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Fuzzy annihilator ideals in distributive lattices

Berhanu Assaye Alaba, Wondwosen Zemene Norahun*

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Abstract. In this paper, we introduce the notion of relative fuzzy annihilator ideals in a distributive lattice. It is proved that the set of all fuzzy annihilator ideals of a distributive lattice L forms a complete Boolean algebra. The concept of fuzzy annihilator preserving homomorphism in a distributive lattice also presented in this paper.

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1. Introduction

J. Hashimoto [3] developed a theory of lattice ideals making effort to evolve this algebraic theory like that of rings. Latter G. Gratzer and E. T. Schmidit [2] have studied on ideal theory for latices. In 1970, M. Mandelker [6] studied relative annihilators and characterized distributive lattices with the help of these annihilators.

On the other side, L. A. Zadeh [12] mathematically formulated the fuzzy subset concept. A. Rosenfeld has developed the concept of fuzzy subgroup [8], W. J. Liu initiated the study of fuzzy subrings, and fuzzy ideals of a ring [5]. B. Yuan and W. Wu [11] introduced the notion of fuzzy ideals and fuzzy congruences of distributive lattices. Latter, U. M. Swamy and D. V. Raju studied properties of fuzzy ideals and congruences of lattices [10]. In recent time, H. K. Saika and M. C. Kalita studied on annihilator of fuzzy subsets of modules [9].

In this paper, we introduce the notion of fuzzy annihilator ideals in distributive lattices and study basic properties of fuzzy annihilators. It is proved that the set of all fuzzy annihilator ideals of a distributive lattice L forms a complete Boolean algebra. The concept of fuzzy annihilator preserving homomorphism in a distributive lattice also presented in this paper. A sufficient condition for a homomorphism to be fuzzy annihilator preserving is derived. In addition, we prove that the image and pre-image of fuzzy annihilator ideals are also fuzzy annihilator ideals.

2. Preliminaries

Throughout this paper L stands for the distributive lattice with least element 0 unless it is specified. We refer to G. Birkhoff [1] for the elementary properties of lattices.

Remember that, for any set A, a function $\mu : A \to ([0,1], \wedge, \vee)$ is called a fuzzy subset of A, where $[0,1]$ is a unit interval, $\alpha \wedge \beta = min(\alpha, \beta)$ and $\alpha \vee \beta = max(\alpha, \beta)$ for all $\alpha, \beta \in [0,1]$.

Definition 2.1 ([8])**.** Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of A as follows: for each $x \in A$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x).$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_i : i \in I\}$ of fuzzy subsets of X, where I is a nonempty index set, the least upper bound $\bigcup_{i \in I} \mu_i$ and the greatest lower bound $\bigcap_{i \in I} \mu_i$ of the μ_i's are given by for each $x \in X$,

$$(\bigcup_{i \in I} \mu_i)(x) = \bigvee_{i \in I} \mu_i(x) \text{ and } (\bigcap_{i \in I} \mu_i)(x) = \bigwedge_{i \in I} \mu_i(x),$$

respectively.

We define the binary operations "$\vee$" and "$\wedge$" on the set of all fuzzy subsets of L as:

$$(\mu \vee \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L,\ y \vee z = x\} \text{ and}$$
$$(\mu \wedge \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L,\ y \wedge z = x\}.$$

If μ and θ are fuzzy ideals of L, then $\mu \wedge \theta = \mu \cap \theta$ and $\mu \vee \theta$ is a fuzzy ideal generated by $\mu \cup \theta$.

For each $t \in [0,1]$, the set

$$\mu_t = \{x \in A : \mu(x) \geq t\}$$

is called the level subset of μ at t [12].

Note that a fuzzy subset μ of L is nonempty if there exists $x \in L$ such that $\mu(x) \neq 0$.

Definition 2.2. [8] Let f be a function from X into Y; μ be a fuzzy subset of X; and θ be a fuzzy subset of Y. The image of μ under f, denoted by $f(\mu)$, is a fuzzy subset of Y and

$$f(\mu)(y) = \begin{cases} Sup\{\mu(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases},$$

where $y \in Y$. The preimage of θ under f, symbolized by $f^{-1}(\theta)$, is a fuzzy subset of X and

$$f^{-1}(\theta)(x) = \theta(f(x)) \text{ for all } x \in X.$$

Theorem 2.3 ([7])**.** *Let f be a function from X into Y. Then the following assertions hold.*

(1) *For all fuzzy subset μ_i of X, $i \in I$, $f(\bigcup_{i \in I} \mu_i) = \bigcup_{i \in I} f(\mu_i)$ and so $\mu_1 \subseteq \mu_2 \Rightarrow f(\mu_1) \subseteq f(\mu_2)$.*

(2) *For all fuzzy subset* θ_j *of* Y, $j \in J$, $f^{-1}(\bigcup_{j\in J}\theta_j) = \bigcup_{j\in J} f^{-1}(\theta_j)$, $f^{-1}(\bigcap_{j\in J}\theta_j) = \bigcap_{j\in J} f^{-1}(\theta_j)$ *and therefore* $\theta_1 \subseteq \theta_2 \Rightarrow f^{-1}(\theta_1) \subseteq f^{-1}(\theta_2)$.
(3) $\mu \subseteq f^{-1}(f(\mu))$. *In particular, if* f *is an injection, then* $\mu = f^{-1}(f(\mu))$, *for all fuzzy subset* μ *of* X.
(4) $f(f^{-1}(\theta)) \subseteq \theta$. *In particular, if* f *is a surjection, then* $f(f^{-1}(\theta)) = \theta$, *for all fuzzy subset* θ *of* Y.
(5) $f(\mu) \subseteq \theta \Leftrightarrow \mu \subseteq f^{-1}(\theta)$, *for all fuzzy subsets* μ *and* θ *of* X *and* Y *respectively.*

Definition 2.4. [10] A fuzzy subset μ of a lattice L is said to be a fuzzy ideal of L, if for all $x, y \in L$,
(i) $\mu(0) = 1$,
(ii) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,
(iii) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

In [10], Swamy and Raju observed that, a fuzzy subset μ of a lattice L is a fuzzy ideal of L if and only if

$$\mu(0) = 1 \text{ and } \mu(x \vee y) = \mu(x) \wedge \mu(y) \text{ for all } x, y \in L.$$

The set of all fuzzy ideals of L is denoted by $FI(L)$.

Let μ be a fuzzy subset of a lattice L. The smallest fuzzy ideal of L containing μ is called a fuzzy ideal of L induced by μ and denoted by $(\mu]$ and

$$(\mu] = \bigcap\{\theta \in FI(L) : \mu \subseteq \theta\}.$$

This definition can be stated as follows.

Theorem 2.5 ([4])**.** *Let* μ *be a fuzzy subset of* L. *The fuzzy subset* $(\mu]$ *of* L *define by*

$$(\mu](x) = Sup\{\alpha \in [0,1] : x \in (\mu_\alpha]\}, \text{ for all } x \in L$$

is the fuzzy ideal induced by μ.

3. Relative fuzzy annihilators

In this section, we give the definition of relative fuzzy annihilator of a lattice L . We also prove that the set of fuzzy annihilator ideals of L forms a complete Boolean algebra.

Definition 3.1. Let μ be a nonempty fuzzy subset of L and θ be a fuzzy ideal of L. The fuzzy annihilator $\langle\mu, \theta\rangle$ of μ relative to θ is defined:

$$\langle\mu, \theta\rangle = Sup\{\eta : \eta \in [0,1]^L,\ \eta \wedge \mu \subseteq \theta\}.$$

Lemma 3.2. *For any two fuzzy subsets* μ *and* θ *of a distributive lattice* L, *we have*

$$(\mu \wedge \theta] = (\mu] \wedge (\theta].$$

Proof. Let μ and θ be fuzzy subsets of L. We need to show $(\mu \wedge \theta] = (\mu] \wedge (\theta]$. We know that $(\mu \wedge \theta]$ is the smallest fuzzy ideal containing $\mu \wedge \theta$. Clearly we have that $\mu \wedge \theta \subseteq (\mu] \wedge (\theta]$. Since $(\mu] \wedge (\theta]$ is a fuzzy ideal containing $\mu \wedge \theta$, we get $(\mu \wedge \theta] \subseteq (\mu] \wedge (\theta]$. Now we proceed to show $(\mu] \wedge (\theta] \subseteq (\mu \wedge \theta]$. For any $x \in L$, the value of the fuzzy ideal generated by a fuzzy subset μ is

$$(\mu](x) = Sup\{t \in [0,1] : x \in (\mu_t]\}.$$

In a distributive lattice L, for any subsets A and B of L, we have $(A\wedge B] = (A]\wedge(B]$, where $A\wedge B = \{a\wedge b : a\in A,\ b\in B\}$.
Since $(\mu]$ and $(\theta]$ are fuzzy ideals, we have $(\mu]\wedge(\theta] = (\mu]\cap(\theta]$. Now,

$$\begin{aligned}((\mu]\wedge(\theta])(x) &= (\mu](x)\wedge(\theta](x)\\ &= Sup\{t_1\in[0,1] : x\in(\mu_{t_1}]\}\wedge Sup\{t_2\in[0,1] : x\in(\theta_{t_2}]\}\\ &= Sup\{t_1\wedge t_2 : x\in(\mu_{t_1}],\ x\in(\theta_{t_2}]\}\\ &\leq Sup\{t : x\in(\mu_t]\wedge(\theta_t]\}\\ &\leq Sup\{t : x\in((\mu\wedge\theta)_t]\}\ \ \texttt{Since}\ \mu_t\wedge\theta_t\subseteq(\mu\wedge\theta)_t\\ &= (\mu\wedge\theta](x)\end{aligned}$$

Thus $(\mu]\wedge(\theta]\subseteq(\mu\wedge\theta]$. So $(\mu]\wedge(\theta] = (\mu\wedge\theta]$. □

Now we prove the following lemma.

Lemma 3.3. *Let μ be a nonempty fuzzy subset of L and θ be a fuzzy ideal of L. Then*

$$\langle\mu,\theta\rangle = Sup\{\eta : \eta\in FI(L),\ \eta\wedge\mu\subseteq\theta\}.$$

Proof. Clearly, $Sup\{\eta : \eta\in FI(L),\ \eta\wedge\mu\subseteq\theta\}\leq Sup\{\delta : \delta\in[0,1]^L,\ \delta\wedge\mu\subseteq\theta\}$. Again,

$$\begin{aligned}\langle\mu,\theta\rangle(x) &= Sup\{\eta(x) : \eta\in[0,1]^L,\ \eta\wedge\mu\subseteq\theta\}\\ &\leq Sup\{(\eta](x) : (\eta]\in FI(L),\ (\eta]\wedge\mu\subseteq\theta\},\ \texttt{since}\ (\eta\wedge\mu] = (\eta]\wedge(\mu].\end{aligned}$$

Thus $\langle\mu,\theta\rangle\subseteq Sup\{\eta : \eta\in FI(L),\ \eta\wedge\mu\subseteq\theta\}$. So

$$\langle\mu,\theta\rangle = Sup\{\eta : \eta\in FI(L),\ \eta\wedge\mu\subseteq\theta\}.$$

□

Theorem 3.4. *Let μ be a nonempty fuzzy subset of L and θ be a fuzzy ideal of L. Then $\langle\mu,\theta\rangle$ is a fuzzy ideal of L.*

Proof. Let μ be a nonempty fuzzy subset of L and θ be a fuzzy ideal of L. Since $\theta\wedge\mu\subseteq\theta$ and $\theta(0) = 1$, we get

$$\langle\mu,\theta\rangle(0) = Sup\{\eta(0) : \eta\in FI(L),\ \eta\wedge\mu\subseteq\theta\}\geq\theta(0) = 1.$$

Again for any $x, y\in L$,

$\langle\mu,\theta\rangle(x)\wedge\langle\mu,\theta\rangle(y)$
$= Sup\{\eta(x) : \eta\in FI(L),\ \eta\wedge\mu\subseteq\theta\}\wedge\ Sup\{\sigma(y) : \sigma\in FI(L),\ \sigma\wedge\mu\subseteq\theta\}$
$= Sup\{\eta(x)\wedge\sigma(y) : \eta,\sigma\in FI(L),\ \eta\wedge\mu\subseteq\theta,\ \sigma\wedge\mu\subseteq\theta\}$
$\leq Sup\{(\eta\vee\sigma)(x)\wedge(\eta\vee\sigma)(y) : \eta,\sigma\in FI(L),\ \eta\wedge\mu\subseteq\theta,\ \sigma\wedge\mu\subseteq\theta\}$.

For each $\eta,\sigma\in FI(L)$ such that $\eta\wedge\mu\subseteq\theta$ and $\sigma\wedge\mu\subseteq\theta$, $\eta\vee\sigma\in FI(L)$ and $(\eta\vee\sigma)\wedge\mu\subseteq\theta$. Then

$$\begin{aligned}\langle\mu,\theta\rangle(x)\wedge\langle\mu,\theta\rangle(y) &\leq Sup\{\lambda(x)\wedge\lambda(y) : \lambda\in FI(L),\ \lambda\wedge\mu\subseteq\theta\}\\ &= Sup\{\lambda(x\vee y) : \lambda\in FI(L),\ \lambda\wedge\mu\subseteq\theta\}\\ &= \langle\mu,\theta\rangle(x\vee y).\end{aligned}$$

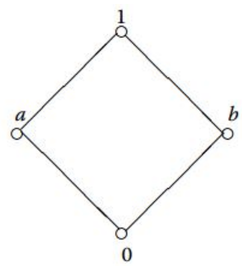
Thus, $\langle\mu,\theta\rangle(x\vee y)\geq\langle\mu,\theta\rangle(x)\wedge\langle\mu,\theta\rangle(y)$.

Now we show $\langle\mu,\theta\rangle(x) \leq \langle\mu,\theta\rangle(x\wedge y)$ and $\langle\mu,\theta\rangle(y) \leq \langle\mu,\theta\rangle(x\wedge y)$.

$$\begin{aligned}\langle\mu,\theta\rangle(x) &= Sup\{\eta(x) : \eta \in FI(L),\ \eta\wedge\mu \subseteq \theta\}\\ &\leq Sup\{\eta(x\wedge y) : \eta \in FI(L),\ \eta\wedge\mu \subseteq \theta\}\\ &= \langle\mu,\theta\rangle(x\wedge y)\end{aligned}$$

Similarly, $\langle\mu,\theta\rangle(y) \leq \langle\mu,\theta\rangle(x\wedge y)$. So $\langle\mu,\theta\rangle(x\wedge y) \geq \langle\mu,\theta\rangle(x) \vee \langle\mu,\theta\rangle(y)$. Hence $\langle\mu,\theta\rangle$ is a fuzzy ideal of L. □

Example 3.5. Consider the distributive lattice $L = \{0, a, b, 1\}$ whose Hasse diagram is given below.



A fuzzy subset θ of L defined by $\theta(0) = 1,\ \theta(a) = \frac{1}{5},\ \theta(b) = \frac{1}{3},\ \theta(1) = \frac{1}{5}$ is a fuzzy ideal. Let μ be a fuzzy subset of L defined as: $\mu(0) = \frac{1}{4},\ \mu(a) = \frac{1}{7},\ \mu(b) = \frac{1}{7},\ \mu(1) = \frac{1}{4}$. Next we can easily find the value of $\langle\mu,\theta\rangle(x)$ for each $x \in L$. For any $\eta \in [0,1]^L$ with $\eta\wedge\mu \subseteq \theta$, we can determine the value of $\eta(x)$. Then we have $\langle\mu,\theta\rangle(a) = \langle\mu,\theta\rangle(1) = \frac{1}{5},\ \langle\mu,\theta\rangle(0) = \langle\mu,\theta\rangle(b) = 1$. Thus, $\langle\mu,\theta\rangle$ is a fuzzy ideal of L.

The characteristics function of any set A is defined as:

$$\chi_A(x) = \begin{cases} 1\ ,\ \text{if}\ \ x \in A \\ 0\ ,\ \text{if}\ \ x \notin A \end{cases}$$

Lemma 3.6. *Let η and δ be fuzzy subsets and μ,θ and λ fuzzy ideals of L. Then*

(1) $\langle\eta,\mu\rangle = \chi_L \Leftrightarrow \eta \subseteq \mu$,
(2) $\theta \subseteq \langle\eta,\theta\rangle$,
(3) $\eta \subseteq \delta \Rightarrow \langle\delta,\mu\rangle \subseteq \langle\eta,\mu\rangle$,
(4) $\mu \subseteq \theta \Rightarrow \langle\delta,\mu\rangle \subseteq \langle\delta,\theta\rangle$,
(5) $\langle\eta,\mu\cap\theta\rangle = \langle\eta,\mu\rangle \cap \langle\eta,\theta\rangle$,
(6) $\langle(\eta],\mu\rangle = \langle\eta,\mu\rangle$,
(7) $\langle\eta\cup\delta,\mu\rangle = \langle\eta,\mu\rangle \cap \langle\delta,\mu\rangle$,
(8) $\langle\mu\vee\theta,\lambda\rangle = \langle\mu,\lambda\rangle \cap \langle\theta,\lambda\rangle$,
(9) $\langle\mu,\theta\rangle = \langle\mu\vee\theta,\theta\rangle = \langle\mu,\mu\wedge\theta\rangle$,
(10) $(\eta] \cap \theta \subseteq \mu \Rightarrow \theta \subseteq \langle\eta,\mu\rangle$.

Proof. Let η and δ be fuzzy subsets and μ,θ and λ fuzzy ideals of L.

(1) Let $\langle\eta,\mu\rangle = \chi_L$. We need to show $\eta \subseteq \mu$. Suppose not. Then there is $x \in L$ such that $\eta(x) > \mu(x)$. This implies that $\gamma(x) \leq \mu(x)$, for each γ such that $\gamma\wedge\eta \subseteq \mu$. Thus $\mu(x)$ is an upper bound of $\{\gamma(x) : \gamma\wedge\eta \subseteq \mu\}$. Which implies $1 = Sup\{\gamma(x) : \gamma\wedge\eta \subseteq \mu\} \leq \mu(x)$. So $\mu(x) \geq \eta(x)$ which is a contradiction. Hence $\eta \subseteq \mu$.

Conversely, assume that $\eta \subseteq \mu$. Then clearly, $\chi_L \wedge \eta \subseteq \eta$. This implies that $\chi_L \wedge \eta \subseteq \mu$. Thus $\langle \eta, \mu \rangle = \chi_L$.

The proof of (2),(3) and (4) is straightforward.

(5) By (4), we have $\langle \eta, \mu \cap \theta \rangle \subseteq \langle \eta, \mu \rangle \cap \langle \eta, \theta \rangle$. On the other hand,

$$\begin{aligned} &\langle \eta, \mu \rangle \cap \langle \eta, \theta \rangle \\ &= Sup\{\gamma_1 : \gamma_1 \in FI(L),\ \gamma_1 \wedge \eta \subseteq \mu\} \wedge\ Sup\{\gamma_2 : \gamma_2 \in FI(L),\ \gamma_2 \wedge \eta \subseteq \theta\} \\ &= Sup\{\gamma_1 \wedge \gamma_2 : \gamma_1,\ \gamma_2 \in FI(L),\ \gamma_1 \wedge \eta \subseteq \mu,\ \gamma_2 \wedge \eta \subseteq \theta\}. \end{aligned}$$

Since $\gamma_1 \wedge \eta \subseteq \mu$ and $\gamma_2 \wedge \eta \subseteq \theta$, we can find a fuzzy ideal γ of L contained in γ_1 and γ_2 such that $\gamma \wedge \eta \subseteq \mu$ and $\gamma \wedge \eta \subseteq \theta$. Based on this fact, we have that

$$\langle \eta, \mu \rangle \cap \langle \eta, \theta \rangle \leq Sup\{\gamma : \gamma \in FI(L),\ \gamma \wedge \eta \subseteq \mu \cap \theta\} = \langle \eta, \mu \cap \theta \rangle.$$

Then $\langle \eta, \mu \rangle \cap \langle \eta, \theta \rangle = \langle \eta, \mu \cap \theta \rangle$.

(6) Since $(\gamma \wedge \eta] = (\gamma] \wedge (\eta] = \gamma \wedge (\eta]$, for every $\gamma \in FI(L)$, we have

$$\langle \eta, \mu \rangle = Sup\{\gamma : \gamma \in FI(L),\ \gamma \wedge \eta \subseteq \mu\} = Sup\{\gamma : \gamma \wedge (\eta] \subseteq \mu\} = \langle (\eta], \mu \rangle.$$

Then $\langle \eta, \mu \rangle = \langle (\eta], \mu \rangle$.

(7) Clearly we have that $\langle \eta \cup \delta, \mu \rangle \subseteq \langle \eta, \mu \rangle \cap \langle \delta, \mu \rangle$. On the other hand,

$$\begin{aligned} \langle \eta, \mu \rangle \cap \langle \delta, \mu \rangle &= \langle (\eta], \mu \rangle \cap \langle (\delta], \mu \rangle \\ &= Sup\{\gamma_1 : \gamma_1 \in FI(L),\ \gamma_1 \wedge (\eta] \subseteq \mu\} \\ &\quad \wedge Sup\{\gamma_2 : \gamma_2 \in FI(L),\ \gamma_2 \wedge (\delta] \subseteq \mu\} \\ &= Sup\{\gamma_1 \wedge \gamma_2 : \gamma_1,\ \gamma_2 \in FI(L),\ \gamma_1 \wedge (\eta] \subseteq \mu,\ \gamma_2 \wedge (\delta] \subseteq \mu\}. \end{aligned}$$

Since $\gamma_1 \wedge (\eta] \subseteq \mu$ and $\gamma_2 \wedge (\delta] \subseteq \mu$, we can find a fuzzy ideal γ of L contained in γ_1 and γ_2 such that $\gamma \wedge (\eta] \subseteq \mu$ and $\gamma \wedge (\delta] \subseteq \mu$. This implies that $(\gamma \wedge ((\eta] \vee (\delta])) \subseteq \mu$. This shows that

$$\begin{aligned} \langle \eta, \mu \rangle \cap \langle \delta, \mu \rangle &\leq Sup\{\gamma : \gamma \wedge ((\eta] \vee (\delta]) \subseteq \mu\} \\ &\leq Sup\{\gamma : \gamma \wedge (\eta \cup \delta) \subseteq \mu\} \\ &= \langle \eta \cup \delta, \mu \rangle. \end{aligned}$$

Using property (6), we can simply prove the remaining properties. □

Let μ be a nonempty fuzzy subset of L. Since $\{0\}$ is an ideal of L, $\langle \mu, \chi_{\{0\}} \rangle$ is a fuzzy ideal of L. Let us denote $\langle \mu, \chi_{\{0\}} \rangle$ by μ^*. Then μ^* is a fuzzy annihilator of μ relative to $\chi_{\{0\}}$. Thus

$$\mu^* = Sup\{\eta : \eta \in [0,1]^L,\ \eta \wedge \mu \subseteq \chi_{\{0\}}\}.$$

The following lemmas can be verified easily.

Lemma 3.7. *Let μ be a nonempty fuzzy subset of L. Then*

(1) $\chi_{\{0\}} \subseteq \mu^*$,
(2) $\mu \wedge \mu^* \subseteq \chi_{\{0\}}$,
(3) $\mu \wedge \mu^* = \chi_{\{0\}}$, *whenever* $\mu(0) = 1$,
(4) $\mu^* \wedge \mu^{**} = \chi_{\{0\}}$.

Lemma 3.8. *Let μ and θ be nonempty fuzzy subsets of L. Then*

(1) $\mu \subseteq \theta \Rightarrow \theta^* \subseteq \mu^*$,
(2) $\theta \wedge \mu \subseteq \chi_{\{0\}} \Leftrightarrow \theta \subseteq \mu^*$,
(3) $\theta \wedge \mu = \chi_{\{0\}} \Leftrightarrow \theta \subseteq \mu^*$, *whenever* $\mu(0) = 1 = \theta(0)$,

(4) $\mu \subseteq \mu^{**}$,
(5) $\mu^* = \mu^{***}$.

Lemma 3.9. *Let μ and θ be fuzzy ideals of L. Then*
(1) $(\chi_{\{0\}})^* = \chi_L$,
(2) $(\chi_L)^* = \chi_{\{0\}}$,
(3) $(\mu \vee \theta)^* = \mu^* \cap \theta^*$,
(4) $(\mu \vee \mu^*)^* = \chi_{\{0\}}$.

Lemma 3.10. *If $\mu_i \in [0,1]^L$ for every $i \in I$, then $(\bigcup_{i\in I} \mu_i)^* = \bigcap_{i\in I} \mu_i^*$.*

Proof. Let $\{\mu_i : i \in I\}$ be family of fuzzy subsets of L. Since $\mu_i \subseteq (\bigcup_{i\in I} \mu_i)$ for each i, $(\bigcup_{i\in I} \mu_i)^* \subseteq \mu_i^*$. Then $(\bigcup_{i\in I} \mu_i)^* \subseteq \bigcap_{i\in I} \mu_i^*$. Again,

$$\begin{aligned}
(\bigcap_{i\in I} \mu_i^*) \wedge (\bigcup_{j\in I} \mu_j) &= (\bigwedge_{i\in I} \mu_i^*) \wedge (\bigvee_{j\in I} \mu_j) \\
&= \bigvee_{j\in I} ((\bigwedge_{i\in I} \mu_i^*) \wedge \mu_j)) \\
&\leq \bigvee_{j\in I} ((\mu_j^* \wedge \mu_j)) \\
&\leq \bigvee_{j\in I} (\chi_{\{0\}}). \\
&= \chi_{\{0\}}
\end{aligned}$$

Thus $(\bigcap_{i\in I} \mu_i^*) \subseteq (\bigcup_{i\in I} \mu_i)^*$. So $(\bigcap_{i\in I} \mu_i^*) = (\bigcup_{i\in I} \mu_i)^*$. □

Now we give the definition of fuzzy annihilator ideal and prove that the set of all fuzzy annihilator ideals is a complete Boolean algebra.

Definition 3.11. A fuzzy ideal μ of L is called a fuzzy annihilator ideal, if $\mu = \theta^*$, for some nonempty fuzzy subset θ of L, or equivalently, $\mu = \mu^{**}$.

We denote the class of all fuzzy annihilator ideals of L by $FAI(L)$.

Example 3.12. Let $L = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on L as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then clearly $(L, \wedge, \vee)$ is a distributive lattice with least element 0. Now define a fuzzy subset μ of L as follows: $\mu(0) = 1 = \mu(a)$ and $\mu(b) = \mu(c) = 0$. Thus, μ is a fuzzy ideal of L and $\mu = \mu^{**}$. So μ is a fuzzy annihilator ideal of L.

Lemma 3.13. *Let $\mu, \theta \in FAI(L)$. Then*
(1) $\mu \cap \theta = (\mu^* \vee \theta^*)^*$,
(2) $\mu \cap \theta = (\mu \cap \theta)^{**}$.

The result (2) of the above lemma can be generalized as given in the following. If $\{\mu_i : i \in \Delta\}$ is a family of fuzzy annihilator ideals of L, then

$$(\bigcap_{i\in\Delta} \mu_i)^{**} = \bigcap_{i\in\Delta} \mu_i.$$

Theorem 3.14. *The set $FAI(L)$ of all fuzzy annihilator ideals of L forms a complete Boolean algebra.*

Proof. For $\mu, \theta \in FAI(L)$, define $\mu \wedge \theta = \mu \cap \theta$ and $\mu \underline{\vee} \theta = (\mu^* \cap \theta^*)^*$. Then clearly, $\mu \cap \theta$, $\mu \underline{\vee} \theta \in FAI(L)$. It can be easily observed that $\langle FAI(L), \cap, \underline{\vee}\rangle$ is a lattice. Since $(\chi_{\{0\}})^* = \chi_L$ and $(\chi_L)^* = \chi_{\{0\}}$, then $\chi_{\{0\}}$ and χ_L are the least and the greatest elements of $FAI(L)$ respectively. Thus, $\langle FAI(L), \cap, \underline{\vee}\rangle$ is a bounded lattice.

Let $\mu \in FAI(L)$. Then $\mu^* \in FAI(L)$ and $\mu \cap \mu^* = \chi_{\{0\}}$, $\mu \underline{\vee} \mu^* = \chi_L$. Thus μ^* is a complement of μ.

Let μ, θ, $\eta \in FAI(L)$. We prove that $\mu \underline{\vee} (\theta \cap \eta) = (\mu \underline{\vee} \theta) \cap (\mu \underline{\vee} \eta)$. We first prove that $(\mu \underline{\vee} \theta) \cap \eta \subseteq \mu \underline{\vee} (\theta \cap \eta)$. We have $\mu \cap \eta \cap (\mu^* \cap (\theta \cap \eta)^*) = \chi_{\{0\}}$, so that $\eta \cap (\mu^* \cap (\theta \cap \eta)^*) \subseteq \mu^*$. Similarly, $\theta \cap \eta \cap (\mu^* \cap (\theta \cap \eta)^*) \subseteq \chi_{\{0\}}$ implies that $\eta \cap (\mu^* \cap (\theta \cap \eta)^*) \subseteq \theta^*$. Then $\eta \cap (\mu^* \cap (\theta \cap \eta)^*) \subseteq \mu^* \cap \theta^*$. Thus

$$\eta \cap (\mu^* \cap (\theta \cap \eta)^*) \cap (\mu^* \cap \theta^*)^* = \chi_{\{0\}}.$$

That is, $(\mu^* \cap (\theta \cap \eta)^*) \cap (\eta \cap (\mu^* \cap \theta^*)^*) = \chi_{\{0\}}$. So $(\mu^* \cap (\theta \cap \eta)^*) \subseteq (\eta \cap (\mu^* \cap \theta^*)^*)^*$. Hence $(\mu \underline{\vee} \theta) \cap \eta \subseteq \mu \underline{\vee} (\theta \cap \eta)$ and so $FAI(L)$ is a distributive lattice. Therefore $\langle FAI(L), \cap, \underline{\vee}\rangle$ is a Boolean algebra.

Next we prove the completeness. Let $\{\mu_i : i \in \Delta\}$ be a family of $FAI(L)$. Then $(\bigcap_{i\in\Delta} \mu_i)^{**} = \bigcap_{i\in\Delta} \mu_i$. Thus $\langle FAI(L), \wedge, \underline{\vee}, *, \chi_{\{0\}}, \chi_L\rangle$ is a complete Boolean algebra. □

4. Fuzzy annihilator preserving homomorphism

In this section we study fuzzy annihilator preserving homomorphism. Throughout this section L and $L^{'}$ denote distributive lattices with least elements 0 and $0^{'}$ respectively and $f : L \longrightarrow L^{'}$ denotes a lattice homomorphism.

Lemma 4.1. *If μ is any nonempty fuzzy subset of L and θ is a fuzzy ideal of L, then $f(\langle \mu, \theta\rangle) \subseteq \langle f(\mu), f(\theta)\rangle$. (In particular, if $\theta = \chi_{\{0\}}$, then $f(\mu^*) \subseteq (f(\mu))^*$).*

Proof.

$$\begin{aligned} f(\langle \mu, \theta\rangle)(y) &= Sup\{\langle \mu, \theta\rangle(x) : x \in f^{-1}(y)\} \\ &= Sup\{Sup\{\eta(x) : \eta \in FI(L),\ \eta \wedge \mu \subseteq \theta\} : x \in f^{-1}(y)\}\} \\ &= Sup\{f(\eta)(y) : \eta \wedge \mu \subseteq \theta\} \\ &\leq Sup\{\lambda(y) : \lambda \in FI(L^{'}),\ \lambda \wedge f(\mu) \subseteq f(\theta)\} \\ &= \langle f(\mu), f(\theta)\rangle(y). \end{aligned}$$

Then $f(\langle \mu, \theta\rangle) \subseteq \langle f(\mu), f(\theta)\rangle$. □

Definition 4.2. For any nonempty fuzzy subset μ of L, f is said to be a fuzzy annihilator preserving if $f(\mu^*) = (f(\mu))^*$.

Theorem 4.3. *If $Kerf = \{0\}$ and f is onto, then f is a fuzzy annihilator preserving.*

Proof. Let μ be a nonempty fuzzy subset of L. We have always $f(\mu^*) \subseteq (f(\mu))^*$. Since $Kerf = \{0\}$ and f is onto, $f^{-1}(\chi_{\{0'\}}) = \chi_{\{0\}}$ and $\theta = f(f^{-1}(\theta))$, for all $\theta \in [0,1]^{L^{'}}$. Let $y \in L^{'}$. Then

$$\begin{aligned}
(f(\mu))^*(y) &= Sup\{\theta(y) : \theta \in FI(L^{'}),\ \theta \wedge f(\mu) \subseteq \chi_{\{0'\}}\} \\
&= Sup\{\theta(y) : f(f^{-1}(\theta)) \wedge f(\mu) \subseteq \chi_{\{0'\}}\} \\
&= Sup\{\theta(y) : f(f^{-1}(\theta) \wedge \mu) \subseteq \chi_{\{0'\}}\} \\
&= Sup\{\theta(y) : f^{-1}(\theta) \wedge \mu \subseteq f^{-1}(\chi_{\{0'\}})\} \\
&= Sup\{\theta(f(x)) : f^{-1}(\theta) \wedge \mu \subseteq \chi_{\{0\}},\ f(x) = y\} \\
&\leq Sup\{Sup\{f^{-1}(\theta)(x) : f^{-1}(\theta) \wedge \mu \subseteq \chi_{\{0\}}\} :\ x \in f^{-1}(y)\} \\
&\leq Sup\{Sup\{\eta(x) : \eta \in FI(L),\ \eta \wedge \mu \subseteq \chi_{\{0\}}\} :\ x \in f^{-1}(y)\} \\
&= f(\mu^*)(y).
\end{aligned}$$

Thus $(f(\mu))^* \subseteq f(\mu^*)$. So f preserves fuzzy annihilator. □

Theorem 4.4. *If $Kerf = \{0\}$, then f^{-1} preserves fuzzy annihilator.*

Proof. Let μ be a nonempty fuzzy subset of $L^{'}$ and $x \in L$. Then

$$\begin{aligned}
f^{-1}(\mu^*)(x) &= (\mu^*)(f(x)) \\
&= Sup\{\theta(f(x)) : \theta \in FI(L^{'}),\ \theta \wedge \mu \subseteq \chi_{\{0'\}}\} \\
&\leq Sup\{\theta(f(x)) : f(f^{-1}(\theta)) \wedge f(f^{-1}(\mu)) \subseteq \chi_{\{0'\}}\} \\
&= Sup\{f^{-1}(\theta)(x) : f^{-1}(\theta) \wedge f^{-1}(\mu) \subseteq \chi_{\{0\}}\} \\
&\leq Sup\{\eta(x) : \eta \in FI(L),\ \eta \wedge f^{-1}(\mu) \subseteq \chi_{\{0\}}\} \\
&= (f^{-1}(\mu))^*(x).
\end{aligned}$$

Thus $f^{-1}(\mu^*) \subseteq (f^{-1}(\mu))^*$. Similarly, $(f^{-1}(\mu))^* \subseteq f^{-1}(\mu^*)$. So $f^{-1}(\mu^*) = (f^{-1}(\mu))^*$. □

Theorem 4.5. *Let f be a fuzzy annihilator preserving epimorphism and $Kerf = \{0\}$. Then we have:*

$$\mu^* = \theta^* \Leftrightarrow (f(\mu))^* = (f(\theta))^*,$$

for any nonempty fuzzy subsets μ, θ of L.

Proof. Let μ and θ be two nonempty fuzzy subsets of L such that $\mu^* = \theta^*$. Since f is annihilator preserving, it gives $(f(\mu))^* = (f(\theta))^*$.

Conversely, suppose that $(f(\mu))^* = (f(\theta))^*$. We know that $\mu \subseteq f^{-1}(f(\mu))$, for each fuzzy subset μ of L. Let $x \in L$. Then $\mu^*(x) \leq (f(\mu))^*(f(x)) = (f(\theta))^*(f(x))$.

This implies

$$\begin{aligned}
\mu^*(x) &\leq (f(\theta))^*(f(x)) \\
&= Sup\{\eta(f(x)) : \eta \in FI(L^{'}),\ \eta \wedge f(\theta) \subseteq \chi_{\{0'\}}\} \\
&= Sup\{\eta(f(x)) : f(f^{-1}(\eta)) \wedge f(\theta) \subseteq \chi_{\{0'\}}\} \\
&= Sup\{f^{-1}(\eta)(x) : f^{-1}(\eta) \wedge \theta \subseteq \chi_{\{0\}}\} \\
&\leq Sup\{\lambda(x) : \lambda \in FI(L),\ \lambda \wedge \theta \subseteq \chi_{\{0\}}\} \\
&= \theta^*(x).
\end{aligned}$$

Thus $\mu^* \subseteq \theta^*$. Similarly, $\theta^* \subseteq \mu^*$. So $\mu^* = \theta^*$ □

The following theorem can be verified easily.

Theorem 4.6. (1) *If f is fuzzy annihilator preserving and onto, then $f(\mu)$ is a fuzzy annihilator ideal of $L^{'}$, for every fuzzy annihilator ideal μ of L.*

(2) *If f^{-1} preserves annihilator, then $f^{-1}(\mu)$ is a fuzzy annihilator ideal of L, for every fuzzy annihilator ideal μ of $L^{'}$.*

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Berhanu Assaye Alaba (berhanu_assaye@yahoo.com)
Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia

Wondwosen Zemene Norahun (wondie1976@gmail.com)
Department of Mathematics, University of Gondar, Gondar, Ethiopia