

## On intuitionistic $L$ -fuzzy prime submodules

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**ABSTRACT.** In this paper the concept of an intuitionistic  $L$ -fuzzy prime submodule of  $M$  is given, and some fundamental lemmas are proved. Also a characterization of an intuitionistic  $L$ -fuzzy prime submodule is given. Finally, we show that an intuitionistic  $L$ -fuzzy prime submodule is inherited by an  $R$ -module epimorphism.

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### 1. INTRODUCTION

Atanassov in [2, 4, 5] introduced the notion of an intuitionistic fuzzy subset  $A$  of a non-empty set  $X$  as a function from  $X$  to  $[0, 1] \times [0, 1]$  as a generalization of fuzzy set given by Zadeh [21] which is a function from  $X$  to  $[0, 1]$ . Atanassov and Stoeva in [3] generalized the notion of intuitionistic fuzzy subset of  $X$  to that of an intuitionistic  $L$ -fuzzy subset, namely a function from  $X$  to lattice  $L \times L$  which is also a generalization of  $L$ -fuzzy set given by Goguen [9]. The development of Algebra in fuzzy setting are very much evident in the book of Kandasamy [13], Mordeson and Malik [15]. Acar in [1] gave the  $L$ -fuzzification of the notion of prime submodules.

In [8], Biswas considered the intuitionistic fuzzification of algebraic structures and introduced the notion of intuitionistic fuzzy subgroup of a group. Hur et al. [10], introduced and examined the notion of an intuitionistic fuzzy ideal of a ring. Rahman, Saikia in [17], Isaac, John in [11], and Sharma in [18] studied some aspects of intuitionistic fuzzy submodules. Since then several authors have obtained interesting results on intuitionistic  $L$ -fuzzy subgroup of the group  $G$ , intuitionistic  $L$ -fuzzy subring and ideal of the ring  $R$  and BP-Algebras, for example: see [16], [7] and [12].

In [6] the notion of intuitionistic fuzzy prime ideal of a ring over  $[0, 1]$  is given in terms of intuitionistic fuzzy singletons and the intuitionistic fuzzy prime spectrum

of a ring is studied by Sharma and Kaur in [19]. The annihilator of intuitionistic fuzzy prime modules is discussed in [20]. In Section 3 of this paper, we generalize their definition to any complete lattice  $L$  when  $R$  is a commutative ring with identity. In Theorem (3.6) we give a characterization of intuitionistic  $L$ -fuzzy prime submodules which is one of the original results obtained in this paper. In Section 4, we investigate the behaviour of intuitionistic  $L$ -fuzzy prime submodules under  $R$ -module homomorphisms, which constitutes another original result of our work.

## 2. PRELIMINARIES

Throughout the paper  $R$  is a commutative ring with identity,  $M$  a unitary  $R$ -module with zero element  $\theta$ . Let  $(L, \leq)$  be a lattice such that  $(L, \vee, \wedge, ', 0, 1)$  be a complete lattice with least element 0 and greatest element 1, where  $a \vee b = lub\{a, b\}$  and  $a \wedge b = glb\{a, b\}$  for all  $a, b \in L$  and " $'$ " is the order-reversing involution on  $L$ .

Let  $a, b \in L$ . Then  $b$  is called a complement of  $a$  if  $a \vee b = 1$  and  $a \wedge b = 0$ . We write  $b = a'$ . Thus  $1' = 0, 0' = 1$  and  $(a')' = a, \forall a \in L$ . If  $a \leq b$  then  $b' \leq a', \forall a, b \in L$ .

An element  $\alpha \in L, 1 \neq \alpha$ , is called a prime element in  $L$  if for all  $a, b \in L$  if  $a \wedge b \leq \alpha$  implies  $a \leq \alpha$  or  $b \leq \alpha$ .

If  $\mu, \nu$  are  $L$ -fuzzy submodules of an  $R$ -module  $M$  such that  $\nu \subseteq \mu$ . Then  $\nu$  is called an  $L$ -fuzzy prime submodule of  $\mu$  if  $r_t, x_s$  be any two  $L$ -fuzzy point of  $R$  and  $M$  respectively ( $r \in R, x \in M, t, s \in L$ ),  $r_t x_s \in \nu$  implies that either  $x_s \in \nu$  or  $x_t \mu \subseteq \nu$ . In particular if  $\mu = \chi_M$ , then  $\nu$  is called an  $L$ -fuzzy prime submodule of  $M$  ([1]).

Given a nonempty set  $X$ , an intuitionistic  $L$ -fuzzy subset  $A$  of  $X$  is a function  $A = (\mu_A, \nu_A) : X \rightarrow L \times L$  with the condition that  $\nu_A(x) \leq (\mu_A(x))', \forall x \in X$ , where " $'$ " is the order-reversing involution on  $L$ . When  $\nu_A(x) = (\mu_A(x))', \forall x \in X$ , then  $A$  is called an  $L$ -fuzzy subset of  $X$ . We denote by  $ILFS(X)$  the set of all intuitionistic  $L$ -fuzzy subsets of  $X$ . For  $A, B \in ILFS(X)$  we say  $A \subseteq B$  if and only if  $\mu_A(x) \leq \mu_B(x)$  and  $\nu_A(x) \geq \nu_B(x)$  for all  $x \in X$ . Also,  $A \subset B$  if and only if  $A \subseteq B$  and  $A \neq B$ .

Let  $A \in ILFS(X)$  and  $p, q \in L$ . Then the set  $A_{(p,q)} = \{x \in X : \mu_A(x) \geq p \text{ and } \nu_A(x) \leq q\}$  is called the  $(p, q)$ -cut subset of  $X$  with respect to  $A$ . By an intuitionistic  $L$ -fuzzy point ( $ILFP$ )  $x_{(p,q)}$  of  $X, x \in X$  and  $p, q \in L \setminus \{0\}$  such that  $p \vee q \leq 1$ , we mean  $x_{(p,q)} \in ILFS(X)$  defined by

$$x_{(p,q)}(y) = \begin{cases} (p, q), & \text{if } y = x \\ (0, 1), & \text{if otherwise.} \end{cases}$$

If  $x_{(p,q)}$  is an intuitionistic  $L$ -fuzzy point of  $X$  and  $x_{(p,q)} \subseteq A \in ILFS(X)$ , we write  $x_{(p,q)} \in A$ . Let  $A = (\mu_A, \nu_A)$  be an ILFS of  $X$  and  $Y \subseteq X$ . Then the restriction of

$A$  to the set  $Y$  is an ILFS  $A_Y = (\mu_{A_Y}, \nu_{A_Y})$  of  $Y$  and is defined as:

$$\mu_{A_Y}(y) = \begin{cases} \mu_A(y), & \text{if } y \in Y \\ 0, & \text{if otherwise} \end{cases}; \quad \nu_{A_Y}(y) = \begin{cases} \nu_A(y), & \text{if } y \in Y \\ 1, & \text{otherwise.} \end{cases}$$

The following are two very basic definitions given in [14] and [19].

**Definition 2.1.** Let  $A \in ILFS(R)$ . Then  $A$  is called an intuitionistic  $L$ -fuzzy ideal (ILFI) of  $R$ , if for all  $x, y \in R$ , the followings are satisfied:

- (i)  $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ ,
- (ii)  $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ ,
- (iii)  $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$ ,
- (iv)  $\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$ .

**Definition 2.2.** Let  $A \in ILFS(M)$ . Then  $A$  is called an intuitionistic  $L$ -fuzzy module (ILFM) of  $M$ , if for all  $x, y \in M, r \in R$ , the followings are satisfied:

- (i)  $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$ ,
- (ii)  $\mu_A(rx) \geq \mu_A(x)$ ,
- (iii)  $\mu_A(\theta) = 1$ ,
- (iv)  $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$ ,
- (v)  $\nu_A(rx) \leq \nu_A(x)$ ,
- (vi)  $\nu_A(\theta) = 0$ .

Let  $IF_L(M)$  denote the set of all intuitionistic  $L$ -fuzzy  $R$ -modules of  $M$  and  $IF_L(R)$  denote the set of all intuitionistic  $L$ -fuzzy ideals of  $R$ . We note that when  $R = M$ , then  $A \in IF_L(M)$  if and only if  $\mu_A(\theta) = 1, \nu_A(\theta) = 0$  and  $A \in IF_L(R)$ .

**Definition 2.3.** Let  $C \in ILFS(R)$  and  $B \in ILFS(M)$ . Define the composition  $C \circ B$  and product  $CB$  respectively as follows: for all  $w \in M$ ,

$$\mu_{C \circ B}(w) = \begin{cases} Sup[\mu_C(r) \wedge \mu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 0, & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

$$\nu_{C \circ B}(w) = \begin{cases} Inf[\nu_C(r) \vee \nu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 1, & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

and

$$\mu_{CB}(w) = \begin{cases} Sup[Inf_{i=1}^n \{\mu_C(r_i) \wedge \mu_B(x_i)\}] & \text{if } w = \sum_{i=1}^n r_i x_i, r_i \in R, x_i \in M, n \in N \\ 0, & \text{if } w \text{ is not expressible as } w = \sum_{i=1}^n r_i x_i \end{cases}$$

$$\nu_{CB}(w) = \begin{cases} Inf[Sup_{i=1}^n \{\nu_C(r_i) \vee \nu_B(x_i)\}] & \text{if } w = \sum_{i=1}^n r_i x_i, r_i \in R, x_i \in M, n \in N \\ 1, & \text{if } w \text{ is not expressible as } w = \sum_{i=1}^n r_i x_i, \end{cases}$$

where as usual supremum and infimum of an empty set are taken to be 0 and 1 respectively. Clearly,  $C \circ B \subseteq CB$ .

The following lemma can be found in [6, 14]. It gives the basic operations between intuitionistic  $L$ -fuzzy ideals and intuitionistic  $L$ -fuzzy modules where  $L$  is a complete lattice satisfying the infinite distributive law.

**Lemma 2.4.** Let  $C \in IF_L(R), A, B \in IF_L(M)$  and let  $L$  be a complete lattice satisfying the infinite distributive law.

- (1)  $CB \subseteq A$  if and only if  $C \circ B \subseteq A$ .
- (2)  $r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$  be ILFPS. Then  $r_{(s,t)} \circ x_{(p,q)} = (rx)_{(s \wedge p, t \vee q)}$ .
- (3) If  $\mu_C(0) = 1, \nu_C(0) = 0$  then  $CA \in IF_L(M)$ .
- (4) Let  $r_{(s,t)} \in ILFS(R)$  be an ILFP. Then for all  $w \in M$ ,

$$\mu_{r_{(s,t)} \circ B}(w) = \begin{cases} Sup[s \wedge \mu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 0, & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

and

$$\nu_{r_{(s,t)} \circ B}(w) = \begin{cases} Inf[t \vee \nu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 1, & \text{if } w \text{ is not expressible as } w = rx. \end{cases}$$

The following theorem gives a relation between an intuitionistic  $L$ -fuzzy modules on  $M$  and submodules of  $M$ . It is a very practical method to construct an intuitionistic  $L$ -fuzzy module on  $M$ .

**Theorem 2.5.** Let  $A \in ILFS(M)$ . Then  $A$  is an intuitionistic  $L$ -fuzzy module if and only if for all  $\alpha, \beta \in L$  with  $\alpha \vee \beta \leq 1$  such that  $A_{(\alpha, \beta)}$  is an  $R$ -submodules of  $M$ .

*Proof.* Simple proof □

**Definition 2.6** ([6]). For a non-constant  $C \in IF_L(R), C$  is called an intuitionistic  $L$ -fuzzy prime ideal of  $R$ , if for any intuitionistic  $L$ -fuzzy points  $x_{(p,q)}, y_{(r,s)} \in ILFS(R), x_{(p,q)}y_{(r,s)} \in C$  implies that either  $x_{(p,q)} \in C$  or  $y_{(r,s)} \in C$ .

### 3. INTUITIONISTIC L-FUZZY PRIME SUBMODULES

In this section, we will give a characterization of an intuitionistic  $L$ -fuzzy prime submodule of  $M$ .

**Definition 3.1.** For  $A, B \in IF_L(M), B$  is called an intuitionistic  $L$ -fuzzy submodule of  $A$ , if  $B \subseteq A$ .

In particular, if  $A = \chi_M$ , then we say  $B$  is an intuitionistic  $L$ -fuzzy submodule of  $M$ .

**Definition 3.2.** Let  $B$  be an intuitionistic  $L$ -fuzzy submodule of  $A, B$  is called an intuitionistic  $L$ -fuzzy prime submodule of  $A$ , if  $r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$  ( $r \in R, x \in M, s, t, p, q \in L$ ),  $r_{(s,t)}x_{(p,q)} \in B$  implies that either  $x_{(p,q)} \in B$  or  $r_{(s,t)}A \subseteq B$ .

In particular, taking  $A = \chi_M$ , if for  $r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$  we have  $r_{(s,t)}x_{(p,q)} \in B$  implies that either  $x_{(p,q)} \in B$  or  $r_{(s,t)}\chi_M \subseteq B$ , then  $B$  is called an intuitionistic  $L$ -fuzzy prime submodule of  $M$ .

The following theorem says that intuitionistic  $L$ -fuzzy prime submodule and intuitionistic  $L$ -fuzzy prime ideals coincide when  $R$  is considered to be a module over itself.

**Theorem 3.3.** *If  $M = R$ , then  $B \in ILFS(M)$ , is an intuitionistic  $L$ -fuzzy prime submodule of  $M$  if and only if  $B \in IF_L(R)$  is an intuitionistic  $L$ -fuzzy prime ideal.*

*Proof.* Let  $B$  be an intuitionistic  $L$ -fuzzy prime submodule of  $M$ . Since  $B \in IF_L(M)$  and  $R$  is a commutative ring,  $B \in IF_L(R)$ .

For  $a_{(p,q)}, b_{(s,t)} \in ILFS(R)$ ,  $a_{(p,q)}b_{(s,t)} \in B$  implies  $a_{(p,q)} \in B$  or  $b_{(s,t)}\chi_M \subseteq B$ .

If  $a_{(p,q)} \in B$ , then  $B$  is an intuitionistic  $L$ -fuzzy prime ideal.

If  $b_{(s,t)}\chi_M \subseteq B$ , then for each  $m \in M$ ,

$$\mu_{b_{(s,t)}\chi_M}(bm) \leq \mu_B(bm)$$

and

$$\nu_{b_{(s,t)}\chi_M}(bm) \geq \nu_B(bm).$$

Since  $R$  has identity,  $b = b1$  and  $\mu_{b_{(s,t)}\chi_M}(b1) = s \leq \mu_B(b)$  and  $\nu_{b_{(s,t)}\chi_M}(b1) = t \geq \nu_B(b)$ . Thus  $s = \mu_{b_{(s,t)}\chi_M}(b) \leq \mu_B(b)$  and  $t = \nu_{b_{(s,t)}\chi_M}(b) \geq \nu_B(b)$ . So  $b_{(s,t)} \in B$ .

Conversely, let  $B$  be an intuitionistic  $L$ -fuzzy prime ideal of  $R$ . Then  $B \subset \chi_R$  and  $B \in IF_L(M)$ . Now, let  $r_{(s,t)}x_{(p,q)} \in B$ , for any  $r_{(s,t)} \in ILFS(R)$ ,  $x_{(p,q)} \in ILFS(M)$ .

If  $x_{(p,q)} \in B$ , then  $B$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M$ .

If  $x_{(p,q)} \notin B$ , then  $r_{(s,t)} \in B$ . Thus by the definition of intuitionistic  $L$ -fuzzy ideal of  $R$ ,

$$\mu_{r_{(s,t)}\chi_M}(rm) = s \leq \mu_B(r) \leq \mu_B(rm)$$

and

$$\nu_{r_{(s,t)}\chi_M}(rm) = t \geq \nu_B(r) \geq \nu_B(rm).$$

So  $r_{(s,t)}\chi_M \subseteq B$ . □

The following theorem, which relates intuitionistic fuzzy submodule to prime submodules of the module, will be needed in the proof of Theorem 3.6.

**Theorem 3.4.** *Let  $B$  be an intuitionistic  $L$ -fuzzy prime submodule of  $A$ . If  $B_{(\alpha,\beta)} \neq A_{(\alpha,\beta)}$ ,  $\alpha, \beta \in L$ , then  $B_{(\alpha,\beta)}$  is a prime submodule of  $A_{(\alpha,\beta)}$ .*

*Proof.* Let  $B_{(\alpha,\beta)} \neq A_{(\alpha,\beta)}$  and  $rx \in B_{(\alpha,\beta)}$ , for some  $r \in R, x \in M$ . If  $rx \in B_{(\alpha,\beta)}$ , then  $\mu_B(rx) \geq \alpha$  and  $\nu_B(rx) \leq \beta$ . Thus  $(rx)_{(\alpha,\beta)} = r_{(\alpha,\beta)}x_{(\alpha,\beta)} \in B$ . Since  $B$  is an intuitionistic  $L$ -fuzzy prime submodule of  $A$ , either  $x_{(\alpha,\beta)} \in B$  or  $r_{(\alpha,\beta)}A \subseteq B$ .

Case(i): If  $x_{(\alpha,\beta)} \in B$ , then  $\mu_B(x) \geq \alpha$  and  $\nu_B(x) \leq \beta$ . Thus  $x \in B_{(\alpha,\beta)}$ .

Case(ii): If  $r_{(\alpha,\beta)}A \subseteq B$ , then for any  $w \in rA_{(\alpha,\beta)}$ ,  $w = rz$ , for some  $z \in A_{(\alpha,\beta)}$ . Thus  $\mu_A(z) \geq \alpha$  and  $\nu_A(z) \leq \beta$ . On the other hand,

$$\alpha = \alpha \wedge \mu_A(z) \leq \text{Sup}\{\alpha \wedge \mu_A(x) : w = rx\} = \mu_{r_{(\alpha,\beta)}A}(w) \leq \mu_B(w).$$

Similarly, we have

$$\beta = \beta \vee \nu_A(z) \geq \text{Inf}\{\beta \vee \nu_A(x) : w = rx\} = \nu_{r_{(\alpha,\beta)}A}(w) \geq \nu_B(w).$$

So  $w \in B_{(\alpha,\beta)}$ . Hence  $rA_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$ . Therefore  $B_{(\alpha,\beta)}$  is a prime submodule of  $A_{(\alpha,\beta)}$ . □

**Corollary 3.5.** *Let  $B$  be an intuitionistic  $L$ -fuzzy prime submodule of  $M$ . Then*

$$B_* = \{x \in M : \mu_B(x) = \mu_B(\theta) \text{ and } \nu_B(x) = \nu_B(\theta)\}$$

*is a prime submodule of  $M$ .*

*Proof.* Clear from Theorem 3.4 as  $B_{(\alpha,\beta)} = B_*$ , when  $\alpha = \mu_B(\theta)$  and  $\beta = \nu_B(\theta)$ .  $\square$

The following theorem is the main result of section 3. It generalize the work of [10] from  $[0, 1]$  to a complete lattice  $L$ .

**Theorem 3.6.** (1) *Let  $N$  be a prime submodule of  $M$  and  $\alpha$  a prime element in  $L$ . If  $A$  is an ILFS of  $M$  defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } y \in N \\ \alpha', & \text{otherwise,} \end{cases}$$

for all  $x \in M$ , where  $\alpha'$  is complement of  $\alpha$  in  $L$ . Then  $A$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M$ .

(2) *Conversely, any intuitionistic  $L$ -fuzzy prime submodule can be obtained as in (1).*

*Proof.* (1) Since  $N$  is a prime submodule of  $M$ ,  $N \neq M$ , we have that  $A$  is non-constant intuitionistic  $L$ -fuzzy submodule of  $M$ . We show that  $A$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M$ .

Suppose  $r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$  are such that  $r_{(s,t)}x_{(p,q)} \in A$  and  $x_{(p,q)} \notin A$ .

If  $x_{(p,q)} \notin A$ , then  $\mu_A(x) = \alpha$  and  $\nu_A(x) = \alpha'$ . Thus  $x \notin N$ .

If  $r_{(s,t)}x_{(p,q)} \in A$ , then  $\mu_{(rx)_{(s \wedge p, t \vee q)}}(rx) \leq \mu_A(rx)$  and  $\nu_{(rx)_{(s \wedge p, t \vee q)}}(rx) \geq \nu_A(rx)$ . Thus  $s \wedge p \leq \mu_A(rx)$  and  $t \vee q \geq \nu_A(rx)$ .

If  $\mu_A(rx) = 1$  and  $\nu_A(rx) = 0$ , then  $rx \in N$ . Since  $x \notin N$  and  $N$  is a prime submodule of  $M$ , we have  $rM \subseteq N$ . Thus  $\mu_A(rm) = 1$  and  $\nu_A(rm) = 0$ , for all  $m \in M$ . So  $\mu_{r_{(s,t)}\chi_M}(rm) = s \leq \mu_A(rm)$  and  $\nu_{r_{(s,t)}\chi_M}(rm) = t \geq \nu_A(rm)$ .

If  $\mu_A(rx) = \alpha$  and  $\nu_A(x) = \alpha'$ , then  $s \wedge p \leq \alpha$  and  $t \vee q \geq \alpha'$ . As  $\alpha$  is prime element of  $L$ , we have  $s \wedge p \leq \alpha$  and  $p \not\leq \alpha$  implies  $s \leq \alpha$  and  $t \vee q \geq \alpha'$  implies  $t' \vee q' \geq \alpha$  and  $q' \not\leq \alpha$  implies  $t' \leq \alpha$ , i.e.,  $t \geq \alpha'$ . Thus for all  $w \in M$ ,

$$\mu_{r_{(s,t)}\chi_M}(w) = s \leq \alpha \leq \mu_A(w) \text{ and } \nu_{r_{(s,t)}\chi_M}(w) = t \geq \alpha' \geq \nu_A(w).$$

So  $r_{(s,t)}\chi_M \subseteq A$ . Hence  $A$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M$ .

(2) Let  $A$  be an intuitionistic  $L$ -fuzzy prime submodule of  $M$ . We show that  $A$  is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } y \in N \\ \alpha', & \text{otherwise,} \end{cases}$$

for all  $x \in M$ , where  $\alpha'$  is complement of the prime element  $\alpha$  in  $L$ .

Claim (1):  $A_* = \{x \in M : \mu_A(x) = \mu_A(\theta) \text{ and } \nu_A(x) = \nu_A(\theta)\}$  is a prime submodule of  $M$ .

Since  $A$  is a non-constant intuitionistic  $L$ -fuzzy prime submodule of  $M$ ,  $A_* \neq M$ . For all  $r \in R, m \in M$ , if  $rm \in A_*$  implies  $\mu_A(rm) = \mu_A(\theta)$  and  $\nu_A(rm) = \nu_A(\theta)$  so that  $(rm)_{(\mu_A(\theta), \nu_A(\theta))} = r_{(\mu_A(\theta), \nu_A(\theta))}m_{(\mu_A(\theta), \nu_A(\theta))} \in A$ , then  $m_{(\mu_A(\theta), \nu_A(\theta))} \in A$  or  $r_{(\mu_A(\theta), \nu_A(\theta))}\chi_M \subseteq A$ .

Case(i): If  $m_{(\mu_A(\theta), \nu_A(\theta))} \in A$ , then  $\mu_A(\theta) \leq \mu_A(m)$  and  $\nu_A(\theta) \geq \nu_A(m)$  but  $\mu_A(\theta) \geq \mu_A(m)$  and  $\nu_A(\theta) \leq \nu_A(m)$  [by definition of  $ILFSM$ ]. Thus  $\mu_A(m) = \mu_A(\theta)$  and  $\nu_A(m) = \nu_A(\theta)$ . So  $m \in A_*$ .

Case(ii): If  $r_{(\mu_A(\theta), \nu_A(\theta))} \chi_M \subseteq A$ , then  $\mu_A(\theta) \leq \mu_A(rm)$  and  $\nu_A(\theta) \geq \nu_A(rm)$ . Thus  $rm \in A_*$ , for all  $m \in M$ . On the other hand,

$$\theta \in N \text{ and } \mu_A(\theta) = 1, \nu_A(\theta) = 0.$$

So for all  $x \in A_*$ ,  $\mu_A(\theta) = \mu_A(x) = 1$  and  $\nu_A(\theta) = \nu_A(x) = 0$ . Hence  $A_* = N$ .

Claim (2):  $A$  has two values.

Since  $A_*$  is a prime submodule of  $M$ ,  $A_* \neq M$ . Then there exists  $z \in M \setminus A_*$ .

We will show that for all  $y \in M$  such that  $y \in A_*$ ,

$$\mu_A(y) = \mu_A(z) < \mu_A(\theta) \text{ and } \nu_A(y) = \nu_A(z) > \nu_A(\theta).$$

Then  $z \in A_*$ . Thus  $\mu_A(z) < 1 = \mu_A(\theta)$  and  $\nu_A(z) > 0 = \nu_A(\theta)$ . so  $z_{(1,0)} \notin A$  and  $z_{(\mu_A(z), \nu_A(z))} = z_{(1,0)} 1_{(\mu_A(z), \nu_A(z))} \in A$ . Hence  $1_{(\mu_A(z), \nu_A(z))} \chi_M \subseteq A$ . Since  $w = 1.w$ , for all  $w \in M$ , we have  $\mu_A(z) \leq \mu_A(w)$  and  $\nu_A(z) \geq \nu_A(w)$ .

Let  $w = y$ . Then  $\mu_A(z) \leq \mu_A(y)$  and  $\nu_A(z) \geq \nu_A(y)$ . Similarly,  $\mu_A(y) \leq \mu_A(z)$  and  $\nu_A(y) \geq \nu_A(z)$ . Thus  $\mu_A(z) = \mu_A(y)$  and  $\nu_A(z) = \nu_A(y)$ .

Claim (3): Let  $\mu_A(z) = \alpha$  and  $\nu_A(z) = \alpha'$ , where  $\alpha$  is prime element in  $L$  and  $\alpha'$  be its complement in  $L$ . First, let  $s \wedge p \leq \alpha$  and  $t \vee q \geq \alpha'$ , i.e.,  $t' \wedge q' \leq \alpha$  and let  $p \not\leq \alpha$  and  $q' \not\leq \alpha$ .

Suppose  $x \in M \setminus A_*$ . Then  $x_{(p,q)} \notin A$ . Thus  $1_{(s,t)} x_{(p,q)} = x_{(s \wedge p, t \vee q)} \in A$ . So  $1_{(s,t)} \chi_M \subseteq A$ , and for all  $w \in M$ ,  $\mu_{1_{(s,t)} \chi_M}(w) \leq \mu_A(w)$  and  $\nu_{1_{(s,t)} \chi_M}(w) \geq \nu_A(w)$ .

Let  $w = x$ . Then  $s = \mu_{1_{(s,t)} \chi_M}(w) \leq \mu_A(x) = \alpha$  and  $t = \nu_{1_{(s,t)} \chi_M}(w) \geq \nu_A(x) = \alpha'$ . Thus  $s \leq \alpha$  and  $t' \leq \alpha$ . Thus, every intuitionistic  $L$ -fuzzy prime submodule of  $M$  is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in N \\ \alpha', & \text{otherwise,} \end{cases}$$

for all  $x \in M$ , where  $\alpha'$  is complement of the prime element  $\alpha$  in  $L$  and  $N$  is a prime submodule of  $M$ . □

This theorem is particularly useful in deciding whether of not an intuitionistic fuzzy submodule is prime. The following example illustrate this.

**Example 3.7.** Let  $M = Z$  be a module over  $R = Z$ . Then

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in 3Z \\ 0.25, & \text{if otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in 3Z \\ 0.75, & \text{otherwise} \end{cases}$$

is an intuitionistic  $L$ -fuzzy prime submodule of  $Z$ , since  $3Z$  is prime submodule of  $Z$  and  $0.25$  is a prime element in  $[0, 1]$ .

#### 4. INTUITIONISTIC $L$ -FUZZY PRIME SUBMODULES OF HOMOMORPHIC MODULES

In this section, we investigate the behaviour of intuitionistic  $L$ -fuzzy prime submodules under an  $R$ -module epimorphism. Firstly, we recall the definition of image and inverse image of an intuitionistic  $L$ -fuzzy subset under a  $R$ -module homomorphism. From now on,  $M$  and  $M_1$  are  $R$ -modules.

**Definition 4.1.** Let  $f$  be a  $R$ -module homomorphism from  $M$  to  $M_1$ ,  $A \in ILFS(M)$  and  $B \in ILFS(M_1)$ . Then  $f(A) \in ILFS(M_1)$  and  $f^{-1}(B) \in ILFS(M)$  are defined by:  $\forall w \in M_1$  and  $\forall m \in M$ ,

$$f(A)(w) = \begin{cases} (Sup\{\mu_A(m) : m \in f^{-1}(w)\}, Inf\{\nu_A(m) : m \in f^{-1}(w)\}), & \text{if } f^{-1}(w) \neq \phi \\ (0, 1), & \text{otherwise} \end{cases}$$

and  $f^{-1}(B)(m) = (\mu_B(f(m)), \nu_B(f(m)))$ .

In the next two theorems we show that, both the image and inverse image of an intuitionistic  $L$ -fuzzy prime submodules under a  $R$ -module epimorphism are again intuitionistic  $L$ -fuzzy prime submodules. Here we need to assume that the complete lattice  $L$  is distributive.

**Theorem 4.2.** Let  $f$  be an  $R$ -modules epimorphism from  $M$  to  $M_1$ , and suppose that  $L$  is distributive. If  $A$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M$  such that  $\chi_{kerf} \subseteq A$ , then  $f(A)$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M_1$ .

*Proof.* Let  $w_1, w_2 \in M_1$ . Then

$$\begin{aligned} & \mu_{f(A)}(w_1) \wedge \mu_{f(A)}(w_2) \\ &= [Sup\{\mu_A(m_1) : f(m_1) = w_1\}] \wedge [Sup\{\mu_A(m_2) : f(m_2) = w_2\}] \\ &= Sup\{\mu_A(m_1) \wedge \mu_A(m_2) : f(m_1) = w_1, f(m_2) = w_2\} \\ &\leq Sup\{\mu_A(m_1 - m_2) : f(m_1) = w_1, f(m_2) = w_2\} \\ &\leq Sup\{\mu_A(m_1 - m_2) : f(m_1 - m_2) = w_1 - w_2\} \\ &= \mu_{f(A)}(w_1 - w_2). \end{aligned}$$

Thus  $\mu_{f(A)}(w_1 - w_2) \geq \mu_{f(A)}(w_1) \wedge \mu_{f(A)}(w_2)$ . Similarly, we can show that

$$\nu_{f(A)}(w_1 - w_2) \leq \nu_{f(A)}(w_1) \vee \nu_{f(A)}(w_2).$$

Furthermore, for all  $w_1 \in M_1$  and  $r \in R$ , we have

$$\begin{aligned} \mu_{f(A)}(w_1) &= Sup\{\mu_A(m) : f(m) = w_1\} \leq Sup\{\mu_A(rm) : f(m) = w_1\} \\ &= Sup\{\mu_A(rm) : f(rm) = rw_1\} \\ &= \mu_{f(A)}(rw_1). \end{aligned}$$

Thus,  $\mu_{f(A)}(rw_1) \geq \mu_{f(A)}(w_1)$ . Similarly, we can show that  $\nu_{f(A)}(rw_1) \leq \nu_{f(A)}(w_1)$ .

Also, it is clear that  $\mu_{f(A)}(\theta_1) = 1$  and  $\nu_{f(A)}(\theta_1) = 0$ . So,  $f(A)$  is an intuitionistic  $L$ -fuzzy submodule of  $M_1$ .

Next, we show that  $f(A)$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M_1$ . Since  $A$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M$ , so  $A$  is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in N \\ \alpha', & \text{otherwise,} \end{cases}$$

for all  $x \in M$ , where  $\alpha'$  is complement of the prime element  $\alpha$  in  $L$  and  $N = A_*$  is a prime submodule of  $M$ .

We first claim that if  $A_*$  is a prime submodule of  $M$  and  $\chi_{kerf} \subseteq A$ , then  $f(A_*)$  is prime submodule of  $M_1$ .

Let  $x \in \chi_{kerf}$ . Then  $\mu_{\chi_{kerf}}(x) = 1 \leq \mu_A(x)$  and  $\nu_{\chi_{kerf}}(x) = 0 \geq \nu_A(x)$ . Thus  $\mu_A(x) = \mu_A(\theta)$  and  $\nu_A(x) = \nu_A(\theta)$ . So  $x \in A_*$ . Hence  $kerf \subseteq A_*$ .



For all  $r \in R, w \in M_1, rw \in f(A_*)$ , there exists  $z \in A_*$  such that  $rw = f(z)$ . Since  $f$  is an epimorphism, there exists  $m \in M$  such that  $rw = rf(m) = f(rm) = f(z)$ . Now,  $rm \in A_*$  and  $A_*$  is a prime submodule of  $M$ . Then either  $m \in A_*$  or  $rM \subseteq A_*$ .

If  $m \in A_*$ , then  $w = f(m) \in f(A_*)$  and if  $rM \subseteq A_*$ , then  $rM_1 = f(rM) \subseteq f(A_*)$ . Thus  $f(A_*)$  is a prime submodule of  $M_1$ . Since  $\alpha$  is a prime element in  $L$ , by Theorem 3.6, for all  $w \in M_1$ ,

$$\mu_{f(A)}(w) = \begin{cases} 1, & \text{if } w \in f(A_*) \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_{f(A)}(w) = \begin{cases} 0, & \text{if } w \in f(A_*) \\ \alpha', & \text{otherwise.} \end{cases}$$

So  $f(A)$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M_1$ . □

**Example 4.3.** Let  $f$  be a homomorphism from  $Z$  to  $Z$  defined by  $f(x) = 2x$ , and let

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in 3Z \\ 0.25, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in 3Z \\ 0.75, & \text{otherwise} \end{cases}$$

be an intuitionistic  $L$ -fuzzy prime submodule of  $Z$ . Then

$$\begin{aligned} f(A)(0) &= (Sup\{\mu_A(x) : f(n) = 0\}, Inf\{\nu_A(x) : f(n) = 0\}) \\ &= (\mu_A(0), \nu_A(0)) = (1, 0) \end{aligned}$$

and

$$\begin{aligned} f(A)(1) &= (Sup\{\mu_A(x) : f(n) = 1\}, Inf\{\nu_A(x) : f(n) = 1\}) \\ &= (0, 1) \text{ [ As } f^{-1}(1) = \emptyset \text{].} \end{aligned}$$

Similarly, we can find that  $f(A)(3) = f(A)(5) = (0, 1)$  and  $f(A)(2) = f(A)(4) = (0.25, 0.75)$  and so on. Thus we get

$$\mu_{f(A)}(x) = \begin{cases} 1, & \text{if } x \in 6Z \\ 0.25, & \text{if } x \in 2Z - 6Z \\ 0, & \text{if } x \in Z - 2Z \end{cases}; \quad \nu_{f(A)}(x) = \begin{cases} 0, & \text{if } x \in 6Z \\ 0.75, & \text{if } x \in 2Z - 6Z \\ 1, & \text{if } x \in Z - 2Z \end{cases}$$

is not an intuitionistic  $L$ -prime fuzzy submodule of  $Z$ . This shows that the assumption that  $f$  be an epimorphism in Theorem 4.2( cannot be dropped.

**Theorem 4.4.** Let  $f$  be a  $R$ -module epimorphism from  $M$  to  $M_1$ . If  $B$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M_1$ , then  $f^{-1}(B)$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M$ .

*Proof.* Let  $B$  be an intuitionistic  $L$ -fuzzy prime submodule of  $M_1$ . Then

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in B_* \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x \in B_* \\ \alpha', & \text{otherwise,} \end{cases}$$

for all  $x \in M_1$ , where  $\alpha'$  is complement of the prime element  $\alpha$  in  $L$  and  $B_*$  is a prime submodule of  $M_1$ .

We first show that  $f^{-1}(B_*)$  is a prime submodule of  $M$ .

For all  $r \in R, m \in M$ , if  $rm \in f^{-1}(B_*)$ , then  $f(rm) \in B_*$ , i.e.,  $rf(m) \in B_*$ . As  $B_*$  is prime submodule of  $M_1$ , either  $f(m) \in B_*$  or  $rM_1 \subseteq B_*$ .

If  $f(m) \in B_*$ , then  $m \in f^{-1}(B_*)$  and if  $rM_1 \subseteq B_*$ , then  $rf(M) = f(rM) \subseteq B_*$ . Thus  $rM \subseteq f^{-1}(B_*)$ . So

$$\mu_{f^{-1}(B)}(x) = \begin{cases} 1, & \text{if } x \in f^{-1}(B_*) \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_{f^{-1}(B)}(x) = \begin{cases} 0, & \text{if } x \in f^{-1}(B_*) \\ \alpha', & \text{otherwise.} \end{cases}$$

Hence  $f^{-1}(B)$  is an intuitionistic  $L$ -fuzzy prime submodule of  $M$ . □

## 5. CONCLUSION

As the study of modules over a ring  $R$  provides us with an insight into the structure of  $R$ . In the same way the study of intuitionistic  $L$ -fuzzy modules provides us with an insight into the structure of lattice  $L$ . In this paper, we have given a characterization of intuitionistic  $L$ -fuzzy prime submodules and also investigate the behaviour of intuitionistic  $L$ -fuzzy prime submodules under  $R$ -homomorphisms. This is useful for the further study of intuitionistic  $L$ -fuzzy modules.

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