Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Nearness rings

Mehmet Ali Öztürk, Ebubekir İnan

Received 30 October 2018; Revised 16 November 2018; Accepted 5 December 2018

Abstract. In this paper, we consider the problem of how to establish algebraic structures on nearness approximation spaces. Essentially, our approach is to define the nearness ring, nearness ideal and nearness ring of all weak cosets by considering new operations on the set of all weak cosets. Afterwards, our aim is to study homomorphism on nearness approximation spaces, and to investigate some properties of nearness rings and ideals.

2010 AMS Classification: 03E75, 54E17, 16U99, 16W99.

Keywords: Near set, Rough set, Approximation space, Nearness approximation space, Nearness rings.

Corresponding Author: Mehmet Ali Öztürk (maozturk@adiyaman.edu.tr)

1. Introduction

Nearness approximation spaces and near sets were introduced in 2007 as a generalization of rough set theory [13, 15, 20]. More recent work consider generalized approach theory in the study of the nearness of non-empty sets that resemble each other [16] and a topological framework for the study of nearness and apartness of sets [10]. An algebraic approach of rough sets has been given by Iwinski [5]. Afterwards, rough subgroups were introduced by Biswas and Nanda [2]. In 2004 Davvaz investigated the concept of roughness of rings [4] (and other algebraic approaches of rough sets in [1, 9, 12, 19, 21]).

Near set theory begins with the selection of probe functions that provide a basis for describing and discerning affinities between objects in distinct perceptual granules. A probe function is a real-valued function representing a feature of physical objects such as images or collections of artificial organisms, e.g. robot societies.

In the concept of ordinary algebraic structures, such a structure that consists of a nonempty set of abstract points with one or more binary operations, which are required to satisfy certain axioms. For example, a groupoid is an algebraic structure $(A, \circ)$ consisting of a nonempty set A and a binary operation "$\circ$" defined on A [3]. In a groupoid, the binary operation "$\circ$" must be only closed in A, i.e., for all a, b in A, the result of the operation $a \circ b$ is also in A. As for the nearness approximation space, the sets are composed of perceptual objects (non-abstract points) instead of abstract points. Perceptual objects are points that have features. And these points describable with feature vectors in nearness approximation spaces [13]. Upper approximation of a nonempty set is obtained by using the set of objects composed by the nearness approximation space together with matching objects. In the algebraic structures constructed on nearness approximation spaces, the basic tool is consideration of upper approximations of the subsets of perceptual objects. In a groupoid A on nearness approximation space, the binary operation "$\circ$" may be closed in upper approximation of A, i.e., for all a, b in A, $a \circ b$ is in upper approximation of A.

There are two important differences between ordinary algebraic structures and nearness algebraic structures. The first one is working with non-abstract points while the second one is considering of upper approximations of the subsets of perceptual objects for the closeness of binary operations.

In 2012, E. İnan and M. A. Öztürk [6, 7] investigated the concept of groups on nearness approximation spaces. Moreover, in 2013, M. A. Öztürk at all [11] introduced group of weak cosets on nearness approximation spaces. Also in 2015, E. İnan and M. A. Öztürk [8] investigated the nearness semigroups. In this paper, we consider the problem of how to establish and improve algebraic structures of nearness approximation spaces. Essentially, our aim is to obtain algebraic structures such as nearness rings using sets and operations that ordinary are not being algebraic structures. Moreover, we define the nearness ring of all weak cosets by considering operations on the set of all weak cosets. To define this quotient structure we don't need to consider ideals.

2. Preliminaries

2.1. **Nearness approximation spaces** [13]**.** Perceptual objects are points that describable with feature vectors. Let $\mathcal{O}$ be a set of perceptual objects. An object description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in X \subseteq \mathcal{O}$. The important thing to notice is the choice of functions $\varphi_i \in B$ used to describe any object of interest. Assume that $B \subseteq \mathcal{F}$ is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, where $\varphi_i : \mathcal{O} \longrightarrow \mathbb{R}$. In combination, the functions representing object features provide a basis for an object description $\Phi : \mathcal{O} \longrightarrow \mathbb{R}^L$, a vector containing measurements (returned values) associated with each functional value $\varphi_i(x)$, where the description length is $|\Phi| = L$.

Object Description: $\Phi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), ..., \varphi_i(x), ..., \varphi_L(x))$.

Sample objects $X \subseteq \mathcal{O}$ are near to each other if and only if the objects have similar descriptions. Recall that each φ defines a description of an object. Then let Δ_{φ_i} denote $\Delta_{\varphi_i} = |\varphi_i(x') - \varphi_i(x)|$, where $x, x' \in \mathcal{O}$. The difference Δ_φ leads to a definition of the indiscernibility relation "$\sim_B$".

Let $x, x' \in \mathcal{O}$, $B \subseteq \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \forall \varphi_i \in B \ , \ \Delta_{\varphi_i} = 0\}$$

is called the indiscernibility relation on $\mathcal{O}$, where description length $i \leq |\Phi|$.

Symbol	*Interpretation*
B	$B \subseteq \mathcal{F}$,
r	$\binom{\lvert B\rvert}{r}$, i.e. , $\lvert B\rvert$ probe functions $\varphi_i \in B$ taken r at a time,
B_r	$r \leq \lvert B\rvert$ probe functions in B,
$\sim_{B_r}$	Indiscernibility relation defined using B_r,
$[x]_{B_r}$	$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$, equivalence (nearness) class,
$\mathcal{O}/\sim_{B_r}$	$\mathcal{O}/\sim_{B_r} = \left\{[x]_{B_r} \mid x \in \mathcal{O}\right\}$, quotient set,
$\xi_{\mathcal{O},B_r}$	Partition $\xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r}$,
$N_r(B)$	$N_r(B) = \{\xi_{\mathcal{O},B_r} \mid B_r \subseteq B\}$, set of partitions,
ν_{N_r}	$\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \longrightarrow [0,1]$, overlap function,
$N_r(B)_* X$	$N_r(B)_* X = \bigcup_{[x]_{B_r} \subseteq X} [x]_{B_r}$, lower approximation,
$N_r(B)^* X$	$N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$, upper approximation,
$Bnd_{N_r(B)}(X)$	$N_r(B)^* X \setminus N_r(B)_* X = \left\{x \in N_r(B)^* X \mid x \notin N_r(B)_* X\right\}$.

Table 1 : Nearness Approximation Space Symbols

A nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ where the approximation space is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim_{B_r}$ defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbourhoods) $N_r(B)$, and overlap function ν_{N_r}. The subscript r denotes the cardinality of the restricted subset B_r, where we consider $\binom{|B|}{r}$, i.e., $|B|$ functions $\phi_i \in \mathcal{F}$ taken r at a time to define the relation $\sim_{B_r}$. This relation defines a partition of $\mathcal{O}$ into non-empty, pairwise disjoint subsets that are equivalence classes denoted by $[x]_{B_r}$, where $[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$. These classes form a new set called the quotient set $\mathcal{O}/\sim_{B_r}$, where $\mathcal{O}/\sim_{B_r} = \left\{[x]_{B_r} \mid x \in \mathcal{O}\right\}$. In effect, each choice of probe functions B_r defines a partition $\xi_{\mathcal{O},B_r}$ on a set of objects $\mathcal{O}$, namely, $\xi_{\mathcal{O},B_r} = \mathcal{O}/\sim_{B_r}$. Every choice of the set B_r leads to a new partition of $\mathcal{O}$. Let $\mathcal{F}$ denote a set of features for objects in a set X, where each $\phi_i \in \mathcal{F}$ that maps X to some value set V_{ϕ_i} (range of ϕ_i). The value of $\phi_i(x)$ is a measurement associated with a feature of an object $x \in X$. The overlap function ν_{N_r} is defined by $\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \longrightarrow [0,1]$, where $\wp(\mathcal{O})$ is the powerset of $\mathcal{O}$. The overlap function ν_{N_r} maps a pair of sets to a number in $[0,1]$ representing the degree of overlap between sets of objects with their features defined by probe functions $B_r \subseteq B$ [18]. For each subset $B_r \subseteq B$ of probe functions, define the binary relation $\sim_{B_r} = \{(x,x') \in \mathcal{O} \times \mathcal{O} \mid \forall \phi_i \in B_r,\ \phi_i(x) = \phi_i(x')\}$. Since each $\sim_{B_r}$ is, in fact, the usual indiscernibility relation, for $B_r \subseteq B$ and $x \in \mathcal{O}$, let $[x]_{B_r}$ denote the equivalence class containing x. If $(x,x') \in \sim_{B_r}$, then x and x' are said to be B-indiscernible with respect to all feature probe functions in B_r. Then define a collection of partitions $N_r(B)$, where $N_r(B) = \{\xi_{\mathcal{O},B_r} \mid B_r \subseteq B\}$.

2.2. **Descriptively near sets.** We need the notion of nearness between sets, and so we consider the concept of the descriptively near sets. In 2007, descriptively near sets were introduced as a means of solving classification and pattern recognition problems arising from disjoint sets that resemble each other [13, 15].

A set of objects $A \subseteq \mathcal{O}$ is characterized by the unique description of each object in the set.

Set Description**:** [10] Let $\mathcal{O}$ be a set of perceptual objects, Φ an object description and $A \subseteq \mathcal{O}$. Then the *set description* of A is defined as

$$\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}.$$

Descriptive Set Intersection**:** [10, 17] Let $\mathcal{O}$ be a set of perceptual objects, A and B any two subsets of $\mathcal{O}$. Then the descriptive (set) intersection of A and B is defined as

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \ \ and\ \Phi(x) \in \mathcal{Q}(B)\}.$$

If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \varnothing$, then A is called descriptively near B and denoted by $A\delta_\Phi B$ [14].

Descriptive Nearness Collections**:** [14] $\xi_\Phi(A) = \{B \in \mathcal{P}(\mathcal{O}) \mid A\delta_\Phi B\}$.

Let Φ be an object description, A any subset of $\mathcal{O}$ and $\xi_\Phi(A)$ a descriptive nearness collections. Then $A \in \xi_\Phi(A)$ [14].

2.3. **Some algebraic structures on nearness approximation spaces.** A binary operation on a set G is a mapping of $G \times G$ into G, where $G \times G$ is the set of all ordered pairs of elements of G. A groupoid is a system $G(\cdot)$ consisting of a nonempty set G together with a binary operation "$\cdot$" on G [3].

Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space and let "$\cdot$" a binary operation defined on $\mathcal{O}$. A subset G of the set of perceptual objects $\mathcal{O}$ is called a group on nearness approximation spaces or shortly nearness group, if the following properties are satisfied:

(NG_1) For all $x, y \in G$, $x \cdot y \in N_r(B)^* G$,

(NG_2) For all $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^* G$,

(NG_3) There exists $e \in N_r(B)^* G$ such that $x \cdot e = e \cdot x = x$ for all $x \in G$ (e is called the near identity element of G),

(NG_4) There exists $y \in G$ such that $x \cdot y = y \cdot x = e$ for all $x \in G$ (y is called the near inverse of x in G and denoted as x^{-1}) [6].

If in addition, for all $x, y \in G$, $x \cdot y = y \cdot x$ property holds in $N_r(B)^* G$, then G is said to be an abelian nearrness group.

Also, a nonempty subset $S \subseteq \mathcal{O}$ is called a nearrness semigroup, if $x \cdot y \in N_r(B)^* S$ for all $x, y \in S$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in S$ property holds in $N_r(B)^* (S)$.

Theorem 2.1 ([6])**.** *Let G be a nearrness group.*

(1) *There exists a unique near identity element $e \in N_r(B)^* G$ such that $x \cdot e = x = e \cdot x$, for all $x \in G$.*

(2) *For all $x \in G$, there exists a unique $y \in G$ such that $x \cdot y = e = y \cdot x$.*

Theorem 2.2 ([6])**.** *Let G be a nearrness group.*

(1) $(x^{-1})^{-1} = x$*, for all $x \in G$.*

(2) *If $x \cdot y \in G$, then $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, for all $x, y \in G$.*

(3) *If either $x \cdot z = y \cdot z$ or $z \cdot x = z \cdot y$, then $x = y$, for all $x, y, z \in G$.*

H is called a subnearness group of nearness group G if H is a nearness group relative to the operation in G. There is only one guaranteed trivial subnearness group of nearness group G, i.e., G itself. Moreover, $\{e\}$ is a trivial subnearness group of nearness group G if and only if $e \in G$.

Theorem 2.3 ([7]). *Let G be a nearness group, H be a nonempty subset of G and $N_r(B)^* H$ be a groupoid. $H \subseteq G$ is a subnearness group of G if and only if $x^{-1} \in H$, for all $x \in H$.*

Let H_1 and H_2 be two nearness subgroups of the nearness group G and $N_r(B)^* H_1$, $N_r(B)^* H_2$ groupoids. If $\left(N_r(B)^* H_1\right) \cap \left(N_r(B)^* H_2\right) = N_r(B)^* (H_1 \cap H_2)$, then $H_1 \cap H_2$ is a nearness subgroup of nearness group G [7].

Let $G \subset \mathcal{O}$ be a nearness group and H be a subnearness group of G. The left weak equivalence relation (compatible relation) "$\sim_L$" defined as

$$a \sim_L b :\Leftrightarrow a^{-1} \cdot b \in H \cup \{e\}.$$

A weak class defined by relation "$\sim_L$" is called left weak coset. The left weak coset that contains the element a is denoted by $\tilde{a}_L$, i.e.

$$\tilde{a}_L = \{a \cdot h \mid h \in H,\ a \in G,\ a \cdot h \in G\} \cup \{a\} = aH.$$

Let $\left(\mathcal{O}_1, \mathcal{F}_1, \sim_{B_{r_1}}, N_{r_1}(B), \nu_{N_{r_1}}\right)$ and $\left(\mathcal{O}_2, \mathcal{F}_2, \sim_{B_{r_2}}, N_{r_2}(B), \nu_{N_{r_2}}\right)$ be two nearness approximation spaces and "$\cdot$", "$\circ$" binary operations over $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively.

Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be two nearness groups and σ a mapping from $N_{r_1}(B)^* G_1$ onto $N_{r_2}(B)^* G_2$. If $\sigma(x \cdot y) = \sigma(x) \circ \sigma(y)$ for all $x, y \in G_1$, then σ is called a nearness homomorphism and also, G_1 is called nearness homomorphic to G_2.

Let $G_1 \subset \mathcal{O}_1$, $G_2 \subset \mathcal{O}_2$ be nearness homomorphic groups, H_1 a nearness subgroup and $N_{r_1}(B)^* H_1$ a groupoid. If $\sigma\left(N_{r_1}(B)^* H_1\right) = N_{r_2}(B)^* \sigma(H_1)$, then $\sigma(H_1)$ is a nearness subgroup of G_2 [7].

The kernel of σ is defined to be the set $Ker\sigma = \{x \in G_1 \mid \sigma(x) = e'\}$, where e' is the nearness identity element of G_2.

Theorem 2.4 ([7]). *Let $G_1 \subset \mathcal{O}_1$,$G_2 \subset \mathcal{O}_2$ be nearness groups that are nearness homomorphic, $Ker\sigma = N$ be nearness homomorphism kernel and $N_r(B)^* N$ be a groupoid. Then N is a nearness normal subgroup of G_1.*

Definition 2.5 ([11]). Let $\mathcal{O}$ be a set of perceptual objects, $G \subset \mathcal{O}$ a nearness group and H a subnearness group of G. Let $G/_{\sim_L}$ be a set of all left weak cosets of G by H, $\xi_\Phi(A)$ a descriptive nearness collections and $A \in \mathcal{P}(\mathcal{O})$. Then

$$N_r(B)^* (G/_{\sim_L}) = \bigcup_{\xi_\Phi(A) \underset{\Phi}{\cap} G/_{\sim_L} \neq \varnothing} \xi_\Phi(A)$$

is called upper approximation of $G/_{\sim_L}$.

Theorem 2.6 ([11]). *Let G be a nearness group, H a subnearness group of G and $G/_{\sim_L}$ a set of all left weak cosets of G by H. If $\left(N_r(B)^* G\right)/_{\sim_L} \subseteq N_r(B)^* (G/_{\sim_L})$, then $G/_{\sim_L}$ is a nearness group under the operation given by $aH \odot bH = (a \cdot b)H$ for all $a, b \in G$.*

Let G be a nearness group and H a subnearness group of G. The nearness group $G/_{\sim_L}$ is called a nearness group of all left weak cosets of G by H and denoted by $G/_wH$ [11].

3. Nearness rings

Definition 3.1. Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space and "+" and "$\cdot$" binary operations defined on $\mathcal{O}$. A subset R of the set of perceptual objects $\mathcal{O}$ is called a ring on nearness approximation spaces or shortly nearness ring if the following properties are satisfied:

(NR_1) R is an abelian nearness group with binary operation "+",
(NR_2) R is a nearness semigroup with binary operation "$\cdot$",
(NR_3) For all $x, y, z \in R$,
$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and
$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ properties hold in $N_r(B)^* R$.

If in addition:

(NR_4) $x \cdot y = y \cdot x$, for all $x, y \in R$,

then R is said to be a commutative nearness ring.

(NR_5) If $N_r(B)^* R$ contains an element 1_R such that $1_R \cdot x = x \cdot 1_R = x$, for all $x \in R$,

then R is said to be a nearness ring with identity.

These properties have to hold in $N_r(B)^* R$. Sometimes they may be hold in $\mathcal{O} / N_r(B)^* R$, then R is not a nearness ring. Multiplying or sum of finite number of elements in R may not always belongs to $N_r(B)^* R$. Therefore always we can not say that $x^n \in N_r(B)^* R$ or $nx \in N_r(B)^* R$, for all $x \in R$ and some positive integer n. If $\left(N_r(B)^* R, +\right)$ and $\left(N_r(B)^* R, \cdot\right)$ are groupoids, then we can say that $x^n \in N_r(B)^* R$ for all positive integer n or $nx \in N_r(B)^* R$ all integer n, for all $x \in R$.

An element x in nearness ring R with identity is said to be left (resp. right) invertible, if there exists $y \in N_r(B)^* R$ (resp. $z \in N_r(B)^* R$) such that $y \cdot x = 1_R$ (resp. $x \cdot z = 1_R$). The element y (resp. z) is called a left (resp. right) inverse of x. If $x \in R$ is both left and right invertible, then x is said to be nearness invertible or nearness unit. The set of nearness units in a nearness ring R with identity forms is a nearness group on with multiplication.

A nearness ring R is a nearness division ring iff $(R \backslash \{0\}, \cdot)$ is a nearness group, i.e., every nonzero elements in R is a nearness unit. Similarly, a nearness ring R is a nearness field iff $(R \backslash \{0\}, \cdot)$ is a commutative nearness group.

Some elementary properties of elements in nearness rings are not always provided as in ordinary rings. If we consider $N_r(B)^* R$ as a ordinary ring, then elementary properties of elements in nearness ring are provided.

Lemma 3.2. *All ordinary rings on nearness approximation space are nearness rings.*

Example 3.3. Let $\mathcal{O} = \{o, p, r, s, t, v, w, x\}$ be a set of perceptual objects and $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ a set of probe functions. Values of the probe functions

$$\varphi_1 : \mathcal{O} \longrightarrow V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$
$$\varphi_2 : \mathcal{O} \longrightarrow V_2 = \{\beta_1, \beta_2, \beta_3\},$$

are given in Table 2.

	o	p	r	s	t	v	w	x
φ_1	α_4	α_2	α_1	α_2	α_1	α_3	α_4	α_3
φ_2	β_1	β_3	β_2	β_3	β_2	β_3	β_1	β_3

Table 2.

Let "+" and "·" be binary operations of perceptual objects on $\mathcal{O}$ as in Tables 3 and 4.

$+$	o	p	r	s	t	v	w	x
o	o	p	r	s	t	v	w	x
p	p	r	s	t	v	w	x	o
r	r	s	t	v	w	x	o	p
s	s	t	v	w	x	o	p	r
t	t	v	w	x	o	p	r	s
v	v	w	x	p	p	r	s	t
w	w	x	o	p	r	s	t	v
x	x	o	p	r	s	t	v	w

Table 3.

$\cdot$	o	p	r	s	t	v	w	x
o	o	o	o	o	o	o	o	o
p	o	p	r	s	t	v	w	x
r	o	r	t	w	o	r	t	w
s	o	s	w	p	t	o	r	v
t	o	t	o	t	o	t	o	t
v	o	v	r	x	t	p	w	s
w	o	w	t	r	o	w	t	r
x	o	x	w	v	t	s	r	p

Table 4.

Since $r + (s + s) \neq (r + s) + s$, $(\mathcal{O}, +)$ is not a group, i.e., $(\mathcal{O}, +, \cdot)$ is not a ring. Let $R = \{r, t, w\}$ be a subset of perceptual objects. Let "+" and "·" be operations of perceptual objects on $R \subseteq \mathcal{O}$ as in Tables 5 and 6.

$+$	r	t	w
r	t	w	o
t	w	o	r
w	o	r	t

Table 5.

$\cdot$	r	t	w
r	t	o	t
t	o	o	o
w	t	o	t

Table 6.

$$[o]_{\varphi_1} = \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(o) = \alpha_4\} = \{o, w\},$$

$$\begin{aligned}[p]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(p) = \alpha_2\} \\ &= \{p, s\} \\ &= [s]_{\varphi_1},\end{aligned}$$

$$\begin{aligned}[r]_{\varphi_1} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(r) = \alpha_1\} \\ &= \{r, t\} \\ &= [t]_{\varphi_1},\end{aligned}$$

$$[v]_{\varphi_1} = \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(v) = \alpha_3\} = \{v, x\} = [x]_{\varphi_1}.$$

Then we have that $\xi_{\varphi_1} = \left\{[o]_{\varphi_1}, [r]_{\varphi_1}, [v]_{\varphi_1}, [w]_{\varphi_1}\right\}$.

$$[o]_{\varphi_2} = \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(o) = \beta_1\} = \{o, w\} = [w]_{\varphi_2},$$

$$[p]_{\varphi_2} = \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(p) = \beta_3\} = \{p, s, v, x\} = [s]_{\varphi_2} = [v]_{\varphi_2} = [x]_{\varphi_2},$$

$$[r]_{\varphi_2} = \{x' \in \mathcal{O} \mid \varphi_2(x') = \varphi_2(r) = \beta_2\} = \{r, t\} = [t]_{\varphi_2}.$$

Thus we obtain that $\xi_{\varphi_2} = \left\{[o]_{\varphi_2}, [p]_{\varphi_2}, [r]_{\varphi_2}\right\}$. So, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}\}$.

In this case, we can write

$$N_1(B)^* R = \bigcup_{[x]_{\varphi_i} \cap R \neq \varnothing} [x]_{\varphi_i} = \{r, t\} \cup \{o, w\} \cup \{o, w\} \cup \{r, t\} = \{o, r, t, w\} \neq \mathcal{O}.$$

From Definition 3.1, since

(NR_1) R is an abelian nearness group with binary operation "$+$",
(NR_2) R is a nearness semigroup with binary operation "$\cdot$" and
(NR_3) For all $x, y, z \in R$,
$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and
$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ properties hold in $N_r(B)^* R$.

conditions hold, R is a nearness ring.

Proposition 3.4. *Let R be a nearness ring and $0 \in R$. If $0 \cdot x, x \cdot 0 \in R$, then for all $x, y \in R$*
(1) $x \cdot 0 = 0 \cdot x = 0$,
(2) $x \cdot (-y) = (-x) \cdot y = -(x \cdot y)$,
(3) $(-x) \cdot (-y) = x \cdot y$.

Definition 3.5. Let R be a nearness ring and S a nonempty subset of R. S is called subnearness ring of R, if S is a nearness ring with binary operations "$+$" and "$\cdot$" on nearness ring R.

Definition 3.6. Let we consider nearness field R and a nonempty subset S of R. S is called subnearness field of R if S is a nearness field.

Theorem 3.7. *Let R be a nearness ring and $(N_r(B)^* S, +)$, $(N_r(B)^* S, \cdot)$ groupoids. S is a subnearness ring of R iff $-x \in S$, for all $x \in S$.*

Proof. Suppose that S is a subnearness ring of R. Then S is a nearness ring and $-x \in S$, for all $x \in S$.

Conversely, suppose $-x \in S$, for all $x \in S$. Since $(N_r(B)^* S, +)$ is a groupoid, from Theorem 2.3 $(S, +)$ is a commutative nearness group. By the hypothesis, since $(N_r(B)^* S, \cdot)$ is a groupoid and $S \subseteq R$, associative property holds in $N_r(B)^* S$. Then $(S, \cdot)$ is a nearness semigroup. For all $x, y, z \in S \subseteq R$, $y + z \in N_r(B)^* S$ and $x \cdot (y + z) \in N_r(B)^* S$. Also $x \cdot y + x \cdot z \in N_r(B)^* S$. Since R is a nearness ring, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ property holds in $N_r(B)^* S$. Similarly, we can show that $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ property holds in $N_r(B)^* S$. Thus S is a subnearness ring of nearness ring R. □

Example 3.8. From Example 3.3, let we consider the nearness ring $R = \{r, t, w\}$. Let $S = \{r, w\}$ be a subset of nearness ring R. Then, "+" and "·" are binary operations of perceptual objects on $S \subseteq R$ as in Tables 7 and 8.

$+$	r	w
r	t	o
w	o	t

Table 7.

$\cdot$	r	w
r	t	t
w	t	t

Table 8.

We know from Example 3.3, for $r = 1$, a classification of $\mathcal{O}$ is $N_1(B) = \{\xi_{(\varphi_1)}, \xi_{(\varphi_2)}\}$. Then, we can obtain $N_1(B)^* S = \{o, r, t, w\}$. Thus we can observe that $(N_r(B)^* S, +)$, $(N_r(B)^* S, \cdot)$ are groupoids and $-r = w, -w = r \in N_r(B)^* S$. So from Theorem 3.7, S is a subnearness ring of nearness ring R.

Theorem 3.9. *Let R be a nearness ring, S_1 and S_2 two subnearness rings of R and $N_r(B)^* S_1$, $N_r(B)^* S_2$ groupoids with the binary operations "+" and "·". If*

$$\left(N_r(B)^* S_1\right) \cap \left(N_r(B)^* S_2\right) = N_r(B)^* (S_1 \cap S_2),$$

then $S_1 \cap S_2$ is a subnearness ring of R.

Corollary 3.10. *Let R be a nearness ring, $\{S_i : i \in \Delta\}$ a nonempty family of subnearness rings of R and $N_r(B)^* S_i$ groupoids for all $i \in \Delta$. If*

$$\bigcap_{i \in \Delta} \left(N_r(B)^* S_i\right) = N_r(B)^* \left(\bigcap_{i \in \Delta} S_i\right),$$

then $\bigcap_{i \in \Delta} S_i$ is a subnearness ring of R.

4. Nearness ideals

Definition 4.1. Let R be a nearness ring and I be a nonempty subset of R. I is a left (right) nearness ideal of R provided for all $x, y \in I$ and for all $r \in R$, $x - y \in N_r(B)^* I$, $r \cdot x \in N_r(B)^* I$ ($x - y \in N_r(B)^* I$, $x \cdot r \in N_r(B)^* I$).

A nonempty set I of a nearness ring R is called a nearness ideal of R, if I is both a left and a right nearness ideal of R.

There is only one guaranteed trivial nearness ideal of nearness ring R, i.e., R itself. Furthermore, $\{0\}$ is a trivial nearness ideal of nearness ring R iff $0 \in R$.

Lemma 4.2. *Every nearness ideal is a subnearness ring of nearness ring* R.

Example 4.3. From Example 3.3 and 3.8, let we consider the nearness ring $R = \{r, t, w\}$ and subnearness ring $S = \{r, w\}$ of R. We can observe that $x - y \in N_r(B)^* S$, $r \cdot x \in N_r(B)^* S$ and $x \cdot r \in N_r(B)^* S$, for all $x, y \in S$ and for all $r \in R$. Then, from Definition 4.1, S is a nearness ideal of R.

Theorem 4.4. *Let* R *be a nearness ring,* I_1 *and* I_2 *two nearness ideals of* R *and* $N_r(B)^* I_1$, $N_r(B)^* I_2$ *groupoids with the binary operations "+" and "·". If*

$$\left(N_r(B)^* I_1\right) \cap \left(N_r(B)^* I_2\right) = N_r(B)^* (I_1 \cap I_2),$$

then $I_1 \cap I_2$ *is a nearness ideal of* R.

Proof. Suppose I_1 and I_2 are two nearness ideals of the nearness ring R. It is obvious that $I_1 \cap I_2 \subset R$. Consider $x, y \in I_1 \cap I_2$. Since I_1 and I_2 are nearness ideals, we have $x - y, r \cdot x \in N_r(B)^* I_1$ and $x - y, r \cdot x \in N_r(B)^* I_2$, i.e., $x - y, r \cdot x \in \left(N_r(B)^* I_1\right) \cap \left(N_r(B)^* I_2\right)$, for all $x, y \in I_1, I_2$ and $r \in R$. Assuming $\left(N_r(B)^* I_1\right) \cap \left(N_r(B)^* I_2\right) = N_r(B)^* (I_1 \cap I_2)$, we have $x - y, r \cdot x \in N_r(B)^* (I_1 \cap I_2)$. From Definition 4.1, $I_1 \cap I_2$ is a nearness ideal of R. □

Corollary 4.5. *Let* R *be a nearness ring,* $\{I_i : i \in \Delta\}$ *a nonempty family of nearness ideals of* R *and* $N_r(B)^* I_i$ *groupoids with the binary operations "+" and "·". If*

$$\bigcap_{i \in \Delta} \left(N_r(B)^* I_i\right) = N_r(B)^* \left(\bigcap_{i \in \Delta} I_i\right),$$

then $\bigcap_{i \in \Delta} I_i$ *is a nearness ideal of* R.

5. Nearness rings of weak cosets

Let R be a nearness ring and S a subnearness ring of R. The left weak equivalence relation (compatible relation) "$\sim_L$" defined as

$$x \sim_L y :\Leftrightarrow -x + y \in S \cup \{e\}.$$

A weak class defined by relation "$\sim_L$" is called left weak coset. The left weak coset that contains the element $x \in R$ is denoted by $\tilde{x}_L$, i.e.,

$$\tilde{x}_L = \{x + s \mid s \in S,\ x \in R,\ x + s \in R\} \cup \{x\}.$$

Similarly, we can define the right weak coset that contains the element $x \in R$ is denoted by $\tilde{x}_R$, i.e.,

$$\tilde{x}_R = \{s + x \mid s \in S,\ x \in R,\ s + x \in R\} \cup \{x\}.$$

We can easily show that $\tilde{x}_L = x + S$ and $\tilde{x}_R = S + x$. Since $(R, +)$ is a abelian nearness group, $\tilde{x}_L = \tilde{x}_R$ and so we use only notation $\tilde{x}$. Then

$$R/_\sim = \{x + S \mid x \in R\}$$

is a set of all weak cosets of R by S. In this case, if we consider $N_r(B)^* R$ instead of nearness ring R

$$\left(N_r(B)^* R\right)/_\sim = \left\{x + S \mid x \in N_r(B)^* R\right\}.$$

Definition 5.1 ([11]). Let R be a nearness ring and S be a subnearness ring of R. For $x, y \in R$, let $x + S$ and $y + S$ be two weak cosets that determined the elements x and y, respectively. Then sum of two weak cosets that determined by $x + y \in N_r(B)^* R$ can be defined as

$$(x+y)+S = \{(x+y)+s \mid s \in S,\ x+y \in N_r(B)^* R,\ (x+y)+s \in R\} \cup \{x+y\}$$

and denoted by

$$(x+S) \oplus (y+S) = (x+y)+S.$$

Definition 5.2. Let R be a nearness ring and S be a subnearness ring of R. For $x, y \in R$, let $x+S$ and $y+S$ be two weak cosets that determined the elements x and y, respectively. Then product of two weak cosets that determined by $x \cdot y \in N_r(B)^* R$ can be defined as

$$(x \cdot y)+S = \{(x \cdot y)+s \mid s \in S,\ x \cdot y \in N_r(B)^* R,\ (x \cdot y)+s \in R\} \cup \{x \cdot y\}$$

and denoted by

$$(x+S) \odot (y+S) = (x \cdot y)+S.$$

Definition 5.3. Let $R/_\sim$ be a set of all weak cosets of R by S, $\xi_\Phi(A)$ a descriptive nearness collections and $A \in \mathcal{P}(\mathcal{O})$. Then

$$N_r(B)^* (R/_\sim) = \bigcup_{\xi_\Phi(A) \underset{\Phi}{\cap} R/_\sim \neq \varnothing} \xi_\Phi(A)$$

is called upper approximation of $R/_\sim$.

Theorem 5.4. *Let R be a nearness ring, S a subnearness ring of R and $R/_\sim$ be a set of all weak cosets of R by S. If $\left(N_r(B)^* R\right)/_\sim \subseteq N_r(B)^* (R/_\sim)$, then $R/_\sim$ is a nearness ring under the operations given by $(x+S) \oplus (y+S) = (x+y)+S$ and $(x+S) \odot (y+S) = (x \cdot y)+S$ for all $x, y \in R$.*

Proof. $(NR1)$ Let $\left(N_r(B)^* R\right)/_\sim \subseteq N_r(B)^* (R/_\sim)$. Since R is a nearness ring from Theorem 2.6, $(R/_\sim, \oplus)$ is a abelian nearness group of all weak cosets of R by S.

$(NR2)$ Since $(R, \cdot)$ is a nearness semigroup;

$(NS1)$ We have that $x \cdot y \in N_r(B)^* R$ and $(x+S) \odot (y+S) = (x \cdot y)+S \in \left(N_r(B)^* R\right)/_\sim$, for all $(x+S), (y+S) \in R/_\sim$. From the hypothesis,

$$(x+S) \odot (y+S) = (x \cdot y)+S \in N_r(B)^* (R/_\sim),$$

for all $(x+S), (y+S) \in R/_\sim$.

$(NS2)$ For all $x, y, z \in R/_\sim$, associative property hols in $N_r(B)^* R$. Then for all $(x+S), (y+S), (z+S) \in R/_\sim$,

$$\begin{aligned}((x+S) \odot (y+S)) \odot (z+S) &= ((x \cdot y)+S) \odot (z+S)\\ &= ((x \cdot y) \cdot z)+S\\ &= (x \cdot (y \cdot z))+S\\ &= (x+S) \odot ((y \cdot z)+S)\\ &= (x+S) \odot ((y+S) \odot (z+S))\end{aligned}$$

holds in $\left(N_r(B)^* R\right)/_\sim$. From the hypothesis, for all $(x+S)$, $(y+S)$, $(z+S) \in R/_\sim$, associative property holds in $N_r(B)^* (R/_\sim)$. Thus $(R/_\sim, \odot)$ is a nearness semigroup of all left weak cosets of R by S.

$(NR3)$ Since R is a nearness ring, left distributive law holds in $N_r(B)^* R$. For all $(x+S),(y+S),(z+S) \in R/_\sim$,

$$\begin{aligned}(x+S)\odot((y+S)\oplus(z+S)) &= (x+S)\odot((y+z)+S)\\ &= (x\cdot(y+z))+S\\ &= ((x\cdot y)+(x\cdot z))+S\\ &= ((x\cdot y)+S)\oplus((x\cdot z)+S)\\ &= ((x+S)\odot(y+S))\oplus((x+S)\odot(z+S)).\end{aligned}$$

So left distributive law holds in $\left(N_r(B)^* R\right)/_\sim$.

Similarly, we can show that right distributive law holds in $\left(N_r(B)^* R\right)/_\sim$,

$$((x+S)\oplus(y+S))\odot(z+S) = ((x+S)\odot(z+S))\oplus((x+S)\odot(z+S)),$$

for all $(x+S),(y+S),(z+S) \in R/_\sim$.

From the hypothesis, distributive laws hold in $N_r(B)^*(R/_\sim)$. Consequently, $R/_\sim$ is a nearness ring. □

Definition 5.5. Let R be a nearness ring and S be a subnearness ring of R. The nearness ring $R/_\sim$ is called a nearness ring of all weak cosets of R by S and denoted by $R/_w S$.

Example 5.6. Let $S=\{r,w\}$ be a subset of $R=\{r,t,w\}$. From Example 3.8, S is a subnearness ring of nearness ring R.

Now, we can compute the all weak cosets of R by S. Then by using the definition of weak coset,

$$\begin{aligned}&r+S=\{r\}\cup\{r\}=\{r\},\ t+S=\{w,r\}\cup\{t\}=\{w,r,t\},\\ &w+S=\{t\}\cup\{w\}=\{t,w\}.\end{aligned}$$

Thus we have that $R/_w S=\{r+S,t+S,w+S\}$. Since $N_1(B)^* R=\{o,r,t,w\}$, we can write the all weak cosets of $N_1(B)^* R$ by S. In this case,

$$o+S=\{r,w\}\cup\{o\}=\{r,w,o\}.$$

So $\left(N_1(B)^* R\right)/_\sim=\{o+S,r+S,t+S,w+S\}\subset\mathcal{P}(\mathcal{O})$.

Let "$\oplus$" and "$\odot$ " be operations on $R/_w S$, by using the Definition 5.1 and 5.2, as in Tables 9 and 10.

$\oplus$	$r+S$	$t+S$	$w+S$
$r+S$	$t+S$	$w+S$	$o+S$
$t+S$	$w+S$	$o+S$	$r+S$
$w+S$	$o+S$	$r+S$	$t+S$

Table 9.

$\odot$	$r+S$	$t+S$	$w+S$
$r+S$	$t+S$	$o+S$	$t+S$
$t+S$	$o+S$	$o+S$	$o+S$
$w+S$	$t+S$	$o+S$	$t+S$

Table 10.

It is enough to show that every element of $\left(N_1(B)^* R\right)/_\sim$ is also an element of $N_1(B)^*(R/_w S)$ in order to ensure $\left(N_r(B)^* R\right)/_\sim\subseteq N_r(B)^*(R/_w S)$.

$$\begin{aligned}\mathcal{Q}(R/_wS) &= \{\Phi(A) \mid A \in R/_wS\}\\ &= \{\Phi(r+S), \Phi(t+S), \Phi(w+S)\}\\ &= \{\{\Phi(r)\}, \{\Phi(w), \Phi(r), \Phi(t)\}, \{\Phi(t), \Phi(w)\}\}\\ &= \{\{(\alpha_1, \beta_2)\}, \{(\alpha_4, \beta_1), (\alpha_2, \beta_1), (\alpha_1, \beta_2)\}, \{(\alpha_1, \beta_2), (\alpha_4, \beta_1)\}\}.\end{aligned}$$

For $r+S \in R/_wS$, we get that

$$\begin{aligned}&\mathcal{Q}(r+S) = \{\Phi(r)\} = \{(\alpha_1, \beta_2)\},\\ &\mathcal{Q}(o+S) = \{\Phi(r), \Phi(w), \Phi(o)\} = \{(\alpha_1, \beta_2), (\alpha_4, \beta_1), (\alpha_4, \beta_1)\}.\end{aligned}$$

Since $\mathcal{Q}(r+S) \cap \mathcal{Q}(o+S) = \{(\alpha_1, \beta_2)\} \neq \varnothing$, it follows that $o+S \in \xi_\Phi(r+S)$. Then $\xi_\Phi(r+S) \underset{\Phi}{\cap} R/_wS \neq \varnothing$ and $r+S, o+S \in N_1(B)^*(R/_wS)$ by Definition 5.3. For $t+S \in R/_wS$, $w+S$ we get that

$$\begin{aligned}&\mathcal{Q}(t+S) = \{\{\Phi(w), \Phi(r), \Phi(t)\}\} = \{(\alpha_4, \beta_1), (\alpha_2, \beta_1), (\alpha_1, \beta_2)\},\\ &\mathcal{Q}(w+S) = \{\{\Phi(t), \Phi(w)\}\} = \{(\alpha_1, \beta_2), (\alpha_4, \beta_1)\}.\end{aligned}$$

Since $\mathcal{Q}(t+S) \cap \mathcal{Q}(t+S) = \{(\alpha_4, \beta_1), (\alpha_2, \beta_1), (\alpha_1, \beta_2)\} \neq \varnothing$ and $\mathcal{Q}(w+S) \cap \mathcal{Q}(w+S) = \{(\alpha_1, \beta_2), (\alpha_4, \beta_1)\} \neq \varnothing$, it follows that $t+S \in \xi_\Phi(t+S), w+S \in \xi_\Phi(w+S)$. Thus $\xi_\Phi(t+S) \underset{\Phi}{\cap} R/_wS \neq \varnothing$, $\xi_\Phi(w+S) \underset{\Phi}{\cap} R/_wS \neq \varnothing$ and $t+S, w+S \in N_1(B)^*(R/_wS)$ by Definition 5.3.

Consequently, $\left(N_r(B)^* R\right)/_{\sim_L} \subseteq N_r(B)^*(R/_wS)$. So, from the Theorem 5.4, $R/_wS$ is a nearness ring of all weak cosets of R by S with the operations given by Tables 9 and 10.

6. Nearness ring homomorphisms

Definition 6.1. Let $R_1, R_2 \subset \mathcal{O}$ be two nearness rings and η a mapping from $N_r(B)^* R_1$ onto $N_r(B)^* R_2$. If $\eta(x+y) = \eta(x) + \eta(y)$ and $\eta(x \cdot y) = \eta(x) \cdot \eta(y)$ for all $x, y \in R_1$, then η is called a nearness ring homomorphism and also, R_1 is called nearness homomorphic to R_2, denoted by $R_1 \simeq_n R_2$.

A nearness ring homomorphism η of $N_r(B)^* R_1$ into $N_r(B)^* R_2$ is called

(i) a nearness momomorphism, if η is one-one,

(ii) a nearness epimorphism, if η is onto $N_r(B)^* R_2$ and

(iii) a nearness isomorphism, if η is one-one and maps $N_r(B)^* R_1$ onto $N_r(B)^* R_2$.

Theorem 6.2. *Let R_1, R_2 be two nearness rings and η a nearness homomorphism of $N_r(B)^* R_1$ into $N_r(B)^* R_2$. Then the following properties hold.*

(1) $\eta(0_{R_1}) = 0_{R_2}$, where $0_{R_2} \in N_r(B)^* R_2$ is the nearness zero of R_2.

(2) $\eta(-x) = -\eta(x)$ for all $x \in R_1$.

Proof. (1) Since η is a nearness homomorphism,

$$\eta(0_{R_1}) + \eta(0_{R_1}) = \eta(0_{R_1} + 0_{R_1}) = \eta(0_{R_1}) = \eta(0_{R_1}) + 0_{R_2}.$$

Then we have that $\eta(0_{R_1}) = 0_{R_2}$ by the Theorem 2.2 (3).

(2) Let $x \in R_1$. Then $\eta(x) + \eta(-x) = \eta(x-x) = \eta(0_{R_1}) = 0_{R_2}$. Similarly, we can obtain that $\eta(-x) + \eta(x) = 0_{R_2}$, for all $x \in R_1$. From Theorem 2.1 (2), since $\eta(x)$ has a unique inverse, $\eta(-x) = -\eta(x)$, for all $x \in R_1$. □

Theorem 6.3. *Let R_1, R_2 be two nearness rings and η a nearness homomorphism of $N_r(B)^* R_1$ into $N_r(B)^* R_2$ and $N_r(B)^* S$ a groupoid. Then the following properties hold.*

(1) *If S is a subnearness ring of nearness ring R_1 and $\eta\left(N_r(B)^* S\right) = N_r(B)^* \eta(S)$, then $\eta(S) = \{\eta(x) : x \in S\}$ is a subnearness ring of R_2.*

(2) *If S is a commutative subnearness ring R_1 and $\eta\left(N_r(B)^* S\right) = N_r(B)^* \eta(S)$, then $\eta(S)$ is a commutative nearness ring of R_2.*

Proof. (1) Let S be a subnearness ring of nearness ring R_1. Then $0_S \in N_r(B)^* S$ and by Theorem 6.2 (1), $\eta(0_S) = 0_{R_2}$, where $0_{R_2} \in N_r(B)^* R_2$. Thus

$$0_{R_2} = \eta(0_S) \in \eta\left(N_r(B)^* S\right) = N_r(B)^* \eta(S).$$

This means that $\eta(S) \neq \varnothing$. Let $\eta(x) \in \eta(S)$, where $x \in S$. Since S is a subnearness ring of R_1, $-x \in N_r(B)^* S$, for all $x \in S$. So for all $\eta(x) \in \eta(S)$,

$$-\eta(x) = \eta(-x) \in \eta\left(N_r(B)^* S\right) = N_r(B)^* \eta(S).$$

Hence by Theorem 3.7, $\eta(S)$ is subnearness ring of R_2.

(2) Let S be a commutative subnearness ring and $\eta(x), \eta(y) \in \eta(S)$. We have that $\eta(S)$ is a subnearness ring of R_2 by (1), i.e., $\eta(S)$ is a nearness ring. Then $\eta(x) \cdot \eta(y) = \eta(x \cdot y) = \eta(y \cdot x) = \eta(y) \cdot \eta(x)$, for all $\eta(x), \eta(y) \in \eta(R_1)$. Thus $\eta(S)$ is commutative subnearness ring of R_2. □

Definition 6.4. Let R_1, R_2 be two nearness rings and η be a nearness homomorphism of $N_r(B)^* R_1$ into $N_r(B)^* R_2$. The kernel of η, denoted by $Ker\eta$, is defined to be the set

$$Ker\eta = \{x \in R_1 : \eta(x) = 0_{R_2}\}.$$

Theorem 6.5. *Let R_1, R_2 be two nearness rings, η a nearness homomorphism of $N_r(B)^* R_1$ into $N_r(B)^* R_2$ and $N_r(B)^* Ker\eta$ a groupoid with binary operations "+" and "·". Then $\varnothing \neq Ker\eta$ is a nearness ideal of R_1.*

Proof. Let $x, y \in Ker\eta$. Then $f(x-y) = f(x) - f(y) = 0_{R_2} - 0_{R_2} = 0_{R_2} \in N_r(B)^* R_2$ and Thus $x - y \in N_r(B)^* (Ker\eta)$. Let $r \in R_1$. Then $f(r \cdot x) = f(r) \cdot f(x) = f(r) \cdot 0_{R_2} = 0_{R_2} \in N_r(B)^* R_2$ and thus $r \cdot x \in N_r(B)^* (Ker\eta)$. Similarly, $x \cdot r \in N_r(B)^* (Ker\eta)$. So, from Definition 4.1, $Ker\eta$ is a nearness ideal of R_1. □

Theorem 6.6. *Let R be a nearness ring and S a subnearness ring of R. Then the mapping $\Pi : N_r(B)^* R \to N_r(B)^* (R/_w S)$ defined by $\Pi(x) = x + S$, for all $x \in N_r(B)^* R$ is a nearness homomorphism.*

Proof. From the definition of Π, Π is a mapping from $N_r(B)^* R$ into $N_r(B)^* (R/_w S)$. By using the Definition 5.2,

$$\Pi(x+y) = (x+y) + S = (x+S) \oplus (y+S) = \Pi(x) \oplus \Pi(y),$$
$$\Pi(x \cdot y) = (x \cdot y) + S = (x+S) \odot (y+S) = \Pi(x) \odot \Pi(y),$$

for all $x, y \in R$. Thus Π is a nearness homomorphism from Definition 6.1. □

Definition 6.7. The near homomorphism Π is called a nearness natural homomorphism from $N_r(B)^* R$ into $N_r(B)^* (R/_w S)$.

Example 6.8. From Example 5.6, we consider the nearness ring of all weak cosets $R/_w S$. Define

$$\begin{array}{rcl} \Pi : N_r(B)^* R & \longrightarrow & N_r(B)^* (R/_w S) \\ x & \longmapsto & \Pi(x) = x + S \end{array}$$

for all $x \in N_r(B)^* R$. By using the Definitions 5.1 and 5.2, we have that

$$\Pi(x+y) = (x+y) + S = (x+S) \oplus (y+S) = \Pi(x) \oplus \Pi(y),$$

$$\Pi(x \cdot y) = (x \cdot y) + S = (x+S) \odot (y+S) = \Pi(x) \odot \Pi(y),$$

for all $x, y \in R$. Hence, Π is a nearness natural homomorphism from $N_r(B)^* R$ into $N_r(B)^* (R/_w S)$.

Definition 6.9. Let R_1, R_2 be two nearness rings and S be a non-empty subset of R_1. Let

$$\chi : N_r(B)^* R_1 \longrightarrow N_r(B)^* R_2$$

be a mapping and

$$\chi_S = {}^{\chi}\big|_S : S \longrightarrow N_r(B)^* R_2$$

a restricted mapping. If $\chi(x+y) = \chi_S(x+y) = \chi_S(x) + \chi_S(y) = \chi(x) + \chi(y)$ and $\chi(x \cdot y) = \chi_S(x \cdot y) = \chi_S(x) \cdot \chi_S(y) = \chi(x) \cdot \chi(y)$ for all $x, y \in S$, then χ is called a restricted nearness homomorphism and also, R_1 is called restricted nearness homomorphic to R_2, denoted by $R_1 \simeq_{rn} R_2$.

Theorem 6.10. *Let R_1, R_2 be two nearness rings and χ be a nearness homomorphism from $N_r(B)^* R_1$ into $N_r(B)^* R_2$. Let $\left(N_r(B)^* Ker\chi, +\right)$ and $\left(N_r(B)^* Ker\chi, \cdot\right)$ be groupoids and $\left(N_r(B)^* R_1\right)/_\sim$ be a set of all weak cosets of $N_r(B)^* R_1$ by $Ker\chi$. If $\left(N_r(B)^* R_1\right)/_\sim \subseteq N_r(B)^* (R_1/_w Ker\chi)$ and $N_r(B)^* \chi(R_1) = \chi\left(N_r(B)^* R_1\right)$, then*

$$R_1/_w Ker\chi \simeq_{rn} \chi(R_1).$$

Proof. Since $\left(N_r(B)^* Ker\chi, +\right)$ and $\left(N_r(B)^* Ker\chi, \cdot\right)$ are groupoids, from Theorem 6.5 $Ker\chi$ is a subnearness ring of R_1. Since $Ker\chi$ is a subnearness ring of R_1 and $\left(N_r(B)^* R_1\right)/_\sim \subseteq N_r(B)^* (R_1/_w Ker\chi)$, then $R_1/_w Ker\chi$ is a nearness ring of all weak cosets of R_1 by $Ker\chi$ from Theorem 5.4. Since $N_r(B)^* \chi(R_1) = \chi\left(N_r(B)^* R_1\right)$, $\chi(R_1)$ is a subnearness ring of R_2. Define

$$\begin{array}{rcl} \eta : N_r(B)^* (R_1/_w Ker\chi) & \longrightarrow & N_r(B)^* \chi(R_1) \\ A & \longmapsto & \eta(A) = \begin{cases} \eta_{R_1/_w Ker\chi}(A) & , A \in \left(N_r(B)^* R_1\right)/_\sim \\ e_{\chi(R_1)} & , A \notin \left(N_r(B)^* R_1\right)/_\sim \end{cases} \end{array}$$

where

$$\begin{array}{rcl} \eta_{R_1/_w Ker\chi} : {}^{\eta}\big|_{R_1/_w Ker\chi} & \longrightarrow & N_r(B)^* \chi(R_1) \\ x + Ker\chi & \longmapsto & \eta_{R_1/_w Ker\chi}(x + Ker\chi) = \chi(x), \end{array}$$

for all $x + Ker\chi \in R_1/_w Ker\chi$.
Since

$$x + Ker\chi = \{x + k \mid k \in Ker\chi, x + k \in R_1\} \cup \{x\},$$

$$y + Ker\chi = \{y + k' \mid k' \in Ker\chi, y + k' \in R_1\} \cup \{y\},$$

and the mapping χ is a nearness homomorphism,

$$\begin{array}{ll} & x+Ker\chi = y+Ker\chi \\ \Rightarrow & x\in y+Ker\chi \\ \Rightarrow & x\in\{y+k' \mid k'\in Ker\chi, y+k'\in R_1\} \text{ or } x\in\{y\} \\ \Rightarrow & x=y+k',\ k'\in Ker\chi,\ y+k'\in R_1 \text{ or } x=y \\ \Rightarrow & -y+x=(-y+y)+k',\ k'\in Ker\chi \text{ or } \chi(x)=\chi(y) \\ \Rightarrow & -y+x=k',\ k'\in Ker\chi \\ \Rightarrow & -y+x\in Ker\chi \\ \Rightarrow & \chi(-y+x)=e_{\chi(R_1)} \\ \Rightarrow & \chi(-y)+\chi(x)=e_{\chi(R_1)} \\ \Rightarrow & -\chi(y)+\chi(x)=e_{\chi(R_1)} \\ \Rightarrow & \chi(x)=\chi(y) \\ \Rightarrow & \eta_{R_1/_w Ker\chi}(x+Ker\chi)=\eta_{R_1/_w Ker\chi}(y+Ker\chi) \end{array}$$

Then $\eta_{R_1/_w Ker\chi}$ is well defined.

For $A,B\in N_r(B)^*(R_1/_w Ker\chi)$, we suppose that $A=B$. Since the mapping $\eta_{R_1/_w Ker\chi}$ is well defined,

$$\begin{aligned} \eta(A) &= \begin{cases} \eta_{R_1/_w Ker\chi}(A) & ,A\in\left(N_r(B)^* R_1\right)/_\sim \\ e_{\chi(R_1)} & ,A\notin\left(N_r(B)^* R_1\right)/_\sim \end{cases} \\ &= \begin{cases} \eta_{R_1/_w Ker\chi}(B) & ,B\in\left(N_r(B)^* R_1\right)/_\sim \\ e_{\chi(R_1)} & ,B\notin\left(N_r(B)^* R_1\right)/_\sim \end{cases} \\ &= \eta(B). \end{aligned}$$

Consequently, η is well defined.

For all $x+Ker\chi, y+Ker\chi\in R_1/_w Ker\chi\subset N_r(B)^*(R_1/_w Ker\chi)$,

$$\begin{array}{ll} & \eta((x+Ker\chi)\oplus(y+Ker\chi)) \\ = & \eta_{R_1/_w Ker\chi}((x+Ker\chi)\oplus(y+Ker\chi)) \\ = & \eta_{R_1/_w Ker\chi}((x+y)+Ker\chi) \\ = & \chi(x+y) \\ = & \chi(x)+\chi(y) \\ = & \eta_{R_1/_w Ker\chi}(x+Ker\chi)+\eta_{R_1/_w Ker\chi}(y+Ker\chi) \\ = & \eta(x+Ker\chi)+\eta(y+Ker\chi). \end{array}$$

and

$$\begin{array}{ll} & \eta((x+Ker\chi)\odot(y+Ker\chi)) \\ = & \eta_{R_1/_w Ker\chi}((x+Ker\chi)\odot(y+Ker\chi)) \\ = & \chi_{R_1/_w Ker\chi}((x\cdot y)+Ker\chi) \\ = & \chi(x\cdot y) \\ = & \chi(x)\cdot\chi(y) \\ = & \eta_{R_1/_w Ker\chi}(x+Ker\chi)\cdot\eta_{R_1/_w Ker\chi}(y+Ker\chi) \\ = & \eta(x+Ker\chi)\cdot\eta(y+Ker\chi). \end{array}$$

Thus η is a restricted nearness homomorphism by Definition 6.9. So $R_1/_w Ker\chi\simeq_{rn}\chi(R_1)$. □

Acknowledgements. This paper is a part of Ebubekir İnan's PhD thesis which approved on 20.03.2015 by İnönü University Graduate School of Natural and Applied Sciences, Turkey.

References

[1] M. Banrjee and M. K. Chakraborty, Rough Sets Through Algebraic Logic, Fund. Inform. 28 (3-4) (1996) 211–221.
[2] R. Biswas and S. Nanda, Rough groups and rough subgroups, Bull. Pol. AC. Math. 42 (1994) 251–254.
[3] A. Clifford and G. Preston, The Algebraic Theory of Semigroups I, Amer. Math. Soc. Providence, RI, Mathematical Surveys 1961.
[4] B. Davvaz, Roughness in rings, Inform. Sci. 164 (2004) 147–163.
[5] T. B. Iwinski, Algebraic approach to rough sets, Bull. Pol. AC. Math. 35 (1987) 673–683.
[6] E. İnan and M. A. Öztürk, Near groups in nearness approximation spaces, Hacet. J. Math. Stat. 41 (4) (2012) 545–558.
[7] E. İnan and M. A. Öztürk, Erratum and notes for near groups on nearness approximation spaces, Hacet. J. Math. Stat. 43 (2) (2014) 279–281.
[8] E. İnan and M. A. Öztürk, Near semigroups on nearness approximation spaces, Ann. Fuzzy Math. Inform. 10 (2) (2015) 287–297.
[9] J. Jarvinen, On the structure of rough approximations, Fund. Inform. 53 (2) (2002) 135–153.
[10] S. A. Naimpally and J. F. Peters, Topology with Applications, Topological Spaces via Near and Far World Scientific 2013.
[11] M. A. Öztürk, M. Uçkun and E. İnan, Near group of weak cosets on nearness approximation spaces, Fund. Inform. 133 (2014) 433–448.
[12] P. Pagliani, Rough sets and Nelson algebras, Fund. Inform. 27 (2-3) (1996) 205–219.
[13] J. F. Peters, Near Sets. General Theory About Nearness of Objects, Applied Math. Sci. 1 (53-56) (2007) 2609–2629.
[14] J. F. Peters, Near sets: An introduction, Math. Comput. Sci. 7 (1) (2013) 3–9.
[15] J. F. Peters, Near sets: Special Theory about Nearness of Objects, Fund. Inform. 75 (1-4) (2007) 407–433.
[16] J. F. Peters and S. Tiwari, Approach merotopies and near filters, Gen. Math. Notes 3 (1) (2011) 1–15.
[17] J. F. Peters and S. A. Naimpally, Applications of near sets, Notices Amer. Math. Soc. 59 (4) (2012) 536–542.
[18] A.Skowron and J. Stepaniuk, Tolerance Approximation Spaces, Fund. Inform. 27 (2-3) (1996) 245–253.
[19] S. Rasouli and B. Davvaz, Roughness in MV-algebras, Inform. Sci. 180 (2010) 737–747.
[20] M. Wolski, Perception and classification. A note on near sets and rough Sets, Fund. Inform. 101 (1-2) (2010) 143–155.
[21] S. Yamak, O. Kazancı and B. Davvaz, Generalized lower and upper approximations in a ring, Inform. Sci. 180 (2010) 1759–1768.

Mehmet Ali Öztürk (maozturk@adiyaman.edu.tr)
Adıyaman University, Faculty of Arts and Sciences, Department of Mathematics, 02040 Adıyaman, Turkey

Ebbekir İnan (einan@adiyaman.edu.tr)
Adıyaman University, Faculty of Arts and Sciences, Department of Mathematics, 02040 Adıyaman, Turkey