)^{-1}(A)) &= \bigwedge\{\nu_{\tau_1}(B) : B \in I^X,\ (f\ |_Z)^{-1}(A) = B\ |_Z\}\\ &\leq \nu_{\tau_1}(f^{-1}(A))\\ &\leq \nu_{\tau_2}(A).\end{aligned}$$

Thus $f\ |_Z$ is intuitionistic smooth continuous. □

5. Neighborhood structures in intuitionistic smooth topological spaces

For a mapping $M : I^X \to I \oplus I$ and $(\lambda, \mu) \in I_1 \oplus I_0$, let us define the family

$$[M]^{(\lambda,\mu)} = \{A \in I^X : \mu_M(A) > \lambda \text{ and } \nu_M(A) < \mu\}.$$

Definition 5.1. Let (X, τ) be an *ists*. Then a mapping $\beta : I^X \to I \oplus I$ is called an intuitionistic smooth base of τ, if for each $(\lambda, \mu) \in I_1 \oplus I_0$, $[\tau]^{(\lambda,\mu)} = \bigcup\{\beta' : \beta' \subset [\beta]^{(\lambda,\mu)}\}$.

Theorem 5.2. *Let (X, τ) be an ists. Then a mapping $\beta : I^X \to I \oplus I$ is an intuitionistic smooth base of τ if and only if for each $(\lambda, \mu) \in I_1 \oplus I_0$, if $x_\lambda \in \mathrm{F_P(X)}$ and $A \in \tau^{(\lambda,\mu)}$ such that $x_\lambda q A$, then there exists a $B \in [\beta]^{(\lambda,\mu)}$ such that $x_\lambda q B \subset A$.*

Proof. ($\Rightarrow$): Suppose a mapping $\beta : I^X \to I \oplus I$ is an intuitionistic smooth base of τ. Then, by Definition 5.1,

$$[\tau]^{(\lambda,\mu)} = \bigcup\{\beta' : \beta' \subset [\beta]^{(\lambda,\mu)}\},\ \text{for each } (\lambda, \mu) \in I_1 \oplus I_0.$$

Let $x_\lambda \in \mathrm{F_P(X)}$ and let $A \in I^X$ such that $x_\lambda q A$ and $A \in [\tau]^{(\lambda,\mu)}$. Then there exists a $\beta_0 \subset [\beta]^{(\lambda,\mu)}$ such that $A = \bigcup \beta_0$. Thus $x_\lambda q(\bigcup \beta_0)$. So, by Result 2.9, $x_\lambda q B$, for some $B \in \beta_0$. Hence the necessary condition holds.

($\Leftarrow$): Suppose the necessary condition holds. Assume that the mapping $\beta : I^X \to I \oplus I$ is not an intuitionistic smooth base of τ. Then there exist $(\lambda_0, \mu_0) \in I_1 \oplus I_0$ and $A \in [\tau]^{(\lambda_0,\mu_0)}$ such that $A \neq \bigcup \beta'$ for all $\beta' \subset [\beta]^{(\lambda_0,\mu_0)}$. Let $\beta^* = \{B \in [\beta]^{(\lambda_0,\mu_0)} : B \subset A\}$ and let $G = \bigcup \beta^*$. Then clearly $A \neq G$. Thus there exists a $x \in X$ such that $G(x) < A(x)$. Let $\alpha = 1 - G(x)$. Then clearly $x_\alpha \in F_P(X)$. Moreover, $1 = G(x) + \alpha < A(x) + \alpha$. Thus $x_\alpha q A$.

On the other hand, $B \subset G$ for each $B \in \beta^*$. Then

$$B(x) + \alpha \leq G(x) + \alpha = 1.$$

Thus $x_\alpha \overline{q} B$ for each $B \subset A$ and $B \in [\beta]^{(\lambda_0,\mu_0)}$. This is a contradiction. So β is an intuitionistic smooth base of τ. □

Theorem 5.3. *Let (X,τ) be an ists. Then a mapping $\beta : I^X \to I \oplus I$ is an intuitionistic smooth base of τ if and only if for each $A \in I^X$ and $p \in \mathrm{F_P(X)}$ such that pqA, $\mu_\tau(A) \leq \bigvee\{\mu_\beta(B) : B \in I^X \text{ and } pqB \subset A\}$ and $\nu_\tau(A) \geq \bigwedge\{\nu_\beta(B) : B \in I^X \text{ and } pqB \subset A\}$.*

Proof. (⇒): Suppose a mapping $\beta : I^X \to I \oplus I$ is an intuitionistic smooth base of τ. Let $p \in \mathrm{F_P(X)}$ and let $A \in I^X$ such that pqA.

Suppose $\tau(A) = (0,1)$. Then clearly the required inequalities hold.

Suppose $\tau(A) \neq (0,1)$, i.e., $\mu_\tau(A) = \lambda > 0$ and $\nu_\tau(A) = \mu < 1$, where $\lambda + \mu \leq 1$. Let ϵ_1 and ϵ_2 be arbitrary $0 < \epsilon_1 \leq \lambda$ and $\mu \leq \epsilon_2 < 1$. Then

$$\mu_\tau(A) > \lambda - \epsilon_1,\ \nu_\tau(A) < \mu + \epsilon_2 \text{ and } (\lambda - \epsilon_1) + (\mu + \epsilon_2) \leq 1.$$

Thus $A \in [\tau]^{(\lambda-\epsilon_1,\mu+\epsilon_2)}$. By Theorem 5.2, there exists $B \in [\beta]^{(\lambda-\epsilon_1,\mu+\epsilon_2)}$ such that $pqB \subset A$. So

$$\bigvee\{\mu_\beta(B) : B \in I^X, pqB \subset A\} > \lambda - \epsilon_1$$

and

$$\bigwedge\{\nu_\beta(B) : B \in I^X, pqB \subset A\} < \mu + \epsilon_2.$$

Since ϵ_1 and ϵ_2 are arbitrary,

$$\bigvee\{\mu_\beta(B) : B \in I^X, pqB \subset A\} \geq \lambda = \mu_\tau(A)$$

and

$$\bigwedge\{\nu_\beta(B) : B \in I^X, pqB \subset A\} \leq \mu = \nu_\tau(A).$$

So, in either cases, the required inequalities hold.

(⇐): Suppose the necessary condition holds. Let $e \in \mathrm{F_P(X)}$ and let $U \in [\tau]^{(\lambda,\mu)}$ such that eqU, where $(\lambda,\mu) \in I_1 \oplus I_0$. Then $\mu_\tau(U) > \lambda$ and $\nu_\tau(U) < \mu$. By the hypothesis,

$$\lambda < \mu_\tau(U) \leq \bigvee\{\mu_\beta(B) : B \in I^X \text{ and } eqB \subset U\}$$

and

$$\mu > \nu_\tau(U) \geq \bigwedge\{\nu_\beta(B) : B \in I^X \text{ and } eqB \subset U\}.$$

Thus there exists a $B \in I^X$ such that $eqB \subset U$, $\mu_\beta(U) > \lambda$ and $\nu_\beta(U) < \beta$. So $B \in [\beta]^{(\lambda,\mu)}$ and $eqB \subset U$. Hence, by Theorem 5.2, β is an intuitionistic smooth base of τ. □

Definition 5.4. Let (X,τ) be an *ists* and let $p \in \mathrm{F_P(X)}$ be fixed.

(i) A mapping $\mathcal{N}_p : I^X \to I \oplus I$ is called the intuitionistic smooth neighborhood system of p w.r.t. τ, if for each $(\lambda,\mu) \in I_1 \oplus I_0$,

$$[\mathcal{N}_p]^{(\lambda,\mu)} = \{A \in I^X : (\exists\, T \in [\tau]^{(\lambda,\mu)})\ (p \in T \subset A)\}.$$

The real number $\mu_{\mathcal{N}_p}(A)$ [resp. $\nu_{\mathcal{N}_p}(A)$] is called the degree of neighborhoodness [resp. non-neighborhoodness] of A to p.

(ii) A mapping $Q_p : I^X \to I \oplus I$ is called the intuitionistic smooth $Q-$neighborhood system of p w.r.t. τ, if for each $(\lambda,\mu) \in I_1 \oplus I_0$,

$$[Q_p]^{(\lambda,\mu)} = \{A \in I^X : (\exists\, T \in [\tau]^{(\lambda,\mu)})\ (pqT \subset A)\}.$$

The real number $\mu_{Q_p}(A)$ [resp. $\nu_{Q_p}(A)$] is called the degree of $Q-$neighborhoodness [resp. non-Q-neighborhoodness] of A to p.

Theorem 5.5. *Let (X,τ) be an ists and let $p \in F_P(X)$ be fixed.*

(1) A mapping $\mathcal{N}_p : I^X \to I \oplus I$ is the intuitionistic smooth neighborhood system of p w.r.t. τ if and only if for each $A \in I^X$,

$$\mathcal{N}_p(A) = \begin{cases} (\bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\}, & \text{if } p \in A, \\ \bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\}) & \\ (0,1) & \text{if } p \notin A. \end{cases}$$

(2) A mapping $Q_p : I^X \to I \oplus I$ is the intuitionistic smooth $Q-$neighborhood system of p w.r.t. τ if and only if for each $A \in I^X$,

$$Q_p(A) = \begin{cases} (\bigvee\{\mu_\tau(V) : V \in I^X, pqV \subset A\}, & \text{if } pqA, \\ \bigwedge\{\nu_\tau(V) : V \in I^X, pqV \subset A\}) & \\ (0,1) & \text{if } p\bar{q}A. \end{cases}$$

Proof. (1) ($\Rightarrow$): Suppose a mapping $\mathcal{N}_p : I^X \to I \oplus I$ is the intuitionistic smooth neighborhood system of p w.r.t. τ. Let $A \in I^X$.

Case(i): Suppose $p \notin A$. Assume that $\mathcal{N}_p(A) \neq (0,1)$, i.e., $\mu_{\mathcal{N}_p}(A) > 0$ or $\nu_{\mathcal{N}_p}(A) < 1$. Then, by the hypothesis and Definition 5.4 (i), there exists a $T \in [\tau]^{(0,1)}$ such that $p \in T \subset A$. Thus $p \in A$. This is a contradiction. So $\mathcal{N}_p(A) = (0,1)$.

Case(ii): Suppose $p \in A$. Then we may have $\mathcal{N}_p(A) = (0,1)$ or $\mathcal{N}_p(A) \neq (0,1)$.
If $\mathcal{N}_p(A) = (0,1)$, then it is obvious that

$$\mu_{\mathcal{N}_p}(A) = 0 \leq \bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\}$$

and

$$\nu_{\mathcal{N}_p}(A) = 1 \geq \bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\}.$$

Furthermore, suppose that

$$\bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\} = \lambda > 0$$

and

$$\bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\} = \mu < 1.$$

Then there exists a $V \in I^X$ such that $\mu_\tau(V) > 0, \nu_\tau(V) < 1$ and $p \in V \subset A$.
Thus, by the hypothesis and Definition 5.4 (i), $A \in [\mathcal{N}_p]^{(0,1)}$. So $\mathcal{N}_p(A) \neq (0,1)$. This is a contradiction. Hence

$$\begin{aligned} &\mathcal{N}_p(A) \\ &= (\bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\}, \bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\}) \\ &= (0,1). \end{aligned}$$

Suppose $\mathcal{N}_p(A) \neq (0,1)$, i.e., $\mu_{\mathcal{N}_p}(A) = \lambda > 0$ or $\nu_{\mathcal{N}_p}(A) = \mu < 1$. Let ϵ_1 and ϵ_2 be arbitrary $0 < \epsilon_1 \leq \lambda$ and $\mu \leq \epsilon_2 < 1$. Then $\mu_{\mathcal{N}_p}(A) > \lambda - \epsilon_1$ and $\nu_{\mathcal{N}_p}(A) < \mu + \epsilon_2$. Thus $A \in [\mathcal{N}_p]^{(\lambda-\epsilon_1, \mu+\epsilon_2)}$. By the hypothesis, there exists a $T \in [\tau]^{(\lambda-\epsilon_1,\mu+\epsilon_2)}$ such that $p \in T \subset A$. So

$$\bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\} > \lambda - \epsilon_1$$

and

$$\bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\} < \mu + \epsilon_2.$$

Since ϵ_1 and ϵ_2 are arbitrary,

$$\bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\} = \mu_{\mathcal{N}_p}(A) \tag{5.5.1}$$

and

$$(5.5.2) \qquad \bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\} < \mu = \nu_{\mathcal{N}_p}(A).$$

On the other hand, let

$$\bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\} = \alpha > 0$$

and

$$\bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\} = \beta < 1.$$

Let ϵ_1 and ϵ_2 be arbitrary $0 < \epsilon_1 \leq \alpha$ and $\beta \leq \epsilon_2 < 1$. Then there exists a $V \in I^X$ such that $\mu_\tau(V) > \alpha - \epsilon_1, \nu_\tau(V) < \beta + \epsilon_2$ and $p \in V \subset A$. Thus $V \in [\tau]^{(\alpha-\epsilon_1,\beta+\epsilon_2)}$ and $p \in V \subset A$. So by the hypothesis, $A \in [\mathcal{N}_p]^{(\alpha-\epsilon_1,\beta+\epsilon_2)}$, i.e., $\mu_{\mathcal{N}_p}(A) > \alpha - \epsilon_1$ and $\nu_{\mathcal{N}_p}(A) < \beta + \epsilon_2$. Since ϵ_1 and ϵ_2 be arbitrary,

$$(5.5.3) \qquad \mu_{\mathcal{N}_p}(A) \geq \alpha = \bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\}$$

and

$$(5.5.4) \qquad \nu_{\mathcal{N}_p}(A) \leq \beta = \bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\}.$$

Hence, by (5.5.1), (5.5.2), (5.5.3) and (5.5.4),

$$\mathcal{N}_p(A) = (\bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\}, \bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\}).$$

This completes the proof of the necessity.

($\Leftarrow$): Suppose the necessary condition holds. For each $(\lambda, \mu) \in I_1 \oplus I_0$, $U \in [\mathcal{N}_p]^{(\lambda,\mu)}$, i.e., $\mu_{\mathcal{N}_p}(U) > \lambda$ and $\nu_{\mathcal{N}_p}(U) < \mu$. Then, by the hypothesis,

$$\lambda < \mu_{\mathcal{N}_p}(U) = \bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\}$$

and

$$\mu > \nu_{\mathcal{N}_p}(U) = \bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\}.$$

Thus there exists a $V \in I^X$ such that $\mu_\tau(V) > \lambda, \nu_\tau(V) < \mu$ and $p \in V \subset U$. So $V \in [\tau]^{(\lambda,\mu)}$ and $p \in V \subset A$. Hence $\mathcal{N}_p{}^{(\lambda,\mu)} \subset \{U \in I^X : (\exists\, T \in \tau^{(\lambda,\mu)})\ (p \in T \subset U)\}$.

On the other hand, suppose there exist $(\lambda, \mu) \in I_1 \oplus I_0$ and $T \in [\tau]^{(\lambda,\mu)}$ such that $p \in T \subset U$. Then

$$\mu_{\mathcal{N}_p}(U) = \bigvee\{\mu_\tau(V) : V \in I^X, p \in V \subset A\} > \lambda$$

and

$$\nu_{\mathcal{N}_p}(U) = \bigwedge\{\nu_\tau(V) : V \in I^X, p \in V \subset A\} < \mu.$$

Thus $U \in [\mathcal{N}_p]^{(\lambda,\mu)}$. So $\{U \in I^X : (\exists\, T \in [\tau]^{(\lambda,\mu)})\ (p \in T \subset U)\} \subset [\mathcal{N}_p]^{(\lambda,\mu)}$.
Hence $\mathcal{N}_p{}^{(\lambda,\mu)} = \{U \in I^X : (\exists\, T \in [\tau]^{(\lambda,\mu)})\ (p \in T \subset U)\}$. Therefore $\mathcal{N}_p$ is the intuitionistic smooth neighborhood system of p w.r.t. τ.

(2) The proof is similar to (1). □

In the following result, we show that the intuitionistic smooth $Q-$neighborhood system of a fuzzy point w.r.t. an ist τ can be given exactly in terms of an intuitionistic smooth base of such τ.

Theorem 5.6. *Let (X,τ) be an ists, let a mapping $\beta : I^X \to I \oplus I$ be an intuitionistic smooth base of τ and let $p \in \mathrm{F_P(X)}$ be fixed. Then a mapping $Q_p : I^X \to I \oplus I$ is the intuitionistic smooth $Q-$neighborhood system of p w.r.t. τ if and only if*

$$Q_p(U) = \begin{cases} (\bigvee\{\mu_\beta(B) : B \in I^X, pqB \subset U\}, & \text{if } pqU \\ \ \bigwedge\{\nu_\beta(B) : B \in I^X, pqB \subset U\}) & \\ (0,1) & \text{if } \ p\bar{q}U, \end{cases}$$

for each $U \in I^X$.

Proof. By Theorem 5.2 and Definition 5.4 (ii), Q_p is the intuitionistic smooth $Q-$ neighborhood system of p w.r.t. τ if and only if for each $(\lambda,\mu) \in I_1 \oplus I_0$, $[Q_p]^{(\lambda,\mu)} = \{U \in I^X : (\exists\ B \in [\beta]^{(\lambda,\mu)})\ (pqB \subset U)\}$. By using this fact, we complete the proof in a way similar to that of Theorem 5.5 (2). □

Proposition 5.7. *Let (X,τ) be an ists and let $p \in \mathrm{F_P(X)}$ be fixed. If a mapping $\mathcal{N}_p : I^X \to I \oplus I$ is the intuitionistic smooth neighborhood system of p w.r.t. τ, then the followings hold:*

(IN1) *if $\mu_{\mathcal{N}_p}(U) > 0$ and $\nu_{\mathcal{N}_p}(U) < 1$, where $U \in I^X$, then $p \in U$,*

(IN2) $(\bigvee\{\mu_{\mathcal{N}_p}(U) : U \in I^X\},\ \bigwedge\{\nu_{\mathcal{N}_p}(U) : U \in I^X\}) = (1,0)$.

(IN3 *for any $U_1, U_2 \in I^X$,*

$$\mu_{\mathcal{N}_p}(U_1 \cap U_2) \geq \mu_{\mathcal{N}_p}(U_1) \wedge \mu_{\mathcal{N}_p}(U_2)$$

and

$$\nu_{\mathcal{N}_p}(U_1 \cap U_2) \leq \nu_{\mathcal{N}_p}(U_1) \vee \nu_{\mathcal{N}_p}(U_2),$$

(IN4) *If $U_1 \subset U_2$ and $U_1, U_2 \in I^X$, then $\mu_{\mathcal{N}_p}(U_1) \leq \mu_{\mathcal{N}_p}(U_2)$ and $\nu_{\mathcal{N}_p}(U_1) \geq \nu_{\mathcal{N}_p}(U_2)$.*

(IN5) *for each $U \in I^X$,*

$$\mu_{\mathcal{N}_p}(U) \leq \bigvee\{\mu_{\mathcal{N}_p}(V) \wedge (\bigwedge_{e \in V} \mu_{\mathcal{N}_e}(V)) : V \in I^X, V \subset U\}$$

and

$$\nu_{\mathcal{N}_p}(U) \geq \bigwedge\{\nu_{\mathcal{N}_p}(V) \vee (\bigvee_{e \in V} \nu_{\mathcal{N}_e}(V)) : V \in I^X, V \subset U\}.$$

Proof. (IN1), (IN2) and (IN4) follow directly from Theorem 5.5 (1).

(IN3) Let $U_1, U_2 \in I^X$. If $\mathcal{N}_p(U_1) = (0,1)$ or $\mathcal{N}_p(U_2) = (0,1)$, then the required inequalities are obvious. Now let us suppose $\mu_{\mathcal{N}_p}(U_1) = \lambda_1 > 0$, $\nu_{\mathcal{N}_p}(U_1) = \mu_1 < 1$ and $\mu_{\mathcal{N}_p}(U_2) = \lambda_2 > 0$, $\nu_{\mathcal{N}_p}(U_2) = \mu_2 < 1$. Let $\epsilon_1 > 0$ and $\epsilon_2 > 0$ be arbitrary such that $\epsilon_1 \leq \lambda_1 \wedge \lambda_2$ and $\epsilon_2 \geq \mu_1 \vee \mu_2$. Then

$$\mu_{\mathcal{N}_p}(U_1) > \lambda_1 - \epsilon_1 \geq 0,\ \nu_{\mathcal{N}_p}(U_1) < \mu_1 + \epsilon_2 \leq 1$$

and

$$\mu_{\mathcal{N}_p}(U_2) > \lambda_2 - \epsilon_1 \geq 0,\ \nu_{\mathcal{N}_p}(U_2) < \mu_2 + \epsilon_2 \leq 1.$$

Thus, by Definition 5.4 (i), there exist $T_1, T_2 \in I^X$ such that

$$\mu_\tau(T_1) > \lambda_1 - \epsilon_1,\ \nu_\tau(T_1) < \mu_1 + \epsilon_2,\ p \in T_1 \subset U_1$$

and

$$\mu_\tau(T_2) > \lambda_2 - \epsilon_1,\ \nu_\tau(T_2) < \mu_2 + \epsilon_2,\ p \in T_2 \subset U_2.$$

So,

$$\mu_\tau(T_1 \cap T_2) \geq \mu_\tau(T_1) \wedge \mu_\tau(T_2) \text{ [Since } \tau \in \mathrm{IST}(X)]$$

$$\begin{aligned} &> (\lambda_1 - \epsilon_1) \wedge (\lambda_2 - \epsilon_1) \\ &= (\lambda_1 \wedge \lambda_2) - \epsilon_1, \\ \nu_\tau(T_1 \cap T_2) &\leq \nu_\tau(T_1) \vee \nu_\tau(T_2) \\ &< (\mu_1 + \epsilon_2) \vee (\mu_2 + \epsilon_2) \\ &= (\mu_1 \vee \mu_2) + \epsilon_2, \end{aligned}$$

and

$$p \in T_1 \cap T_2 \subset U_1 \wedge U_2.$$

By the hypothesis,

$$\mu_{\mathcal{N}_p}(U_1 \cap U_2) > (\lambda_1 \wedge \lambda_2) - \epsilon_1$$

and

$$\nu_{\mathcal{N}_p}(U_1 \cap U_2) < (\mu_1 \vee \mu_2) + \epsilon_2.$$

Since ϵ_1 and ϵ_2 are arbitrary,

$$\mu_{\mathcal{N}_p}(U_1 \cap U_2) \geq \lambda_1 \wedge \lambda_2 = \mu_{\mathcal{N}_p}(U_1) \wedge \mu_{\mathcal{N}_p}(U_2)$$

and

$$\nu_{\mathcal{N}_p}(U_1 \cap U_2) \leq \mu_1 \vee \mu_2 = \nu_{\mathcal{N}_p}(U_1) \vee \nu_{N_p}(U_2).$$

(IN5) Let $U \in I^X$. If $\mathcal{N}_p(U) = (0, 1)$, then the required inequalities are obvious. Let us suppose $\mathcal{N}_p(U) \neq (0, 1)$, i.e., $\mu_{\mathcal{N}_p}(U) = \lambda > 0$ and $\nu_{\mathcal{N}_p}(U) = \mu < 1$. Let $\epsilon_1 > 0$ and $\epsilon_2 > 0$ be arbitrary such that $\epsilon_1 \leq \lambda$ and $\epsilon_2 \geq \mu$. Then

$$\mu_{\mathcal{N}_p}(U) > \lambda - \epsilon_1 \text{ and } \nu_{\mathcal{N}_p}(U) < \mu + \epsilon_2.$$

Thus, by Definition 5.4 (i), there exists $V_0 \in I^X$ such that

$$\mu_\tau(V_0) > \lambda - \epsilon_1,\ \nu_\tau(V_0) < \mu + \epsilon_2 \text{ and } p \in V_0 \subset U.$$

Since $V_0 \subset V_0$, $\mu_{\mathcal{N}_e}(V_0) > \lambda - \epsilon_1$ and $\nu_{\mathcal{N}_e}(V_0) < \mu + \epsilon_2$ for each $e \in V_0$. So

$$\bigwedge_{e \in V_0} \mu_{\mathcal{N}_e}(V_0) \geq \lambda - \epsilon_1 \text{ and } \bigvee_{e \in V_0} \nu_{\mathcal{N}_e}(V_0) \geq \lambda + \epsilon_2.$$

On the other hand, in particular, $\mu_{\mathcal{N}_p}(V_0) > \lambda - \epsilon_1$ and $\nu_{\mathcal{N}_p}(V_0) < \mu + \epsilon_2$. Thus

$$\begin{aligned} &\bigvee\{\mu_{\mathcal{N}_p}(V) \wedge (\bigwedge_{e \in V} \mu_{\mathcal{N}_e}(V)) : V \in I^X, V \subset U\} \\ &\geq \mu_{\mathcal{N}_p}(V_0) \wedge (\bigwedge_{e \in V_0} \mu_{\mathcal{N}_e}(V_0)) \\ &\geq \lambda - \epsilon_1 \end{aligned}$$

and

$$\begin{aligned} &\bigwedge\{\nu_{\mathcal{N}_p}(V) \vee (\bigvee_{e \in V} \nu_{\mathcal{N}_e}(V)) : V \in I^X, V \subset U\} \\ &\leq \nu_{\mathcal{N}_p}(V_0) \vee (\bigvee_{e \in V_0} \nu_{\mathcal{N}_e}(V_0)) \\ &\leq \mu + \epsilon_2. \end{aligned}$$

Since ϵ_1 and ϵ_2 are arbitrary, the required inequalities follow. This complete the proof. □

Proposition 5.8. *Let (X, τ) be an ists and let $p \in F_P(X)$ be fixed. If a mapping $Q_p : I^X \to I \oplus I$ is the intuitionistic smooth $Q-$neighborhood system of p w.r.t. τ, then the following hold:*

(IQ1) *for each $U \in I^X$, if $\mu_{Q_p}(U) > 0$ and $\nu_{Q_p}(U) < 1$, then pqU,*

(IQ2) $(\bigvee\{\mu_{Q_p}(U) : U \in I^X\}, \bigwedge\{\nu_{Q_p}(U) : U \in I^X\}) = (1, 0)$,
(IQ3) *For any* $U_1, U_2 \in I^X$,

$$\mu_{Q_p}(U_1 \cap U_2) \geq \mu_{Q_p}(U_1) \wedge \mu_{Q_p}(U_2)$$

and

$$\nu_{Q_p}(U_1 \cap U_2) \leq \nu_{Q_p}(U_1) \vee \nu_{Q_p}(U_2),$$

(IQ4) *if* $U_1 \subset U_2$ *and* $U_1, U_2 \in I^X$, *then*

$$\mu_{Q_p}(U_1) \leq \mu_{Q_p}(U_2) \text{ and } \nu_{Q_p}(U_1) \geq \nu_{Q_p}(U_2),$$

(IQ5) *for each* $U \in I^X$,

$$\mu_{Q_p}(U) \leq \bigvee\{\mu_{Q_p}(V) \wedge (\bigwedge_{eqV})\nu_{Q_e}(U) : V \in I^X, V \subset U\}$$

and

$$\nu_{Q_p}(U) \geq \bigwedge\{\nu_{Q_p}(V) \vee (\bigvee_{eqV})\nu_{Q_e}(U) : V \in I^X, V \subset U\}.$$

Proof. The proof is similar to Proposition 5.7. □

Proposition 5.9. *Let the mapping* $Q_p : I^X \to I \oplus I$ *satisfies the conditions* (IQ1)-(IQ5). *We define the mapping* $\tau : I^X \to I \oplus I$ *as follows: for each* $U \in I^X$,

$$\tau(U) = \begin{cases} (\bigwedge_{eqV} \mu_{Q_e}(U), \bigvee_{eqV} \nu_{Q_e}(U)) & \text{if } U \in I^X \setminus \{\boldsymbol{\alpha} : \alpha \in I_1\}, \\ (1, 0) & \text{if } U \in \{\boldsymbol{\alpha} : \alpha \in I_1\}. \end{cases}$$

Then $\tau \in$*IST(X). Furthermore, the mapping* Q_p *is unique the intuitionistic smooth* $Q-$*neighborhood system of* p *w.r.t.* τ.

Proof. From the definition of τ, it is clear that $\mu_\tau(U) + \nu_\tau(U) \leq 1$, for each $U \in I^X$. Then $\tau : I^X \to I \oplus I$ is a mapping. It is obvious that $\tau(\boldsymbol{\alpha}) = (1, 0)$, for each $\alpha \in I_1$. By (IQ1) and (IQ4), for each $e \in F_P(X)$,

$$\mu_\tau(\mathbf{1}) = \bigwedge_{eqU} \mu_{Q_e}(U) = \mu_{Q_e}(\mathbf{1}) = 1$$

and

$$\nu_\tau(\mathbf{1}) = \bigvee_{eqU} \nu_{Q_e}(U) = \nu_{Q_e}(\mathbf{1}) = 0.$$

Thus $\tau(\mathbf{1}) = (1, 0)$. So the condition (IST1) holds.

Let $U_1, U_2 \in I^X$. If $U_1 \cap U_2 = \mathbf{0}$, then it is obvious that

$$\mu_\tau(U_1 \cap U_2) = 1 \geq \mu_\tau(U_1) \wedge \mu_\tau(U_2)$$

and

$$\nu_\tau(U_1 \cap U_2) = 0 \leq \nu_\tau(U_1) \vee \nu_\tau(U_2).$$

Let us suppose $U_1 \cap U_2 \neq \mathbf{0}$. Then, by the proof of Proposition 2.7 in [14],

$$\mu_\tau(U_1 \cap U_2) \geq \mu_\tau(U_1) \wedge \mu_\tau(U_2).$$

On the other hand,

$$\nu_\tau(U_1 \cap U_2) = \bigvee_{eq(U_1 \cap U_2)} \nu_{Q_e}(U_1 \cap U_2)$$

$$\leq \bigvee_{eq(U_1\cap U_2)} [\nu_{Q_e}(U_1) \vee \nu_{Q_e}(U_2)] \text{ [By the condition (IQ3)]}$$
$$= (\bigvee_{eq(U_1\cap U_2)} \nu_{Q_e}(U_1)) \vee (\bigvee_{eq(U_1\cap U_2)} \nu_{Q_e}(U_2))$$
$$\leq (\bigvee_{eqU_1} \nu_{Q_e}(U_1)) \vee (\bigvee_{eqU_2} \nu_{Q_e}(U_2))$$
$$= \nu_\tau(U_1) \vee \nu_\tau(U_2).$$

So the condition (IST2) holds.

Now let $\{U_\alpha\}_{\alpha\in\Gamma} \subset I^X$. If $\bigcup_{\alpha\in\Gamma} U_\alpha = \mathbf{0}$, then it is obvious that

$$\mu_\tau(\bigcup_{\alpha\in\Gamma} U_\alpha) = 1 \geq \bigwedge_{\alpha\in\Gamma} \mu_\tau(U_\alpha)$$

and

$$\nu_\tau(\bigcup_{\alpha\in\Gamma} U_\alpha) = 0 \leq \bigvee_{\alpha\in\Gamma} \nu_\tau(U_\alpha).$$

Suppose $\bigcup_{\alpha\in\Gamma} U_\alpha \neq \mathbf{0}$. Then, by the proof of Proposition 2.7 in [14],

$$\mu_\tau(\bigcup_{\alpha\in\Gamma} U_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mu_\tau(U_\alpha).$$

On the other hand,

$$\nu_\tau(\bigcup_{\alpha\in\Gamma} U_\alpha) = \bigvee_{eq(\bigcup_{\alpha\in\Gamma} U_\alpha)} \nu_{Q_e}(\bigcup_{\alpha\in\Gamma} U_\alpha)$$
$$\leq \bigvee_{eqU_{\alpha_0}} \nu_{Q_e}(U_{\alpha_0}) \text{ [By the condition (IQ4) and Result 2.6]}$$
$$= \nu_\tau(U_{\alpha_0}).$$

Thus $\nu_\tau(\bigcup_{\alpha\in\Gamma} U_\alpha) \leq \bigvee_{\alpha\in\Gamma} \nu_\tau(U_\alpha)$.

So the condition (IST3) holds. Hence $\tau \in$IST(X).

Now we show the mapping $Q_p : I^X \to I \oplus I$ satisfying the condition (IQ1)-(IQ5) is unique the intuitionistic smooth $Q-$neighborhood system of p w.r.t. τ.

Let a mapping $M_p : I^X \to I \oplus I$ be the another intuitionistic smooth $Q-$ neighborhood system of p w.r.t. τ. Then, by the proof of Proposition 2.7 in [14], $\mu_{Q_p} = \mu_{M_p}$. Thus it is sufficient to show that $\nu_{Q_p} = \nu_{M_p}$.

Let $U \in I^X$. If $p\bar{q}U$, then, by Theorem 5.5 (2) and (IQ1),

$$\nu_{M_p}(U) = 0 = \nu_{Q_p}(U). \tag{5.9.1}$$

Let us suppose pqU. Then

$$\nu_{M_p}(U) = \bigwedge\{\nu_\tau(V) : V \in I^X, pqV \subset U\} \text{ [By Theorem 5.5 (2)]}$$
$$= \bigwedge\{\bigvee_{eqV} \nu_{Q_e}(V) : V \in I^X, pqV \subset U\} \text{ [By the definition of } \tau]$$
$$\geq \bigwedge\{\mu_{Q_p}(V) : V \in I^X, pqV \subset U\} \text{ [Since } \bigvee_{eqV} \nu_{Q_e}(V) \geq \nu_{Q_P}(V)]$$
$$\geq \nu_{Q_e}(U). \text{ [By the condition (IQ4)]}$$

Thus

$$\nu_{M_p}(U) \geq \nu_{Q_p}(U), \text{ for each } U \in I^X \text{ such that } pqU. \tag{5.9.2}$$

On the other hand,

$$\begin{aligned}
&\nu_{Q_p}(U)\\
&\geq \bigwedge\{\nu_{Q_p}(V)\vee(\bigvee_{eqV}\nu_{Q_e}(V)) : V\in I^X, V\subset U\} \text{ [By the condition (IQ5)]}\\
&= [\bigwedge\{\nu_{Q_p}(V)\vee(\bigvee_{eqV}\nu_{Q_e}(V)) : V\in I^X, pqV, V\subset U\}]\\
&\quad \wedge[\bigwedge\{\nu_{Q_p}(V)\vee(\bigvee_{eqV}\nu_{Q_e}(V)) : V\in I^X, p\overline{q}V, V\subset U\}]\\
&= \bigwedge\{\nu_{Q_p}(V)\vee(\bigvee_{eqV}\nu_{Q_e}(V)) : V\in I^X, pqV, V\subset U\}\\
&\quad \text{[By the condition (IQ1), } \nu_{Q_P}(V)=1 \text{ for } p\overline{q}V]\\
&\geq \bigwedge\{\bigvee_{eqV}\nu_{Q_e}(V) : V\in I^X, pqV, V\subset U\}\\
&= \nu_{M_p}(U).
\end{aligned}$$

So

(5.9.3) $$\nu_{Q_p}(U)\geq\nu_{M_p}(U), \text{ for each } U\in I^X \text{ with } pqU.$$

By (5.9.1), (5.9.2) and (5.9.3), $\nu_{Q_p}=\nu_{M_p}$. Hence $M_p=Q_p$. Therefore Q_p is unique. □

6. Conclusions

We introduced the concept of an intuitionistic smooth continuity and obtained some results (in particular, see Theorems 3.8, 3.9 and 3.10). Next, we defined the intuitionistic smooth subspace of an *ists* and obtained two properties. Finally, we introduced the concept of an intuitionistic smooth base and obtained its characterizations (See Theorems 5.2 and 5.2). Furthermore, we define the intuitionistic smooth neighborhood system and the intuitionistic smooth Q-neighborhood system, and obtained their characterizations (See Theorem 5.5).

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