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# Application of weighted average cardinality measure of intuitionistic fuzzy sets in ranking of intuitionistic fuzzy sets

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ABSTRACT. In this paper, we have explored different aspects of weighted average cardinality of finite intuitionistic fuzzy sets with aid of necessity and possibility operators. Also we have defined the concept of scalar valued relative sigma count of intuitionistic fuzzy sets, which in turn is utilized to define a ranking technique for intuitionistic fuzzy sets and the concept of intuitionistic fuzzy quantifiers.

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## 1. INTRODUCTION

Fuzzy sets introduced in 1965 express the level of belongingness to a set [34], and are used to describe the situations and objects which are unsharply defined in every day language. Several higher order extensions of fuzzy sets and, their possible equivalences have been debated [3, 4, 7, 12, 19, 20]. Among these generalizations, the notion of $IFS$ proposed by Atanassov, which specifies a degree of membership, a degree of non membership and a degree of hesitancy has gained an inevitable attention of researchers due to its wide range of applications in dealing with real life vagueness and uncertainty [1, 2, 5, 6, 16].

In case of fuzzy sets, there are several approaches to define the cardinality of a fuzzy set: either as a single number called scalar cardinality [13, 14, 17, 21], or alternatively, as fuzzy cardinalities, in which case the cardinality of a fuzzy set is defined as a fuzzy quantity [11, 18].

This paper is concerned with the intuitionistic version of these notions and the resulting theory. Looking behind, in 1995, Atanassova [8] introduced the concept of a scalar cardinality of an intuitionistic fuzzy set while Szmidt [29], presented it in the form of a numerical interval with membership degree as lower bound and the sum of hesitancy and membership degrees as an upper bound. Tripathy [31], re-explored different properties of this numerical interval cardinality and further proposed the concept of relative intuitionistic fuzzy count. The mean value of the cardinality interval was defined as a scalar cardinality of $IFS$ by Vlachos [32] and Král [23], developed an axiomatic approach towards the extensions of scalar cardinalities of fuzzy sets to their intuitionistic versions. In [25], another family of scalar cardinality measures of $IFSs$ called weighted average cardinality measure for intuitionistic fuzzy sets was introduced along with some other new fuzzy measures on $IFSs$.

Intuitionistic fuzzy sets have been successfully used in management sciences especially to deal with decision and game problems. The ranking of $IFSs$ by score function or by other defuzzification methods play the most important role in designing the possible solution of these problems [24].

Commencing from the notion of intuitionistic fuzzy set, the main purpose of this paper is to explore different aspects of a newly defined weighted average cardinality measure for intuitionistic fuzzy sets [25] and, to utilize it as a basic mathematical tool for establishing an efficient ranking technique for finite intuitionistic fuzzy sets in an intuitionistic fuzzy decision making environment. Before stepping into the main problem, we shall provide the reader an overview of the basic definitions and notations involved in this study. This research work will comprise of two parts: In Section 3, we shall present some new results related to the study of different properties of weighted average cardinality measure such as valuation property and complementation rule introduced in [25]. Section 4, will be reserved for the applications where this quantified mathematical tool will be used to define the concept of scalar valued relative sigma count for an intuitionistic fuzzy set. The newly defined relative sigma count is further employed to define an efficient and simple ranking technique of intuitionistic fuzzy sets. The efficiency of the technique will be illustrated by different case studies in the field of organizational management and medical diagnosis. Moreover, the concept of intuitionistic fuzzy quantifiers based on this new scalar valued relative sigma count for an intuitionistic fuzzy set is demonstrated with the aid of examples.

Lastly, taking into consideration the close relation between intuitionistic fuzzy sets and the other generalized fuzzy sets such as interval valued fuzzy sets [19] and vague sets [12], we are in a position to claim that, all the results produced in this work can be easily modified and adapted to the extended frame works of any of the mentioned higher order fuzzy sets and fuzzy graphs as well [26, 27, 28, 30].

## 2. Preliminaries

Before introducing the definition of intuitionistic fuzzy set we recall the complete and bounded lattice $L^*$ which provides the mathematical basis for upcoming definitions and results. The set $L^* = \left\{(a_1,a_2) \in L^2 \mid L = [0,1], a_1 + a_2 \leq 1\right\}$ is a complete and bounded lattice $(L^*, \leq_{L^*})$ equipped with order $\leq_{L^*}$, which is defined as: $(a_1,a_2) \leq_{L^*} (b_1,b_2)$ if and only if $a_1 \leq b_1$ and $a_2 \geq b_2$. The elements $1_{L^*} = (1,0)$ and $0_{L^*} = (0,1)$ are the greatest and the smallest element of the lattice $L^*$ respectively.

The intuitionistic fuzzy set $IFS$ $A$ on $X$ is a mapping $A : X \longrightarrow L^*$ such that for any $x \in X$, $A(x) = (\mu_A(x), \nu_A(x)) = (a_1,a_2) \in L^*$ that is $a_1 + a_2 \leq 1$.

Alternatively as defined by the founder of $IFS$:

**Definition 2.1** ([4]). An $IFS$ on a universe of discourse $X$ is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A(x)$, $\nu_A(x) \in [0,1]$ are the degree of membership and the degree of non membership of $x$ in $A$ respectively with $\mu_A(x)+\nu_A(x)\leq 1$ for all $x\in X$. The class of all intuitionistic fuzzy sets on $X$ is denoted by $IFS(X)$.

Note that a fuzzy set $A$ in $X$ is an intuitionistic fuzzy set for which $\mu_A(x)+\nu_A(x)=1$ holds for every $x\in X$ and the class of all fuzzy sets in $X$ is denoted by $FS(X)$.

The complement of an $IFS\ A=\{(x,\mu_A(x),\nu_A(x))\mid x\in X\}$ is defined as:

$$A^c=\{(x,\nu_A(x),\mu_A(x))\mid x\in X\}.$$

**Definition 2.2** ([4]). For any two $IFSs$ $A$ and $B$ the subsethood $A\subseteq B$ and equality of $A$ and $B$ denoted by $A=B$ are defined as:

$$\begin{aligned} A\subseteq B &\iff (\forall x\in X)(\mu_A(x)\leq\mu_B(x) \text{ and } \nu_A(x)\geq\nu_B(x)),\\ A=B &\iff (\forall x\in X)(\mu_A(x)=\mu_B(x) \text{ and } \nu_A(x)=\nu_B(x)).\end{aligned}$$

**Definition 2.3** ([15]). An intuitionistic fuzzy negator is a decreasing $L^*\longrightarrow L^*$ mapping $\check{N}$ that satisfies $\check{N}(0_{L^*})=1_{L^*}$ and $\check{N}(1_{L^*})=0_{L^*}$. If $\check{N}(\check{N}(x))=x$, $\forall x\in L^*$, then $\check{N}$ is called an involutive intuitionistic fuzzy negator. The mapping $\check{N}_s$ defined as: $\check{N}_s(x_1,x_2)=(x_2,x_1)$ $\forall(x_1,x_2)\in L^*$ will be called the standard intuitionistic fuzzy negator. The involutive negator on $L^*$ can always be related to an involutive negator on $[0,1]$.

**Definition 2.4** ([15]). An intuitionistic fuzzy t-norm is an increasing, commutative, associative $(L^*)^2\longrightarrow L^*$ mapping $\check{T}$ satisfying $\check{T}(1_{L^*},x)=x$, for all $x\in L^*$.

**Definition 2.5** ([15]). An intuitionistic fuzzy t-conorm is an increasing, commutative, associative $(L^*)^2\longrightarrow L^*$ mapping $\check{S}$ satisfying $\check{S}(0_{L^*},x)=x$ for all $x\in L^*$.

**Definition 2.6** ([15]). The dual of an intuitionistic fuzzy t-norm $\check{T}$(t-conorm $\check{S}$) w.r.t. a negator $\check{N}$ is a mapping $\check{T}^*$[respectively $\check{S}^*$] defined by for $x,y\in L^*$,

$$\check{T}^*(x,y)=\check{N}(\check{T}(\check{N}(x),\check{N}(y)))$$
$$[\text{respectively } \check{S}^*(x,y)=\check{N}(\check{S}(\check{N}(x),\check{N}(y)))].$$

It can be verified that $\check{T}^*$ is an intuitionistic fuzzy conorm and $\check{S}^*$ is an intuitionistic fuzzy t-norm.

**Definition 2.7** ([15]). An intuitionistic fuzzy t-norm $\check{T}=\overline{T}$ (respectively t-conorm $\check{S}=\overline{S}$) is called t-representable, if there exists a t-norm $T$ and a t-conorm $S$ on [0,1] (respectively a t-conorm $S'$ and a t-norm $T'$ on [0,1]) such that for $x,y\in L^*$,

$$\overline{T}(x,y)=(T(x_1,y_1),S(x_2,y_2))$$
$$[\text{respectively } \overline{S}(x,y)=(S'(x_1,y_1),T'(x_2,y_2))]$$

where $T$ and $S$ (respectively $S'$ and $T'$) are called the representants of $\check{T}$ (respectively $\check{S}$).

For instance, using the family of Frank t-norms;

$$T^F_\omega(a,b)=\left\{\begin{array}{ll} T_M(a,b), & \text{if } \omega=0\\ T_P(a,b), & \text{if } \omega=1\\ T_L(a,b), & \text{if } \omega=+\infty\\ \log_\omega\left(1+\frac{(\omega^a-1)(\omega^b-1)}{\omega-1}\right), & \text{otherwise}\end{array}\right\}$$

and their dual conorms

$$S^F_\omega(a,b) = \left\{ \begin{array}{ll} S_M(a,b), & \text{if } \omega = 0 \\ S_P(a,b), & \text{if } \omega = 1 \\ S_L(a,b), & \text{if } \omega = +\infty \\ 1 - \log_\omega\left(1 + \frac{(\omega^{1-a}-1)(\omega^{1-b}-1)}{\omega-1}\right), & \text{otherwise} \end{array} \right\}$$

such that $\omega \in [0,+\infty[$ and $a,b \in [0,1]$, we define t-representable intuitionistic fuzzy family of Frank t-norms as;

$$\overline{T}(x,y) = (T^F_\omega(x_1,y_1), S^F_\omega(x_2,y_2))$$

and their dual t-representable intuitionistic fuzzy family of Frank t-conorms as:

$$\overline{S}(x,y) = (S^F_\omega(x_1,y_1), T^F_\omega(x_2,y_2)),$$

where $x = (x_1,x_2)$, $y = (y_1,y_2) \in L^*$.

**Theorem 2.8** ([22])**.** *The family of Frank t-norms $T^F_\omega$ and Frank t-conorms $S^F_\omega$ together with ordinal sums of these are the unique t-norms and t-conorms which satisfy the relation:*

$$T^F_\omega(a,b) + S^F_\omega(a,b) = a+b,$$

*for every $a,b \in [0,1]$.*

**Remark 2.9.** For two intuitionistic fuzzy sets $A,B \in IFS(X)$, the generalized $\cup$ and $\cap$ between two $IFSs$ is modelled by t-representable intuitionistic fuzzy t-norms $\overline{T}$ and conorms $\overline{S}$ defined as:

$$A \cup_{\overline{S}} B = \overline{S}(A(x),B(x)) = (S(\mu_A(x),\mu_B(x)), T(\nu_A(x),\nu_B(x))) \tag{2.1}$$

$$A \cap_{\overline{T}} B = \overline{T}(A(x),B(x)) = (T(\mu_A(x),\mu_B(x)), S(\nu_A(x),\nu_B(x))) \tag{2.2}$$

such that $A(x) = (\mu_A(x),\nu_A(x))$ and $B(x) = (\mu_B(x),\nu_B(x))$ and $T$ and $S$ are t-norm and t-conorm on $[0,1]$ satisfying the following two relations:

$$T(\mu_A(x),\mu_B(x)) + S(1-\mu_A(x), 1-\mu_B(x)) \le 1 \tag{2.3}$$

$$S(\mu_A(x),\mu_B(x)) + T(1-\mu_A(x), 1-\mu_B(x)) \le 1. \tag{2.4}$$

Now if we work with the pair $(T,S)$ which are duals of each other ($T(a,b) = 1 - S(1-a, 1-b)$ for all $a,b \in [0,1]$), then the relations (2.3) and (2.4) are verified by equality.

## 3. The class of weighted average cardinality Measure $Card_\theta(A)$ and its Properties

In intuitionistic fuzzy literature, the notion of measure for an $IFS$ has been extended mainly by Ban [9] in two ways: as a fuzzy measure of intuitionistic fuzzy sets and intuitionistic fuzzy measure of intuitionistic fuzzy sets. Both of these measures were based on $\sigma$-algebra of intuitionistic fuzzy sets in a crisp universe $X$ called an intuitionistic fuzzy $\sigma$-algebra on $X$.

**Definition 3.1** ([10])**.** An intuitionistic fuzzy $\sigma$-algebra on $X$ is a family $\pounds$ of $IFSs$ on $X$ satisfying the properties:

(i) $X \in \pounds$,
(ii) If $A \in \pounds$ then it implies $A^c \in \pounds$,
(iii) $\cup_{n\in N} A_n \in \pounds$, for every sequence $(A_n)_{n\in N}$ of $IFSs$ in $\pounds$.

The sets in $\pounds$ are called the intuitionistic fuzzy measurable sets and the pair $(X,\pounds)$ is called an intuitionistic fuzzy measurable space.

**Definition 3.2** ([9]). Let $(X,\pounds)$ be an intuitionistic fuzzy measurable space. A function $m : \pounds \to [0,\infty]$ is called a fuzzy measure of intuitionistic fuzzy sets, if it satisfies the following conditions:
(i) $m(\phi) = 0$.
(ii) for any $A,B \in \pounds$, $A \subseteq B$ implies $m(A) \leq m(B)$.
The measure $m$ with the boundary condition $m(X) = 1$ is called a normalized or normal fuzzy measure.

In this section, we have investigated a new fuzzy measure defined in terms of weighted average cardinality of intuitionistic fuzzy sets in [25] and found that this new scalar cardinality measure of intuitionistic fuzzy set satisfies properties similar to the ones introduced for fuzzy sets in [33]. Moreover, we have studied some basic characteristics of this family of cardinality measures especially the valuation property, the subadditivity property, the complementary rule defined using t-norms and negations on $L^*$. Also, in a subsection we have explored different aspects of this cardinality measure by employing the logical operators of necessity and possibility. Through out this work we shall take $X = \{x_1,x_2,...x_n\}$ i.e., $|X| = n$.

Before proceeding let us recall the definition of weighted average cardinality measure of intuitionistic fuzzy sets.

**Definition 3.3** ([25]). A mapping $Card_\theta(A) : IFS(X) \longrightarrow [0,\infty)$ is called the weighted average cardinality of intuitionistic fuzzy sets given as:

$$Card_\theta(A) = \sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1-\nu_A(x)) \text{ where } \theta \in [0.5,1]. \tag{3.1}$$

It is easy to observe that $Card_\theta(A)$ satisfies the following properties, for all $\theta \in [0.5,1]$:
(i) (Coincidence) for all $x \in X$, $Card_\theta(A(x)) = 1$, when $A(x) = 1_{L^*}$,
(ii) (Monotonicity) for all $A,B \in IFS(X)$,

$$A \subseteq B \text{ implies } Card_\theta(A) \leq Card_\theta(B), \tag{3.2}$$

(iii) (Additivity) for all $A,B \in IFS(X)$,

$$Supp(A) \cap Supp(B) = \phi \text{ implies } Card_\theta(A \cup_{\check{S}} B) = Card_\theta(A) + Card_\theta(B).$$

Now from Theorem 2.8, we know that if we restrict the pair $(T,S)$ in (2.1) and (2.2) by the family of Frank t-norms $T^F_\omega$ and their dual conorm $S^F_\omega$, then the operators $T^F_\omega$ and $S^F_\omega$ satisfy the valuation property stated as: $T^F_\omega(a,b) + S^F_\omega(a,b) = a+b$ for all $a,b \in [0,1]$. Thus, they are the most suitable choice to study the properties of new defined weighted average cardinality measure for intuitionistic fuzzy sets.

If $\check{T}$ is a t-norm on $L^*$ that does not have zero divisor, i.e., $\check{T}(x,y) = 0_{L^*}$ implies $x = 0_{L^*}$ or $y = 0_{L^*}$ and $\check{S}$ is the dual conorm of $\check{T}$ on $L^*$, then the property (iii) of $Card_\theta$ can be equivalently defined by following statement:
(iv) for $A,B \in$IFS$(X)$, $A \cap_{\check{T}} B = \phi$ implies $Card_\theta(A \cup_{\check{S}} B) = Card_\theta(A) + Card_\theta(B)$.

In this work, we have utilized the following t-representable intuitionistic fuzzy Frank t-norms and t-conorms:

$$\overline{T}_M(A(x),B(x)) = (T_M(\mu_A(x),\mu_B(x)),S_M(\nu_A(x),\nu_B(x))),$$
$$\overline{T}_P(A(x),B(x)) = (T_P(\mu_A(x),\mu_B(x)),S_P(\nu_A(x),\nu_B(x)))),$$
$$\overline{T}_L(A(x),B(x)) = (T_L(\mu_A(x),\mu_B(x)),S_L(\nu_A(x),\nu_B(x)))),$$
$$\overline{S}_M(A(x),B(x)) = (S_M(\mu_A(x),\mu_B(x)),T_M(\nu_A(x),\nu_B(x)))),$$
$$\overline{S}_P(A(x),B(x)) = (S_P(\mu_A(x),\mu_B(x)),T_P(\nu_A(x),\nu_B(x)))),$$
$$\overline{S}_L(A(x),B(x)) = (S_L(\mu_A(x),\mu_B(x)),T_L(\nu_A(x),\nu_B(x)))),$$

where $A(x) = (\mu_A(x),\nu_A(x))$ and $B(x) = (\mu_B(x),\nu_B(x))$.

**Proposition 3.4.** *Let $A,B \in$IFS(X). The following properties hold, for $Card_\theta$:*

(1) *(Valuation property):*

$(1_a)$ $Card_\theta(A \cup_{\overline{S}_M} B) + Card_\theta(A \cap_{\overline{T}_M} B) = Card_\theta(A) + Card_\theta(B)$,

$(1_b)$ $Card_\theta(A \cup_{\overline{S}_P} B) + Card_\theta(A \cap_{\overline{T}_P} B) = Card_\theta(A) + Card_\theta(B)$,

(2) *(Complementary rule): $Card_\theta(A) + Card_\theta(A^c) = Card_\theta(X)$ if and only if $\theta = 0.5$.*

*Proof.* $(1_a)$ For all $A,B \in IFS(X)$,

$$(A \cap_{\overline{T}_M} B)(x) = (T_M(\mu_A(x),\mu_B(x)),S_M(\nu_A(x),\nu_B(x)))$$

and

$$(A \cup_{\overline{S}_M} B)(x) = (S_M(\mu_A(x),\mu_B(x)),T_M(\nu_A(x),\nu_B(x)))$$

implies that

$$Card_\theta(A \cap_{\overline{T}_M} B) = \sum_{x\in X} \theta \min(\mu_A(x),\mu_B(x)) + (1-\theta)\min((1-\nu_A(x)),(1-\nu_B(x)))$$

and

$$Card_\theta(A \cup_{\overline{S}_M} B) = \sum_{x\in X} \theta \max(\mu_A(x),\mu_B(x)) + (1-\theta)\max((1-\nu_A(x)),(1-\nu_B(x)))$$

which together result in

$$Card_\theta(A \cup_{\overline{S}_M} B) + Card_\theta(A \cap_{\overline{T}_M} B) = Card_\theta(A) + Card_\theta(B).$$

$(1_b)$ For all $A,B \in IFS(X)$,

$$(A \cap_{\overline{T}_P} B)(x) = (T_P(\mu_A(x),\mu_B(x)),S_P(\nu_A(x),\nu_B(x)))$$

and

$$(A \cup_{\overline{S}_P} B)(x) = (S_P(\mu_A(x),\mu_B(x)),T_P(\nu_A(x),\nu_B(x)))$$

implies that:

$$Card_\theta(A \cap_{\overline{T}_P} B) = \sum_{x\in X} \theta(\mu_A(x)\mu_B(x)) + (1-\theta)(1-\nu_A(x)-\nu_B(x)+\nu_A(x)\nu_B(x))$$

and

$$Card_\theta(A \cup_{\overline{S}_P} B) = \sum_{x\in X} \theta(\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x)) + (1-\theta)(1-\nu_A(x)\nu_B(x)).$$

Hence

$$Card_\theta(A \cup_{\overline{S}_P} B) + Card_\theta(A \cap_{\overline{T}_P} B) = Card_\theta(A) + Card_\theta(B).$$

(2) Let us take for all $x \in X$. Then

$$\begin{aligned}
& Card_\theta(A) + Card_\theta(A^c) = Card_\theta(X) \\
\iff \quad & \sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1-\nu_A(x)) + \sum_{x\in X} \theta\nu_A(x) + (1-\theta)(1-\mu_A(x)) \\
& = \sum_{x\in X} \theta + (1-\theta) = \sum_{x\in X} 1 \\
\iff \quad & \theta\mu_A(x) + (1-\theta)(1-\nu_A(x)) + \theta\nu_A(x) + (1-\theta)(1-\mu_A(x)) = 1 \\
\iff \quad & 2(1-\theta) = 1 \\
\iff \quad & 1-\theta = 0.5 \\
\iff \quad & \theta = 0.5.
\end{aligned}$$

□

**Theorem 3.5.** *For any $A, B \in IFS(X)$,*
(1) $Card_\theta((A \cup_{\overline{S}_M} B)^c) + Card_\theta((A \cap_{\overline{T}_M} B)^c) = Card_\theta(A^c) + Card_\theta(B^c)$,
(2) $Card_\theta((A \cup_{\overline{S}_P} B)^c) + Card_\theta((A \cap_{\overline{T}_P} B)^c) = Card_\theta(A^c) + Card_\theta(B^c)$.

*Proof.* (1) For all $A, B \in IFS(X)$,

$$(A \cap_{\overline{T}_M} B)^c(x) = (S_M(\nu_A(x), \nu_B(x)), T_M(\mu_A(x), \mu_B(x)))$$

and

$$(A \cup_{\overline{S}_M} B)^c(x) = (T_M(\nu_A(x), \nu_B(x)), S_M(\mu_A(x), \mu_B(x)))$$

implies that

$$Card_\theta(A \cup_{\overline{S}_M} B)^c = \sum_{x\in X} \theta \min(\nu_A(x), \nu_B(x)) + (1-\theta)\min((1-\mu_A(x)), (1-\mu_B(x)))$$

and

$$Card_\theta(A \cap_{\overline{T}_M} B)^c = \sum_{x\in X} \theta \max(\nu_A(x), \nu_B(x)) + (1-\theta)\max((1-\mu_A(x)), (1-\mu_B(x)))$$

which together result in

$$Card_\theta(A \cup_{\overline{S}_M} B)^c + Card_\theta(A \cap_{\overline{T}_M} B)^c = Card_\theta(A^c) + Card_\theta(B^c).$$

(2) For all $A, B \in IFS(X)$,

$$(A \cap_{\overline{T}_P} B)^c(x) = (S_P(\nu_A(x), \nu_B(x)), T_P(\mu_A(x), \mu_B(x)))$$

and

$$(A \cup_{\overline{S}_P} B)^c = (T_P(\nu_A(x), \nu_B(x)), S_P(\mu_A(x), \mu_B(x)))$$

implies that

$$Card_\theta(A \cup_{\overline{S}_P} B)^c = \sum_{x\in X} \theta(\nu_A(x)\nu_B(x)) + (1-\theta)(1-\mu_A(x) - \mu_B(x) + \mu_A(x)\mu_B(x))$$

and

$$Card_\theta(A \cap_{\overline{T}_P} B)^c = \sum_{x\in X} \theta(\nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x)) + (1-\theta)(1-\mu_A(x)\mu_B(x)).$$

Hence

$$Card_\theta(A \cup_{\overline{S}_P} B)^c + Card_\theta(A \cap_{\overline{T}_P} B)^c = Card_\theta(A^c) + Card_\theta(B^c). \qquad \square$$

3.1. **Results of $Card_\theta(A)$ on possibility and necessity operators.** Among several interesting operators introduced in the family of all intuitionistic fuzzy sets [4], the two operators called by Atanassov the necessity and possibility operators have gained a substantial attention of the research community. For any $A \in IFS(X)$, the necessity operator $\square : IFS(X) \longrightarrow IFS(X)$ is defined as:

$$\square A = \{(x, \mu_A(x), 1-\mu_A(x)) \mid x \in X\} \tag{3.3}$$

and the possibility operator $\lozenge : IFS(X) \longrightarrow IFS(X)$ is defined as:

$$\lozenge A = \{(x, 1-\nu_A(x), \nu_A(x)) \mid x \in X\}. \tag{3.4}$$

Both of these operators had exhibited many interesting properties. Besides their connection with the well known possibility theory, the terms possibility and necessity are clear even to those practitioners who are not expert in this field. A natural and straightforward interpretation of these two ideas made them useful in many different fields. For example, let us mention the problem of ranking fuzzy numbers, where hundreds of methods have been proposed but its formulation in the setting of possibility theory have resulted in generally accepted tool; the possibility/necessity indices of dominance; which are successfully applied in many situations.

In this subsection, we shall explore different aspects of the new defined weighted average cardinality measure $Card_\theta(A) = \sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1-\nu_A(x))$ when applied to modal operators $\square A$ and $\lozenge A$.

**Remark 3.6.** For any $A \in IFS(X)$, the following hold:
(1) $Card_\theta(\square A) = Card_\theta((\lozenge A^c)^c)$,
(2) $Card_\theta(\lozenge A) = Card_\theta((\square A^c)^c)$,
(3) $Card_\theta(\square\square A) = Card_\theta(\square A)$,
(4) $Card_\theta(\square\lozenge A) = Card_\theta(\lozenge A)$,
(5) $Card_\theta(\lozenge\square A) = Card_\theta(\square A)$,
(6) $Card_\theta(\lozenge\lozenge A) = Card_\theta(\lozenge A)$,
(7) $Card_\theta(\lozenge A^c) = Card_\theta((\square A)^c)$,
(8) $Card_\theta(\square A^c) = Card_\theta((\lozenge A)^c)$,
(9) $Card_\theta(\square A) \leq Card_\theta(\lozenge A)$.

**Theorem 3.7.** *For any $A, B \in IFS(X)$, we have:*
(1) $Card_\theta\square(A \cup_{\overline{S}_M} B) = Card_\theta(\square A \cup_{\overline{S}_M} \square B)$,
(2) $Card_\theta\lozenge(A \cup_{\overline{S}_M} B) = Card_\theta(\lozenge A \cup_{\overline{S}_M} \lozenge B)$,
(3) $Card_\theta\square(A \cap_{\overline{T}_M} B) = Card_\theta(\square A \cap_{\overline{T}_M} \square B)$,
(4) $Card_\theta\lozenge(A \cap_{\overline{T}_M} B) = Card_\theta(\lozenge A \cap_{\overline{T}_M} \lozenge B)$,
(5) $Card_\theta\square(A \cup_{\overline{S}_P} B) = Card_\theta(\square A \cup_{\overline{S}_P} \square B)$,
(6) $Card_\theta\lozenge(A \cup_{\overline{S}_P} B) = Card_\theta(\lozenge A \cup_{\overline{S}_P} \lozenge B)$,

(7) $Card_\theta\Box(A\cap_{\overline{T}_P}B)=Card_\theta(\Box A\cap_{\overline{T}_P}\Box B)$,
(8) $Card_\theta\Diamond(A\cap_{\overline{T}_P}B)=Card_\theta(\Diamond A\cap_{\overline{T}_P}\Diamond B)$,
(9) $Card_\theta\Box(A\cup_{\overline{S}_L}B)=Card_\theta(\Box A\cup_{\overline{S}_L}\Box B)$,
(10) $Card_\theta\Diamond(A\cup_{\overline{S}_L}B)=Card_\theta(\Diamond A\cup_{\overline{S}_L}\Diamond B)$,
(11) $Card_\theta\Box(A\cap_{\overline{T}_L}B)=Card_\theta(\Box A\cap_{\overline{T}_L}\Box B)$,
(12) $Card_\theta\Diamond(A\cap_{\overline{T}_L}B)=Card_\theta(\Diamond A\cap_{\overline{T}_L}\Diamond B)$.

*Proof.* (1) For any $A,B\in IFS(X)$,

$$\begin{aligned}\Box(A\cup_{\overline{S}_M}B)&=\{(x,\max(\mu_A(x),\mu_B(x)),1-\max(\mu_A(x),\mu_B(x)))\mid x\in X\}\\&=\{(x,\max(\mu_A(x),\mu_B(x)),\min(1-\mu_A(x),1-\mu_B(x)))\mid x\in X\}\\&=\Box A\cup_{\overline{S}_M}\Box B.\end{aligned}$$

Then

$$Card_\theta(\Box(A\cup_{\overline{S}_M}B))=Card_\theta(\Box A\cup_{\overline{S}_M}\Box B).$$

(2) For any $A,B\in IFS(X)$,

$$\begin{aligned}\Diamond(A\cup_{\overline{S}_M}B)&=\{(x,1-\min(\nu_A(x),\nu_B(x)),\min(\nu_A(x),\nu_B(x)))\mid x\in X\}\\&=\{(x,\max(1-\nu_A(x),1-\nu_B(x)),\min(\nu_A(x),\nu_B(x)))\mid x\in X\}\\&\Diamond A\cup_{\overline{S}_M}\Diamond B.\end{aligned}$$

Then

$$Card_\theta\Diamond(A\cup_{\overline{S}_M}B)=Card_\theta(\Diamond A\cup_{\overline{S}_M}\Diamond B).$$

The proofs of the remaining parts can be constructed in a similar manner. □

**Theorem 3.8.** *For any $A,B\in IFS(X)$,*
(1) $Card_\theta\Box(A\cup_{\overline{S}_M}B)+Card_\theta\Box(A\cap_{\overline{T}_M}B)=Card_\theta\Box(A)+Card_\theta\Box(B)$,
(12) $Card_\theta\Diamond(A\cup_{\overline{S}_M}B)+Card_\theta\Diamond(A\cap_{\overline{T}_M}B)=Card_\theta\Diamond(A)+Card_\theta\Diamond(B)$,
(3) $Card_\theta\Box(A\cup_{\overline{S}_P}B)+Card_\theta\Box(A\cap_{\overline{T}_P}B)=Card_\theta\Box(A)+Card_\theta\Box(B)$,
(4) $Card_\theta\Diamond(A\cup_{\overline{S}_P}B)+Card_\theta\Diamond(A\cap_{\overline{T}_P}B)=Card_\theta\Diamond(A)+Card_\theta\Diamond(B)$,
(5) $Card_\theta\Box(A\cup_{\overline{S}_L}B)+Card_\theta\Box(A\cap_{\overline{T}_L}B)=Card_\theta\Box(A)+Card_\theta\Box(B)$,
(6) $Card_\theta\Diamond(A\cup_{\overline{S}_L}B)+Card_\theta\Diamond(A\cap_{\overline{T}_L}B)=Card_\theta\Diamond(A)+Card_\theta\Diamond(B)$.

*Proof.* (1) From (1) and (3) of Theorem 3.7, we have for any $A,B\in IFS(X)$,

$$\begin{aligned}&Card_\theta\Box(A\cup_{\overline{S}_M}B)+Card_\theta\Box(A\cap_{\overline{T}_M}B)\\&=Card_\theta(\Box A\cup_{\overline{S}_M}\Box B)+Card_\theta(\Box A\cap_{\overline{T}_M}\Box B)\\&=\sum_{x\in X}\theta\max(\mu_A(x),\mu_B(x))+(1-\theta)(1-\min((1-\mu_A(x)),(1-\mu_B(x))))\\&\quad+\sum_{x\in X}\theta\min(\mu_A(x),\mu_B(x))+(1-\theta)(1-\max((1-\mu_A(x)),(1-\mu_B(x))))\\&=\sum_{x\in X}\theta\max(\mu_A(x),\mu_B(x))+(1-\theta)\max(\mu_A(x),\mu_B(x))\\&\quad+\sum_{x\in X}\theta\min(\mu_A(x),\mu_B(x))+(1-\theta)\min(\mu_A(x),\mu_B(x))\end{aligned}$$

$= Card_\theta \Box(A) + Card_\theta \Box(B)$.

(2) From (2) and (4) of Theorem 3.7, we have that

$$\begin{aligned}
& Card_\theta \Diamond(A \cup_{\overline{S}_M} B) + Card_\theta \Diamond(A \cap_{\overline{T}_M} B) \\
&= Card_\theta(\Diamond A \cup_{\overline{S}_M} \Diamond B) + Card_\theta(\Diamond A \cap_{\overline{T}_M} \Diamond B) \\
&= \sum_{x\in X} \theta \max(1-\nu_A(x), 1-\nu_B(x)) + (1-\theta)(1-\min(\nu_A(x), \nu_B(x))) \\
&\quad + \sum_{x\in X} \theta(\min(1-\nu_A(x), 1-\nu_B(x))) + (1-\theta)(1-\max(\nu_A(x), \nu_B(x))) \\
&= \sum_{x\in X} \theta \max(1-\nu_A(x), 1-\nu_B(x)) + (1-\theta)(\max(1-\nu_A(x), 1-\nu_B(x))) \\
&\quad + \sum_{x\in X} \theta(\min(1-\nu_A(x), 1-\nu_B(x))) + (1-\theta)(\min(1-\nu_A(x), 1-\nu_B(x))) \\
&= Card_\theta \Diamond(A) + Card_\theta \Diamond(B).
\end{aligned}$$

(3) From (5) and (7) of Theorem 3.7, we have that

$$\begin{aligned}
& Card_\theta \Box(A \cup_{\overline{S}_P} B) + Card_\theta \Box(A \cap_{\overline{T}_P} B) \\
&= Card_\theta(\Box A \cup_{\overline{S}_P} \Box B) + Card_\theta(\Box A \cap_{\overline{T}_P} \Box B) \\
&= \sum_{x\in X} \theta(\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)) + (1-\theta)(1-((1-\mu_A(x))(1-\mu_B(x)))) \\
&+ \sum_{x\in X} \theta(\mu_A(x)\mu_B(x)) + (1-\theta)(1-((1-\mu_A(x)) + (1-\mu_B(x)) - (1-\mu_A(x)(1-\mu_A(x))) \\
&= \sum_{x\in X} \theta\mu_A(x) + \theta\mu_B(x) + (1-\theta)(1-((1-\mu_A(x))) + (1-\theta)(1-((1-\mu_B(x))) \\
&= Card_\theta \Box(A) + Card_\theta \Box(B).
\end{aligned}$$

(4) From (6) and (8) of Theorem 3.7

$$\begin{aligned}
& Card_\theta \Diamond(A \cup_{\overline{S}_P} B) + Card_\theta \Diamond(A \cap_{\overline{T}_P} B) \\
&= Card_\theta(\Diamond A \cup_{\overline{S}_P} \Diamond B) + Card_\theta(\Diamond A \cap_{\overline{T}_P} \Diamond B) \\
&= \sum_{x\in X} \theta((1-\nu_A(x)) + (1-\nu_B(x)) - (1-\nu_A(x))(1-\nu_B(x))) + (1-\theta)(1-\nu_A(x)\nu_B(x)) \\
&\quad + \sum_{x\in X} \theta((1-\nu_A(x))(1-\nu_B(x))) + (1-\theta)(1-(\nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x))) \\
&= \sum_{x\in X} \theta((1-\nu_A(x)) + (1-\nu_B(x)) - (1-\nu_A(x))(1-\nu_B(x))) + (1-\theta)((1-\nu_A(x)) + \\
&\qquad (1-\nu_B(x)) - (1-\nu_A(x))(1-\nu_B(x))) \\
&\quad + \sum_{x\in X} \theta(1-\nu_A(x))(1-\nu_B(x)) + (1-\theta)(1-\nu_A(x))(1-\nu_B(x)) \\
&= Card_\theta \Diamond(A) + Card_\theta \Diamond(B).
\end{aligned}$$

(5) From (9) and (11) of Theorem 3.7,

$$\begin{aligned}
& Card_\theta \Box(A \cup_{\overline{S}_L} B) + Card_\theta \Box(A \cap_{\overline{T}_L} B) \\
&= Card_\theta(\Box A \cup_{\overline{S}_L} \Box B) + Card_\theta(\Box A \cap_{\overline{T}_L} \Box B) \\
&= \sum_{x\in X} \theta \min(1, \mu_A(x) + \mu_B(x)) + (1-\theta)(1-\max(0, (1-\mu_A(x)) + (1-\mu_B(x)) - 1)) \\
&\quad + \sum_{x\in X} \theta \max(0, \mu_A(x) + \mu_B(x) - 1) + (1-\theta)(1-\min(1, (1-\mu_A(x)) + (1-\mu_B(x)))) \\
&= \sum_{x\in X} \theta \min(1, \mu_A(x) + \mu_B(x)) + (1-\theta)\min(1, \mu_A(x) + \mu_B(x))
\end{aligned}$$

$$+\sum_{x\in X}\theta\max(0,\mu_A(x)+\mu_B(x)-1)+(1-\theta)\max(0,\mu_A(x)+\mu_B(x)-1)$$
$$=Card_\theta\Box(A)+Card_\theta\Box(B).$$

(6) From (10) and (12) of Theorem 3.7, we get:

$$Card_\theta\Diamond(A\cup_{\overline{S}_L}B)+Card_\theta\Diamond(A\cap_{\overline{T}_L}B)$$
$$=Card_\theta(\Diamond A\cup_{\overline{S}_L}\Diamond B)+Card_\theta(\Diamond A\cap_{\overline{T}_L}\Diamond B)$$
$$=\sum_{x\in X}\theta\max(0,(1-\nu_A(x))+(1-\nu_B(x))-1)+(1-\theta)(1-\min(1,\nu_A(x)+\nu_B(x)))$$
$$+\sum_{x\in X}\theta\min(1,(1-\nu_A(x))+(1-\nu_B(x)))+(1-\theta)(1-\max(0,\nu_A(x)+\nu_B(x)-1))$$
$$=\sum_{x\in X}\theta\max(0,(1-\nu_A(x))+(1-\nu_B(x))-1)+(1-\theta)\max(0,(1-\nu_A(x))+(1-\nu_B(x))-1)$$
$$+\sum_{x\in X}\theta\min(1,(1-\nu_A(x))+(1-\nu_B(x)))+(1-\theta)\min(1,(1-\nu_A(x))+(1-\nu_B(x)))$$
$$=Card_\theta\Diamond(A)+Card_\theta\Diamond(B).$$ □

## 4. Applications of $Card_\theta$ in Ranking Procedure

In this section, we shall employ the new proposed weighted average cardinality measure $Card_\theta(A)$ for an intuitionistic fuzzy set $A$ to present a new definition of relative sigma count for intuitionistic fuzzy sets and then based on this concept we will introduce a ranking technique for intuitionistic fuzzy sets. The notion of *relative sigma count* of fuzzy sets was introduced by Zadeh [35] as:

$$\sum Count(A/B)=\frac{\sum Count(A\cap B)}{\sum Count(B)}=\frac{\sum_{x\in X}\min(\mu_A(x),\mu_B(x))}{\sum_{x\in X}\mu_B(x)}.$$

The relative sigma count of an intuitionistic fuzzy set was introduced as a numerical interval in [31]. We have presented a new definition of this concept and quantified it to a scalar number which makes it a more feasible object that could be worked upon in any situation.

**Definition 4.1.** Let $A,B\in IFS(X)$. The relative sigma count of set $A$ with respect to set $B$ is defined as:

$$Card_\theta(A/B)=\frac{Card_\theta(A\cap B)}{Card_\theta(B)}=\frac{\sum_{x\in X}\theta\min(\mu_A(x),\mu_B(x))+(1-\theta)\min(1-\nu_A(x),1-\nu_B(x))}{\sum_{x\in X}\theta\mu_B(x)+(1-\theta)(1-\nu_B(x))},$$

where $\theta\in[0.5,1]$.

Suppose that $A$ and $B$ are two fuzzy sets. Then $Card_\theta(A/B)=\sum Count(A/B)$.

Now if $A$ is an intuitionistic fuzzy set and $B$ is a fuzzy set, then

$$Card_\theta(A/B)=\frac{\sum_{x\in X}\theta\min(\mu_A(x),\mu_B(x))+(1-\theta)\min(1-\nu_A(x),\mu_B(x))}{\sum_{x\in X}\mu_B(x)},$$

and if $A$ is a fuzzy set and $B = X$, then $Card_\theta(A/B) = \frac{\sum_{x\in X}\mu_A(x)}{n} = \frac{1}{n}\sum_{x\in X}\mu_A(x)$.

Clearly, $Card_\theta(A/B) \in [0,1]$ and $Card_\theta(A/A) = \frac{\sum_{x\in X}\theta\mu_A(x)+(1-\theta)(1-\nu_A(x))}{\sum_{x\in X}\theta\mu_A(x)+(1-\theta)(1-\nu_A(x))} = 1$.

**Definition 4.2.** For a given class $A_l \in IFS(X), l \in L$ and for any fixed $B \in IFS(X)$, we say $A_p$ dominates $A_q$, written as $A_p \rhd A_q$, if $Card_\theta(A_p/B) \geq Card_\theta(A_q/B)$, where $\theta \in [0.5,1]$.

**Definition 4.3.** For a given class $A_l \in IFS(X), l \in L$, the set $\psi \in IFS(X)$ is said to be the *super IFS* ,if

$$\psi = \left\{(x, \mu_\psi(x), \nu_\psi(x)) \mid x \in X\right\},$$

where $\mu_\psi(x) = \max_{l\in L}(\mu_{A_l}(x))$ and $\nu_\psi(x) = \min_{l\in L}(\nu_{A_l}(x))$.

4.1. **Application of relative sigma count to Medical diagnosis and organizational management.** In this subsection, we shall demonstrate the efficiency and simplicity of the proposed ranking procedure of intuitionistic fuzzy sets by presenting two case studies. The first one is related to the field of medical diagnosis while the second one addresses the employee appraisal problem of organizational management. In the first case study we shall directly utilize the Definition 3.2 in the ranking procedure of intuitionistic fuzzy sets. However, in the second situation we need firstly, construct the *super* set $\psi$ as defined in Definition 3.3 and then we rank the given class of intuitionistic fuzzy sets with respect to the set $\psi$. It must be noted that through out this subsection we shall specify $\theta = 0.5$ in the definition of relative sigma count for intuitionistic fuzzy sets.

**Case Study 4.4.** (Medical Diagnosis) Suppose that a doctor has to make a judgement about a patient disease, who is claiming the presence of symptoms such as Temperature, Cough, Chest pain, Headache, Stomach pain that are collective symptoms of diseases like Viral fever, Malaria, Chest problem, Typhoid, Stomach problem. We propose a diagnosis procedure comprising of the following three simple steps:

**(i)** Determination of symptoms,
**(ii)** Formulation of medical knowledge based on intuitionistic fuzzy sets,
**(iii)** Determination of diagnosis on the basis of ranking procedure proposed in Definition 4.2.

In the above situation the doctor will, firstly, formulate a crisp set of symptoms $\sigma$ =$\{s_1$(temperature), $s_2$(cough), $s_3$(chest pain), $s_4$(headache), $s_5$(stomach pain)$\}$. As a next step he/she will assign to the patient $P$ a degree of membership and a degree of non membership with respect to each of the symptoms $s_i \in \sigma, i = 1,2...5$ on the basis of his/her medical examination and tests. Thus, for the doctor a patient is now an intuitionistic fuzzy set $P$ on the set of symptoms $\sigma$ given as say:

| | | | | | |
|---|---|---|---|---|---|
| $P(Patient)$ | $(0.8,0.0)$ | $(0.6,0.1)$ | $(0.2,0.8)$ | $(0.6,0.1)$ | $(0.1,0.6)$ |

Table 1: Patient w.r.t. symptons-1

Moreover, on the basis of his/her medical knowledge he/she will view each of the possible diagnosis like, Viral Fever($VF$), Malaria($M$), Chest Problem($CP$), Typhoid($T$), Stomach Problem($SP$) as intuitionstic fuzzy set over the set of symptoms $\sigma$.

In the following case study, we formulate a matrix representing all these five diseases as the intuitionistic fuzzy sets over the set of symptoms $\sigma$.

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|
| $VF$ | $(0.4,0.0)$ | $(0.3,0.5)$ | $(0.1,0.7)$ | $(0.4,0.3)$ | $(0.1,0.7)$ |
| $M$ | $(0.7,0.0)$ | $(0.2,0.6)$ | $(0.0,0.9)$ | $(0.7,0.0)$ | $(0.1,0.8)$ |
| $CP$ | $(0.3,0.3)$ | $(0.6,0.1)$ | $(0.2,0.7)$ | $(0.2,0.6)$ | $(0.1,0.9)$ |
| $T$ | $(0.1,0.7)$ | $(0.2,0.7)$ | $(0.2,0.4)$ | $(0.8,0.0)$ | $(0.2,0.7)$ |
| $SP$ | $(0.1,0.8)$ | $(0.0,0.8)$ | $(0.2,0.8)$ | $(0.2,0.8)$ | $(0.8,0.1)$ |

Table 2: Diseases w.r.t. symptoms-1

Then as the final stage he/she will find the relative sigma count for all of the diseases with respect to patient $P$ and adopt the ranking technique defined in Definition 3.2 to make his/her final judgement. The mathematical calculation regarding this scenario is as follow:

$$Card_{0.5}(VF/P) = 0.707,$$
$$Card_{0.5}(M/P) = 0.745,$$
$$Card_{0.5}(CP/P) = 0.672,$$
$$Card_{0.5}(T/P) = 0.490,$$
$$Card_{0.5}(SP/P) = 0.327.$$

Thus, we obtain a ranking as $M \triangleright VF \triangleright CP \triangleright T \triangleright SP$ which indicates that the patient is most likely facing the problem of Malaria.

**Case Study 4.5.** (Organizational Management) Consider the problem of annual selection of the best employee in an organization. The characteristics, which are used to determine this selection are:

$C_1$ : *Progressiveness,*
$C_2$ : *Team work,*
$C_3$ : *Discipline,*
$C_4$ : *Punctuality,*
$C_5$ : *Efficiency;*
$C_6$ : *Stress management.*

The people under consideration are: John, David, Billy and Brown.

We formulate a matrix representing the four employees who are evaluated on the basis of the above mentioned six characteristics. A team of experts has allocated the following scores to the candidates with respect to corresponding criteria. For example experts have allocated John 0.8 with respect to discipline while 0.1 is given to him for non discipline.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| John | $(0.2,0.7)$ | $(0.5,0.2)$ | $(0.8,0.1)$ | $(0.6,0.3)$ | $(0.4,0.5)$ | $(0.3,0.6)$ |
| David | $(0.6,0.2)$ | $(0.2,0.7)$ | $(0.7,0.3)$ | $(0.8,0.2)$ | $(0.5,0.3)$ | $(0.9,0.1)$ |
| Billy | $(0.2,0.7)$ | $(0.4,0.5)$ | $(0.8,0.2)$ | $(0.9,0.1)$ | $(0.6,0.3)$ | $(0.5,0.2)$ |
| Brown | $(0.5,0.4)$ | $(0.3,0.5)$ | $(0.6,0.3)$ | $(0.5,0.3)$ | $(0.7,0.2)$ | $(0.9,0.0)$ |

Table 3: Employees w.r.t. selection characteristics

To apply the proposed technique we deal the data of John as a single IFS. So we are dealing with a class of four IFS's: John, David, Billy and Brown.

First of all, we construct the super intuitionistic fuzzy set $\psi$ of the above mentioned data which is also an IFS given in the form of a row matrix:

| | | | | | | |
|---|---|---|---|---|---|---|
| $\psi$ | $(0.6,0.2)$ | $(0.5,0.2)$ | $(0.8,0.1)$ | $(0.9,0.1)$ | $(0.7,0.2)$ | $(0.9,0.0)$ |

Table 4: Super IFS w.r.t. selection characteristics

A selector may have to consider the above characteristics and formulate their judgement for each employee in a organization utilizing all the given information. We shall use the technique of dominance defined in Definition 3.2 as the factor of performance appraisal. Then we observe that:

$$Card_{0.5}(\text{John}/\psi) = 0.66,$$
$$Card_{0.5}(\text{David}/\psi) = 0.822,$$
$$Card_{0.5}(\text{Billy}/\psi) = 0.77,$$
$$Card_{0.5}(\text{Brown}/\psi) = 0.812.$$

Clearly,

$$Card_{0.5}(\text{David}/\psi) \geq Card_{0.5}(\text{Brown}/\psi) \geq Card_{0.5}(\text{Billy}/\psi) \geq Card_{0.5}(\text{John}/\psi).$$

Thus, we have a ranking:

$$\text{David} \triangleright \text{Brown} \triangleright \text{Billy} \triangleright \text{John}.$$

4.2. **Application of relative sigma count in defining quantification rules.** In this subsection, we shall once again illustrate some applications of relative sigma count for intuitionistic fuzzy sets by defining extended quantified intuitionistic fuzzy propositions and studying their truthfulness with the aid of practical examples. We shall demonstrate the applications by taking $\theta = 0.5$ in the definition of relative sigma count for intuitionistic fuzzy sets.

**Definition 4.6.** For $A, B \in IFS(X)$, we define a *quantified intuitionistic fuzzy proposition* as '$Qx's$ *are* $A's$' and an extended quantified intuitionistic fuzzy proposition by '$QA's$ *are* $B's$', where the intuitionistic quantifier $Q$ is specified by a membership function $\mu_Q(x)$ and a non membership function $\nu_Q(x)$ for any particular $x \in X$. Mathematically speaking, the truth value of the extended quantified intuitionistic fuzzy proposition ($QA's$ *are* $B's$) in a finite universe $X$ is given by $(\mu_Q(r), \nu_Q(r))$ such that:

$$r = Card_{0.5}(B/A) = \frac{Card_{0.5}(A \cap B)}{Card_{0.5}(A)}.$$

In particular ,if $A, B \in FS(X)$, then the proposition '$QA's$ *are* $B's$' is reduced to Zadeh's sense [35].

Moreover, as a special case when $A = X$ and $B \in IFS(X)$, the extended quantified intuitionistic fuzzy proposition '$QA's$ *are* $B's$' becomes a simple quantified intuitionistic fuzzy proposition '$Qx's$ *are* $B's$' and, then the truth value of this intuitionistic quantified proposition is modelled by:

$$truth(Qx's\ are\ B's) = (\mu_Q(r_0), \nu_Q(r_0))$$

such that

$$r_0 = Card_{0.5}(B/X) = \frac{Card_{0.5}(B \cap X)}{Card_{0.5}(X)} = \frac{\frac{1}{2}\sum_{x \in X} \mu_B(x) + (1 - \nu_B(x))}{n}.$$

We demonstrate both the situations by the following case studies.

**Case Study 4.7.** Let us consider the proposition '*most cars are fast.* We want to check its truthfulness. Clearly it is a quantified intuitionistic fuzzy proposition of the form '$Qx's$ *are* $B's$'. To illustrate the problem we may restrict ourself to the given situation where $X = cars = \{x_1, x_2, x_3\}$ is a finite universe of discourse and $B = fast = \{(x_1, 0.1, 0.8), (x_2, 0.6, 0.2), (x_3, 0.8, 0.2)\}$ is the intuitionistic fuzzy set on $X$ giving the idea of fastness of each car $(x \in X)$ and $Q = most$ is the intuitionistic fuzzy quantifier with membership and non membership functions alongwith their graphs defined as:

$$\mu_Q(t) = \left\{ \begin{array}{cc} 0 & 0 \le t \le 0.375 \\ 1-(1+(2.4t-.9)^2)^{-1} & 0.375 \le t \le 1 \\ 1 & 1 \le t \end{array} \right\}, \tag{4.1}$$

$$\nu_Q(t) = \left\{ \begin{array}{cc} 1 & 0 \le t \le 0.25 \\ (1+(2t-.5)^2)^{-1} & 0.25 \le t \le 1 \\ 0 & 1 \le t \end{array} \right\}. \tag{4.2}$$

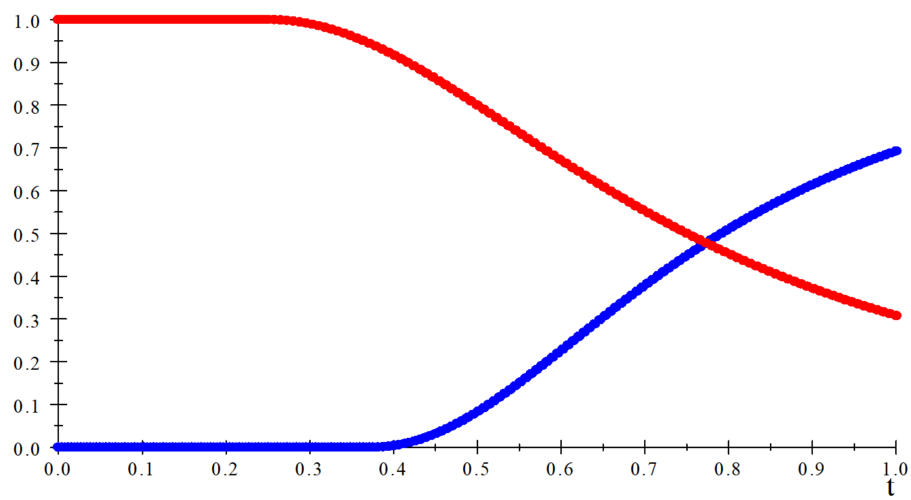


FIGURE 1. Graphs of membership and non membership functions

Now this is the case where the truth value of a quantified intuitionistic fuzzy proposition '$Qx's$ *are* $B's$' is modelled by a situation in which $A = X$ and $B \in IFS(X)$, i.e.,

$$truth(Qx's \text{ are } B's) = (\mu_Q(r_0), \nu_Q(r_0)),$$

where

$$r_0 = Card_{0.5}(B/X) = \frac{\frac{1}{2}\sum_{x \in X} \mu_B(x) + (1-\nu_B(x))}{n}.$$

Then $r_0 = Card_{0.5}(B/X) = \frac{3.3}{6} = 0.55$, where $r_0$ defines the proportion of $B$ in $X$.

Now substituting $r_0 = 0.55$ in (4.1), (4.2) we have $\mu_Q(0.55) = 0.14995$ and $\nu_Q(0.55) = 0.73529$. The truth value may vary depending on how the quantifier $Q(most)$ and the set $B(fast)$ are defined.

**Case Study 4.8.** Let us consider a more general proposition, '*Most fast cars are dangerous*'. Clearly this proposition is an extended quantified intuitionistic fuzzy proposition of the form '$QB's$ *are* $A's$'. Now here we take the data for $Q = most$, $B = fast$ and $X = cars$ from the previous example, but we do need to define the intuitionistic fuzzy set *dangerous* over $X$. Let it be: $A = dangerous = \{(x_1, 0.2, 0.7), (x_2, 0.5, 0.4), (x_3, 0.6, 0.4)\}$. Then the truth value of this extended quantified intuitionistic fuzzy proposition is calculated by:

$$truth(QB's\ are\ A's) = (\mu_Q(r), \nu_Q(r)),$$

where

$$r = Card_{0.5}(A/B) = \frac{\frac{1}{2}\sum_{x\in X} \min(\mu_A(x),\mu_B(x)) + \min(1-\nu_A(x), 1-\nu_B(x))}{\frac{1}{2}\sum_{x\in X} \mu_B(x) + (1-\nu_B(x))} = 0.787.$$

Finally, substituting the value of $r$ in (4.1) and (4.2), we get $\mu_Q(0.787) = 0.494\,37$ and $\nu_Q(0.787) = 0.464\,37$.
Thus the final results may vary depending on how the quantifier $Q(most)$, the set $B(fast)$, and the set $A(dangerous)$ are defined.

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