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$L-$fuzzy ideals in universal algebras

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Abstract. In this paper, the general theory of algebraic fuzzy systems is applied to the more general classes of algebras called universal algebras. We define $L-$fuzzy ideals of universal algebras of a given type (or language) and investigate some of their properties.

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1. Introduction

Swamy and Raju have studied the general theory of algebraic fuzzy systems in a serious of papers [11, 13]. They introduced the concept of a fuzzy $\mathfrak{L}-$subset of a set X corresponding to a given class $\mathfrak{L}$ of subsets of X having truth values in a complete Brouwerian lattice, as suggested by Goguen [5]. This subsumes the notions such as fuzzy subgroup of a group [9, 10], fuzzy ideal of a ring [14], fuzzy ideals and congruences of lattices [12] and fuzzy subspace of a vector space[8].

On the other hand, Gumm and Ursini [7] introduced the theory of ideals in universal algebras having a constant 0 by the use of ideal terms as a generalization of those familiar structures like normal subgroups (in groups), normal subloops (in loops), ideals (in rings), submodule (in modules), subspaces (in vector spaces) and filters (in Implication algebras or Boolean algebras, where 0 is replaced by the unit). Mainly they have characterized ideal determined varieties using the Mal'cev condition. Further investigations on ideals of universal algebras appear in a series of literature [1, 2, 3, 15].

Motivated by all the above results, we applied, in this paper, the general theory of algebraic fuzzy system developed in [11, 13] to the more general classes of algebras called universal algebras. Specifically we define $L-$fuzzy ideals of universal algebras having a constant 0 and investigate some of their properties. Fuzzy ideals generated by a fuzzy set are characterized from the fuzzy point of view as well as from the algebraic point of view. Furthermore, an equivalent condition is given for a variety of algebras to be an ideal determined.

2. Preliminaries

This section contains some notations and preliminary results which will be used in this paper. Throughout this note $A \in \mathcal{K}$, where $\mathcal{K}$ is a class of algebras of a fixed type Ω and assume that there is a distinguished nullary operation, or equationaly definable constant in all algebras of $\mathcal{K}$ which we denote by 0. For a positive integer n, we write $\overrightarrow{a}$ to denote the $n-$tuple $\langle a_1, a_2, ..., a_n\rangle \in A^n$. The following two definitions are due to [7].

Definition 2.1. A term $P(\overrightarrow{u}, \overrightarrow{v})$ is said to be an ideal term in $\overrightarrow{v}$, if $P(\overrightarrow{u}, \overrightarrow{0}) = 0$.

Definition 2.2. A nonempty subset I of A is called an ideal of A, if $P(\overrightarrow{a}, \overrightarrow{b}) \in I$, for all $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in I^m$ and any ideal term $P(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$.

By an $L-$ fuzzy subset of A, we mean a mapping $\mu : A \to L$, where L is a non-trivial complete Brouwerian lattice $(L, \wedge, \vee, 0, 1)$, i.e., a complete lattice satisfying the infinite meet distributive law:

$$\alpha \wedge (\bigvee_{\beta\in M} \beta) = \bigvee_{\beta\in M} (\alpha \wedge \beta),$$

for all $\alpha \in L$ and any $M \subseteq L$. For each $\alpha \in L$, the $\alpha-$level set of μ denoted by μ_α is a subset of A given by:

$$\mu_\alpha = \{x \in A : \alpha \leq \mu(x)\}$$

For further details, we refer to [11, 13, 16]. On the other hand we take [4, 6] into account for those ordinary theories on universal algebras.

3. Fuzzy ideals

In this section we define and characterize $L-$fuzzy ideals of universal algebras.

Definition 3.1. An $L-$fuzzy subset μ of A is said to be an $L-$fuzzy ideal of A (or shortly a fuzzy ideal of A), if the following conditions are satisfied:

(i) $\mu(0) = 1$,

(ii) if $P(\overrightarrow{x}, \overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$ and $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$ implies that

$$\mu(P(\overrightarrow{a}, \overrightarrow{b})) \geq \mu^m(\overrightarrow{b}).$$

The following theorem gives an equivalent condition for fuzzy subsets to be a fuzzy ideal in terms of their level sets.

Theorem 3.2. *A fuzzy subset μ of A is a fuzzy ideal of A if and only if μ_α is an ideal of A, for all $\alpha \in L$.*

Proof. Suppose that μ is a fuzzy ideal of A. For any $\alpha \in L$, let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in (\mu_\alpha)^m$ and $P(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Since $\mu(P(\overrightarrow{a}, \overrightarrow{b})) \geq \mu^m(\overrightarrow{b}) \geq \alpha$, we get $P(\overrightarrow{a}, \overrightarrow{b}) \in \mu_\alpha$. Then each μ_α is an ideal of A.

Conversely, suppose that the level subset μ_α is an ideal of A, for all $\alpha \in L$. In particular, μ_α is an ideal for $\alpha = 1$. Then $\mu(0) = 1$. Let $P(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$ and $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$. Put $\mu^m(\overrightarrow{b}) = \alpha$. Then $\overrightarrow{b} \in (\mu^m)_\alpha = (\mu_\alpha)^m$. Since each μ_α is an ideal we get, $P(\overrightarrow{a}, \overrightarrow{b}) \in \mu_\alpha$. Thus $\mu(P(\overrightarrow{a}, \overrightarrow{b})) \geq \alpha = \mu^m(\overrightarrow{b})$. So μ is a fuzzy ideal of A. □

This theorem confirms that a fuzzy ideal of A is precisely a fuzzy $\mathfrak{L}$-subset of A, where $\mathfrak{L}$ is the set of all ideals of A (see [11]).

Corollary 3.3. *A subset I of A is an ideal of A if and only if its characteristic mapping χ_I is a fuzzy ideal of A.*

Theorem 3.4. *If $\{\mu_i\}_{i\in\Delta}$ is a family of fuzzy ideals of A, then $\cap_{i\in\Delta}\mu_i$ is a fuzzy ideal of A.*

This theorem confirms that, for any fuzzy subset λ of A always there exists a smallest fuzzy ideal containing λ which we call it the fuzzy ideal of A generated by λ and is denoted by $\langle\lambda\rangle$.

Lemma 3.5. *Let S be any subset of A and χ_S its characteristic function. Then $\langle\chi_S\rangle = \chi_{\langle S\rangle}$.*

Proof. We show that $\chi_{\langle S\rangle}$ is the smallest fuzzy ideal containing χ_S. Since $\langle S\rangle$ is an ideal of A, it follows from Corollary 3.3 that $\chi_{\langle S\rangle}$ is a fuzzy ideal of A. It is also clear that $\chi_S \subseteq \chi_{\langle S\rangle}$. Let λ be any fuzzy ideal of A such that $\chi_S \subseteq \lambda$. Then $\lambda(s) = 1$, for all $s \in S$. Let z be any element in A. If $z \notin \langle S\rangle$, then $\chi_{\langle S\rangle}(z) = 0 \leq \lambda(z)$. Let $z \in \langle S\rangle$. Then $z = p(\overrightarrow{a}, \overrightarrow{s})$, for some $\overrightarrow{a} \in A^n$, $\overrightarrow{s} \in S^m$ and some ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ in $\overrightarrow{y}$. Now consider:

$$\begin{aligned}\lambda(z) &= \lambda(p(\overrightarrow{a}, \overrightarrow{s}))\\ &\geq \lambda^m(\overrightarrow{s})\\ &= \lambda^m(s_1, s_2...., s_m) \quad (\text{ where each } s_i \in S)\\ &= \lambda(s_1) \wedge \lambda(s_2) \wedge ... \wedge \lambda(s_m)\\ &= 1.\end{aligned}$$

Thus $\chi_{\langle S\rangle} \subseteq \lambda$. So $\chi_{\langle S\rangle} = \langle\chi_S\rangle$. □

For any fuzzy subset λ of A, it is clear that:

$$\lambda(x) = Sup\{\alpha \in L : x \in \lambda_\alpha\},$$

for all $x \in A$. In the following theorem we characterize a fuzzy ideal generated by a fuzzy set in terms of its level sets.

Theorem 3.6. *For a fuzzy subset λ of A, let $\widehat{\lambda}$ be defined by:*

$$\widehat{\lambda}(x) = Sup\{\alpha \in L : x \in \langle\lambda_\alpha\rangle\}, \text{ for all } x \in A.$$

Then $\widehat{\lambda} = \langle\lambda\rangle$.

Proof. We show that $\widehat{\lambda}$ is the smallest fuzzy ideal of A containing λ. Let us first show that $\widehat{\lambda}$ is a fuzzy ideal.

(i) $\widehat{\lambda}(0) = Sup\{\alpha \in L : 0 \in \langle\lambda_\alpha\rangle\} = 1$.

(ii) Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$ and $P(\overrightarrow{x}, \overrightarrow{y})$ be an ideal term in $\overrightarrow{y}$. Then

$$\begin{aligned}(\widehat{\lambda})^m(\overrightarrow{b}) &= Inf\{\widehat{\lambda}(b_i) : 1 \le i \le m\}\\ &= Inf\{Sup\{\alpha_i \in L : b_i \in \langle\lambda_{\alpha_i}\rangle\} : 1 \le i \le m\}\\ &= Sup\{Inf\{\alpha_i \in L : 1 \le i \le m\} : b_i \in \langle\lambda_{\alpha_i}\rangle\}.\end{aligned}$$

If we put $\beta = Inf\{\alpha_i \in L : 1 \le i \le m\}$, then we get $\lambda_{\alpha_i} \subseteq \lambda_\beta$, for all $1 \le i \le m$. Thus

$$\begin{aligned}(\widehat{\lambda})^m(\overrightarrow{b}) &= Sup\{Inf\{\alpha_i \in L : 1 \le i \le m\} : b_i \in \langle\lambda_{\alpha_i}\rangle\}\\ &\le Sup\{\beta \in L : \overrightarrow{b} \in \langle\lambda_\beta\rangle^m\}\\ &\le Sup\{\beta \in L : P(\overrightarrow{a}, \overrightarrow{b}) \in \langle\lambda_\beta\rangle\}\\ &= \widehat{\lambda}(P(\overrightarrow{a}, \overrightarrow{b})).\end{aligned}$$

Thus $\widehat{\lambda}$ is a fuzzy ideal of A. It is also clear to see that $\lambda \subseteq \widehat{\lambda}$. Suppose that μ is any other fuzzy ideal of A such that $\lambda \subseteq \mu$. Then $\langle\lambda_\alpha\rangle \subseteq \mu_\alpha$, for all $\alpha \in L$. Now for any $x \in A$,

$$\begin{aligned}\widehat{\lambda}(x) &= Sup\{\alpha \in L : x \in \langle\lambda_\alpha\rangle\}\\ &\le Sup\{\alpha \in L : x \in \mu_\alpha\}.\\ &= \mu(x).\end{aligned}$$

So $\widehat{\lambda}$ is the smallest fuzzy ideal containing λ. Hence $\widehat{\lambda} = \langle\lambda\rangle$. □

Corollary 3.7. *For any fuzzy subset μ of A,*

$$\langle\mu_\alpha\rangle = \langle\mu\rangle_\alpha, \text{ for all } \alpha \in L.$$

In the following we give an algebraic characterization of fuzzy ideals generated by fuzzy sets.

Definition 3.8. For a fuzzy subset λ of A, define $\overline{\lambda}$ to be a fuzzy subset of A as follows: $\overline{\lambda}(0) = 1$ *and for* $0 \ne x \in A$,

$$\begin{aligned}\overline{\lambda}(x) = Sup\{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, P(\overrightarrow{a}, \overrightarrow{b}) = x, \text{ where}\\ \overrightarrow{a} \in A^n, P(\overrightarrow{x}, \overrightarrow{y}) \text{ is an ideal term in } \overrightarrow{y}\}\end{aligned}$$

Theorem 3.9. *For any fuzzy subset λ of A, $\overline{\lambda} = \widehat{\lambda}$.*

Proof. For each $0 \ne x \in A$, let us define two sets H_x and G_x as follows:

$$\begin{aligned}H_x &= \{\lambda^m(\overrightarrow{b}) : \overrightarrow{b} \in A^m, P(\overrightarrow{a}, \overrightarrow{b}) = x\\ &\quad \text{where } \overrightarrow{a} \in A^n, P(\overrightarrow{x}, \overrightarrow{y}) \text{ is an ideal term in } \overrightarrow{y}\},\\ G_x &= \{\alpha \in L : x \in \langle\lambda_\alpha\rangle\}.\end{aligned}$$

Clearly both H_x and G_x are subsets of L. Our claim is to see that

$$Sup\{\alpha : \alpha \in H_x\} = Sup\{\alpha : \alpha \in G_x\}.$$

One way, we show that $H_x \subseteq G_x$. For, $\alpha \in H_x$ implies that $\alpha = \lambda^m(\overrightarrow{b})$, for some $\overrightarrow{b} \in A^m$ such that $P(\overrightarrow{a}, \overrightarrow{b}) = x$, for some $\overrightarrow{a} \in A^n$, where $P(\overrightarrow{x}, \overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$. That is $\overrightarrow{b} \in (\lambda_\alpha)^m$. Then $x \in \langle\lambda_\alpha\rangle$. Thus $\alpha \in G_x$. So $H_x \subseteq G_x$.

The other way, we can prove is that for each $\alpha \in G_x$, there exists $\beta \in H_x$ such that $\alpha \leq \beta$. For, let $\alpha \in G_x$. Then $x \in \langle \lambda_\alpha \rangle$, that is, $x = P(\overrightarrow{a}, \overrightarrow{b})$, for some $\overrightarrow{b} \in (\lambda_\alpha)^m$ and $\overrightarrow{a} \in A^n$, where $P(\overrightarrow{x}, \overrightarrow{y})$ is an ideal term in $\overrightarrow{y}$. If we put $\beta = \lambda^m(\overrightarrow{b})$, then we get $\beta \in H_x$ and $\alpha \leq \beta$. This completes the proof. □

Corollary 3.10. *For any fuzzy subset λ of A, $\overline{\lambda} = \langle \lambda \rangle$.*

Theorem 3.11. *The set of all fuzzy ideals of A forms a complete lattice where the infimum and supremum of any family $\{\mu_i : i \in \Delta\}$ of fuzzy ideals of A is given by:*

$$\bigwedge \mu_i = \cap \mu_i \text{ and } \bigvee \mu_i = \langle \cup \mu_i \rangle.$$

4. Fuzzy congruence relations

The following definition is due to [11].

Definition 4.1. An $L-$fuzzy subset Θ of $A \times A$ is called an $L-$fuzzy equivalence relation on A, if it satisfies the following conditions:

(i) $\Theta(x, x) = 1$,
(ii) $\Theta(x, y) = \Theta(y, x)$,
(iii) $\Theta(x, z) \geq \Theta(x, y) \wedge \Theta(y, z)$,

for all $x, y, z \in A$. An $L-$fuzzy congruence relation on A is an $L-$fuzzy equivalence relation Θ on A such that

$$\Theta(f(x_1, x_2, ..., x_n), f(y_1, y_2, ..., y_n)) \geq \bigwedge \{\Theta(x_i, y_i) : 1 \leq i \leq n\},$$

for all $n-$ary $f \in \Omega$ and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in A$. If no confusion appears we simply say a fuzzy congruence Θ on A instead of an $L-$fuzzy congruence Θ on A.

Given a fuzzy congruence Θ on A and $x \in A$, the fuzzy subset Θ_x of A given by:

$$\Theta_x(y) = \Theta(x, y),$$

for all $y \in A$, is called a fuzzy congruence class of A determined by Θ and x.

Theorem 4.2. *If Θ is a fuzzy congruence on A, then the zero fuzzy congruence class Θ_0 is a fuzzy ideal of A.*

Proof. Suppose that Θ is a fuzzy congruence relation on A. Then clearly, $\Theta_0(0) = 1$. Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$. Then

$$\begin{aligned}
\Theta_0(P(\overrightarrow{a}, \overrightarrow{b})) &= \Theta(0, P(\overrightarrow{a}, \overrightarrow{b})) \\
&= \Theta(P(\overrightarrow{a}, \overrightarrow{0}), P(\overrightarrow{a}, \overrightarrow{b})) \\
&= \Theta(P(a_1, a_2, ..., a_n, 0, 0, ..., 0), P(a_1, a_2, ..., a_n, b_1, b_2, ..., b_m) \\
&\geq \Theta(a_1, a_1) \wedge ... \wedge \Theta(a_n, a_n) \wedge \Theta(0, b_1) \wedge ... \wedge \Theta(0, b_m) \\
&= \Theta(0, b_1) \wedge ... \wedge \Theta(0, b_m) \\
&= \Theta_0(b_1) \wedge ... \wedge \Theta_0(b_m) \\
&= (\Theta_0)^m(\overrightarrow{b}).
\end{aligned}$$

Thus Θ_0 is a fuzzy ideal of A. □

Let μ be a fuzzy ideal of A and Θ a fuzzy congruence on A. Define a fuzzy subset $\Theta[\mu]$ of A by:

$$\Theta[\mu](x) = Sup\{\mu(y) \wedge \Theta(x, y) : y \in A\}.$$

Then we have the following results.

Lemma 4.3. *$\Theta[\mu]$ is a fuzzy ideal of A.*

Proof. Let $\overrightarrow{a} \in A^n$, $\overrightarrow{b} \in A^m$. Then

$$\begin{aligned}
&\Theta[\mu]^m(\overrightarrow{b})\\
&= Inf\{\Theta[\mu](b_i) : 1 \leq i \leq m\}\\
&= Inf\{Sup\{\mu(x_i) \wedge \Theta(x_i, b_i) : x_i \in A\} : 1 \leq i \leq m\}\\
&= Sup\{Inf\{\mu(x_i) : 1 \leq i \leq m\} \wedge Inf\{\Theta(x_i, b_i) : 1 \leq i \leq m\} : x_1, ..., x_n \in A\}\\
&= Sup\{\mu^m(\overrightarrow{x}) \wedge Inf\{\Theta(x_i, b_i) : 1 \leq i \leq m\} : x_1, ..., x_m \in A\}\\
&\leq Sup\{\mu(p(\overrightarrow{a}, \overrightarrow{x})) \wedge \Theta(p(\overrightarrow{a}, \overrightarrow{x}), p(\overrightarrow{a}, \overrightarrow{b})) : \overrightarrow{x} \in A^m\},\ \forall \overrightarrow{a} \in A^n\\
&\leq Sup\{\mu(y) \wedge \Theta(y, p(\overrightarrow{a}, \overrightarrow{b})) : y \in A\}\\
&= \Theta[\mu](p(\overrightarrow{a}, \overrightarrow{b})).
\end{aligned}$$

Thus $\Theta[\mu]$ is a fuzzy ideal of A. □

Corollary 4.4. *Let Θ and Φ be fuzzy congruence relations on A. Then*

$$\Theta \vee \Phi(x, 0) = \Theta \circ \Phi(x, 0),$$

where the composition $\Theta \circ \Phi$ is given by:

$$\Theta \circ \Phi(x, y) = Sup\{\theta(x, z) \wedge \phi(z, y) : z \in A\},$$

for all $x, y \in A$.

Proof. For each $x \in A$,

$$\begin{aligned}
\Theta \vee \Phi(x, 0) &= (\Theta \vee \Phi)_0(x)\\
&= \Theta[\Phi_0](x)\\
&= sup\{\Phi_0(y) \wedge \Theta(y, x) : y \in A\}\\
&= sup\{\Phi(0, y) \wedge \Theta(y, x) : y \in A\}\\
&= sup\{\Theta(x, y) \wedge \Phi(y, 0) : y \in A\}\\
&= \Theta \circ \Phi(x, 0).
\end{aligned}$$

□

Definition 4.5. [7] A class $\mathcal{K}$ of algebras is called an ideal determined, if every ideal I is the zero congruence class of a unique congruence relation denoted by I^δ. In this case, the map $I \mapsto I^\delta$ defines an isomorphism between the lattice of ideals and congruences on A.

Theorem 4.6. *A class $\mathcal{K}$ of algebras is an ideal determined if and only if every fuzzy ideal μ is the zero fuzzy congruence class of a unique fuzzy congruence relation denoted by Θ^μ.*

Proof. Suppose that $\mathcal{K}$ is an ideal determined variety. Let μ be any fuzzy ideal of A. Then μ_α is an ideal of A, for all $\alpha \in L$, that is, for each $\alpha \in L$, there is a unique congruence relation on A denoted by $(\mu_\alpha)^\delta$, for which μ_α is its zero congruence class.

Now define a fuzzy relation Θ^μ on A as follows:

$$\Theta^\mu(x,y) = Sup\{\alpha \in L : (x,y) \in (\mu_\alpha)^\delta\},$$

for all $x, y \in A$. We first show that Θ^μ is a fuzzy congruence relation on A. Then clearly, it is reflexive and symmetric.

Let us show that Θ^μ is transitive.

$$\begin{aligned}\Theta^\mu(x,y) \wedge \Theta^\mu(y,z) &= Sup\{\alpha \in L : (x,y) \in (\mu_\alpha)^\delta\} \wedge Sup\{\beta \in L : (y,z) \in (\mu_\beta)^\delta\} \\ &= Sup\{\alpha \wedge \beta : (x,y) \in (\mu_\alpha)^\delta, (y,z) \in (\mu_\beta)^\delta\}.\end{aligned}$$

If we put $\gamma = \alpha \wedge \beta$, then we get $\mu_\alpha \subseteq \mu_\gamma$ and $\mu_\beta \subseteq \mu_\gamma$. It follows from the fact $I \subseteq J \Rightarrow I^\delta \subseteq J^\delta$ that $(\mu_\alpha)^\delta \subseteq (\mu_\alpha)^\delta$. Thus,

$$\begin{aligned}\Theta^\mu(x,y) \wedge \Theta^\mu(y,z) &= Sup\{\alpha \wedge \beta : (x,y) \in (\mu_\alpha)^\delta, (y,z) \in (\mu_\beta)^\delta\} \\ &\leq Sup\{\gamma : (x,y), (y,z) \in (\mu_\gamma)^\delta\} \\ &\leq Sup\{\gamma : (x,z) \in (\mu_\gamma)^\delta\}. \\ &= \Theta^\mu(x,z)\end{aligned}$$

So it is transitive. Hence it is a fuzzy equivalence relation.

Let $x_1, ..., x_n, y_1, ..., y_n \in A$ and f be an $n-$ary operation. Then

$$\begin{aligned}Inf\{\Theta^\mu(x_i,y_i) : 1 \leq i \leq n\} &= Inf\{Sup\{\alpha_i \in L : (x_i,y_i) \in (\mu_{\alpha_i})^\delta\} : 1 \leq i \leq n\} \\ &= Sup\{Inf\{\alpha_i \in L : 1 \leq i \leq n\} : (x_i,y_i) \in (\mu_{\alpha_i})^\delta\}.\end{aligned}$$

If we put $\gamma = Inf\{\alpha_i \in L : 1 \leq i \leq n\}$, then we get $\mu_{\alpha_i} \subseteq \mu_\gamma$, for all $i = 1, 2, ..., n$ which implies that $(\mu_{\alpha_i})^\delta \subseteq (\mu_\alpha)^\delta$, for all $i = 1, 2, ..., n$. Thus

$$\begin{aligned}Inf\{\Theta^\mu(x_i,y_i) : 1 \leq i \leq n\} &= Sup\{Inf\{\alpha_i \in L : 1 \leq i \leq n\} : (x_i,y_i) \in (\mu_{\alpha_i})^\delta\} \\ &\leq Sup\{\gamma \in L : (x_i,y_i) \in (\mu_\gamma)^\delta, \forall\, i = 1, 2, ..., n\} \\ &\leq Sup\{\gamma \in L : (f(x_1, ..., x_n), f(y_1, ..., y_n)) \in (\mu_\gamma)^\delta\} \\ &= \Theta^\mu(f(x_1, ..., x_n), f(y_1, ..., y_n)).\end{aligned}$$

So Θ^μ is a fuzzy congruence relation on A. Moreover, the zero fuzzy congruence class of Θ^μ is precisely μ, for,

$$\begin{aligned}[\Theta^\mu]_0(x) &= \Theta^\mu(x,0) \\ &= Sup\{\alpha \in L : (x,0) \in (\mu_\alpha)^\delta\} \\ &= Sup\{\alpha \in L : (\mu_\alpha)^\delta[x] = (\mu_\alpha)^\delta[0]\} \\ &= Sup\{\alpha \in L : (\mu_\alpha)^\delta[x] = \mu_\alpha\} \\ &= Sup\{\alpha \in L : x \in \mu_\alpha\} \\ &= \mu(x).\end{aligned}$$

To prove the uniqueness of such a fuzzy congruence, let us take any fuzzy congruence Φ on A for which $\Phi_0 = \mu$. Then $\Phi_\alpha[0] = \mu_\alpha$, for all $\alpha \in L$, that is, μ_α is the zero congruence class of the congruence relation Φ_α. By the uniqueness of the congruence $(\mu_\alpha)^\delta$, we get $\Phi_\alpha = (\mu_\alpha)^\delta$, for all $\alpha \in L$. Thus $\Phi = \Theta^\mu$. So Θ^μ is the unique fuzzy congruence on A for which $(\Theta^\mu)_0 = \mu$. In this case, the map $\mu \mapsto \Theta^\mu$ defines an isomorphism between the lattice of fuzzy ideals and the lattice of fuzzy congruence relations on A. We see from Corollary 3.3 that every ideal of A can be identified as a fuzzy ideal by its characteristic mapping. This proofs the converse part. □

Lemma 4.7. *If $\mathcal{K}$ is ideal determined and $A \in \mathcal{K}$, then $\Theta[\mu]$ is the smallest fuzzy ideal of A containing both μ and Θ_0.*

Proof. Let us first see that $\Theta[\mu]$ contains both μ and Θ_0. Then for each $x \in A$,

$$\begin{aligned}\Theta[\mu](x) &= Sup\{\mu(y) \wedge \Theta(y,x) : y \in A\}\\ &\geq \mu(y) \wedge \Theta(y,x), \quad \text{for all } y \in A.\end{aligned}$$

In particular, for $y = x$, that is, $\Theta[\mu](x) \geq \mu(x)$. Thus $\mu \subseteq \Theta[\mu]$. Also, if we take $y = 0$, then we get $\Theta[\mu](x) \geq \Theta(0,x) = \Theta_0(x)$. Thus $\Theta_0 \subseteq \Theta[\mu]$. Suppose that ν is a fuzzy ideal of A such that $\mu \subseteq \nu$ and $\Theta_0 \subseteq \nu$. Choose the binary term d_1 and the quaternary term q_1 from (1.6) of [7]. Then for any $x, y \in A$, we have

$$d_1(x,x) = 0 \quad \text{and } x = q_1(x, y, 0, d_1(x,y)).$$

Thus

$$\nu(x) \geq \mu(x) \geq \mu(y) \text{ and } \nu(x) \geq \nu(d_1(x,y)).$$

So

$$\begin{aligned}\Theta(x,y) &\leq \Theta(d_1(x,x), d_1(x,y))\\ &= \Theta(0, d_1(x,y))\\ &= \Theta_0(d_1(x,y))\\ &\leq \nu(d_1(x,y))\\ &= \nu(x).\end{aligned}$$

Now consider the following:

$$\begin{aligned}\Theta[\mu](x) &= Sup\{\mu(y) \wedge \Theta(x,y) : y \in A\}\\ &\leq Sup\{\nu(x) \wedge \nu(x) : y \in A\} = \nu(x).\end{aligned}$$

Hence $\Theta[\mu]$ is the smallest fuzzy ideal of A containing both μ and Θ_0. □

If $\mathcal{K}$ is an ideal determined class of algebras, then the supremum of two fuzzy ideals is easy to describe. This could be done in the following way. If μ and ν are fuzzy ideals of $A \in \mathcal{K}$ and Θ^ν is the unique fuzzy congruence on A for which $\nu = (\Theta^\nu)_0$, then Lemma 4.7 confirms that $\Theta^\nu[\mu]$ is the supremum of μ and ν.

It is proved by the use of Mal'cev condition in [7] that a class $\mathcal{K}$ of algebras is an ideal determined if and only if for some positive integer m, there are binary terms $d_1, d_2, ..., d_m, d_{m+1}$ such that:

$$d_1(y,z) = d_2(y,z) =, ..., = d_m(y,z) = 0 \Rightarrow y = z \quad \text{and}$$

$$d_{m+1}(y,y) = 0, \quad d_{m+1}(0,y) = y.$$

In this case, for an ideal I, the congruence I^δ is characterized as follows:

$$I^\delta = \{(a,b) \in A \times A : d_i(a,b) \in I, \quad \text{for all } 1 \leq i \leq m\}$$

Similarly for a fuzzy ideal μ of A we characterize the unique fuzzy congruence Θ^μ of A as follows:

$$\Theta^\mu(a,b) = Inf\{\mu(d_i(a,b)) : 1 \leq i \leq m\}.$$

5. Concluding Remarks

The results presented in this paper indicate that many of the basic concepts in fuzzy normal subgroups (respectively fuzzy ideals) of the well known structures; groups (respectively rings) can readily be extended to fuzzy ideals of the wider class of algebras; universal algebras.

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