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Bipolar fuzzy topological spaces

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Abstract.
We define a bipolar fuzzy point and obtain some of its properties. Also we introduce the concepts of a bipolar fuzzy topology, bipolar fuzzy base and subbase and find some properties of each concept. Second, we define a bipolar fuzzy neighborhood and continuity and obtain bipolar fuzzy analogues of many results concerning to classical neighborhood and continuity. Third, we introduce the concepts of a bipolar fuzzy subspace and a bipolar fuzzy quotient space and find some properties of each concept. In particular, we prove the existence of the bipolar fuzzy initial topology (See Proposition 6.2). Finally, we define a compactness in bipolar fuzzy topological spaces and investigate some of its properties. In particular, we obtain 'Alexander Subbase Theorem' in the sense of bipolar fuzzy sets (See Proposition 7.7).

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1. Introduction

In 1994, Zhang [19] introduced the notion of a bipolar fuzzy set (Refer to [11, 12, 18]). After then, Jun and Park [6], Jun et al. [7] and Lee [10] applied bipolar fuzzy sets to BCK/BCI-algebras. Moreover, Akram and Dudek [1] studied bipolar fuzzy graph, and Majumder [14] introduced bipolar fuzzy Γ-semigroup. Moreover, Talebi et al [16] investigated operations on bipolar fuzzy graph. In particular, Azhagappan and Kamaraj [3] investigated bipolar fuzzy topological spaces. Recently, Kim et al. [9] constructed the category consisting of bipolar fuzzy set and preserving mappings between them and studied it in the sense of a topological universe.

In this paper, first, we define a bipolar fuzzy point and obtain some of its properties. Also we introduce the concepts of a bipolar fuzzy topology, bipolar fuzzy base and subbase and find some properties of each concept. Second, we define a bipolar fuzzy neighborhood and continuity and obtain bipolar fuzzy analogues of many results concerning to classical neighborhood and continuity. Third, we introduce the concepts of a bipolar fuzzy subspace and a bipolar fuzzy quotient space and find some properties of each concept. In particular, we prove the existence of the bipolar fuzzy initial topology (See Proposition 6.2). Finally, we define a compactness in bipolar fuzzy topological spaces and investigate some of its properties. In particular, we obtain 'Alexander Subbase Theorem' in the sense of bipolar fuzzy sets (See Proposition 7.7).

2. Preliminaries

In this section, we list some concepts related to bipolar fuzzy sets (for examples, the complement of a bipolar fuzzy set, the inclusion between two bipolar fuzzy sets, the union and the intersection of two bipolar fuzzy sets, the intersection and union of arbitrary bipolar fuzzy sets) and some properties needed later sections.

Definition 2.1 ([11])**.** Let X be a nonempty set. Then A pair $A = (A^+, A^-)$ is called a bipolar-valued fuzzy set (or, bipolar fuzzy set) in X, if $A^+ : X \to [0, 1]$ and $A^- : X \to [-1, 0]$ are mappings.

In particular, the bipolar fuzzy empty set [resp. the bipolar fuzzy whole set] (See [3]), denoted by $\mathbf{0}_{bp} = (\mathbf{0}^+_{bp}, \mathbf{0}^-_{bp})$ [resp. $\mathbf{1}_{bp} = (\mathbf{1}^+_{bp}, \mathbf{1}^-_{bp})$], is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$\mathbf{0}^+_{bp}(x) = 0 = \mathbf{0}^-_{bp}(x) \text{ [resp. } \mathbf{1}^+_{bp}(x) = 1 \text{ and } \mathbf{1}^-_{bp}(x) = -1].$$

We will denote the set of all bipolar fuzzy sets in X as $BPF(X)$.

For each $x \in X$, we use the positive membership degree $A^+(x)$ to denote the satisfaction degree of the element x to the property corresponding to the bipolar fuzzy set A and the negative membership degree $A^-(x)$ to denote the satisfaction degree of the element x to some implicit counter-property corresponding to the bipolar fuzzy set A.

If $A^+(x) \neq 0$ and $A^-(x) = 0$, then it is the situation that x is regarded as having only positive satisfaction for A. If $A^+(x) = 0$ and $A^-(x) \neq 0$, then it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter-property of A. It is possible for some $x \in X$ to be such that $A^+(x) \neq 0$ and $A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of X.

Definition 2.2 ([11])**.** Let X be a nonempty set and let $A,\ B \in BPF(X)$.

(i) We say that A is subset of B, denoted by $A \subset B$, if for each $x \in X$,

$$A^+(x) \le B^+(x) \text{ and } A^-(x) \ge B^-(x).$$

(ii) The complement of A, denoted by $A^c = ((A^c)^+, (A^c)^-)$, is a bipolar fuzzy set in X defined as: for each $x \in X$, $A^c(x) = (1 - A^+(x), -1 - A^-(x))$, i.e.,

$$(A^c)^+(x) = 1 - A^+(x),\ (A^c)^-(x) = -1 - A^-(x).$$

(iii) The intersection of A and B, denoted by $A \cap B$, is a bipolar fuzzy set in X defined as:for each $x \in X$,

$$(A \cap B)(x) = (A^+(x) \wedge B^+(x), A^-(x) \vee B^-(x)).$$

(iv) The union of A and B, denoted by $A \cup B$, is a bipolar fuzzy set in X defined as: for each $x \in X$,

$$(A \cup B)(x) = (A^+(x) \vee B^+(x), A^-(x) \wedge B^-(x)).$$

Definition 2.3 (See [3, 11])**.** Let X be a nonempty set and let $A,\ B \in BPF(X)$. We say that A is equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

Result 2.4 ([9], Proposition 3.5)**.** *Let $A,\ B,\ C \in BPF(X)$. Then*
(1) (Idempotent laws): $A \cup A = A$, $A \cap A = A$,
(2) (Commutative laws): $A \cup B = B \cup A$, $A \cap B = B \cap A$,
(3) (Associative laws): $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws): $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$.
(6) (DeMorgan's laws): $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) $A \cap B \subset A$ *and* $A \cap B \subset B$,
(9) $A \subset A \cup B$ *and* $B \subset A \cup B$,
(10) *if* $A \subset B$ *and* $B \subset C$, *then* $A \subset C$,
(11) *if* $A \subset B$, *then* $A \cap C \subset B \cap C$ *and* $A \cup C \subset B \cup C$.

Result 2.5 ([9], Corollary 3.6)**.** *Let $A,\ B \in BPF(X)$. Then the followings are equivalent:*
(1) $A \subset B$,
(2) $A \cap B = A$,
(3) $A \cup B = B$.

Definition 2.6 ([9])**.** Let X be a nonempty set and let $(A_j)_{j \in J} \subset BPF(X)$.
(i) The intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$, is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$(\bigcap_{j \in J} A_j)(x) = (\bigwedge_{j \in J} A_j^+(x), \bigvee_{j \in J} A_j^-(x)).$$

(ii) The union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} A_j$, is a bipolar fuzzy set in X defined by: for each $x \in X$,

$$(\bigcup_{j \in J} A_j)(x) = (\bigvee_{j \in J} A_j^+(x), \bigwedge_{j \in J} A_j^-(x)).$$

Result 2.7 ([9], Proposition 3.8)**.** *Let $A \in BPF(X)$ and let $(A_j)_{j \in J} \subset BPF(X)$. Then*
(1) (Generalized distributive laws): $A \cup (\bigcap_{j \in J} A_j) = \bigcap_{j \in J} (A \cup A_j)$,

$$A \cap (\bigcup_{j \in J} A_j) = \bigcup_{j \in J} (A \cap A_j),$$

(2) (Generalized DeMorgan's laws): $(\bigcup_{j \in J} A_j)^c = \bigcap_{j \in J} A_j^c$, $(\bigcap_{j \in J} A_j)^c = \bigcup_{j \in J} A_j^c$.

From Results 2.4 and 2.7, it is obvious that $(BPF(X), \cup, \cap, ^c, \mathbf{0}_{bp}, \mathbf{1}_{bp})$ is a complete distributive lattice satisfying the DeMorgan's laws.

Definition 2.8. Let X and Y be a nonempty sets, let $A_X \in BPF(X)$ and $A_Y \in BPF(Y)$ and let $f : X \to Y$ be a mapping. Then

(i) The image of A_X under f, denoted by $f(A_X) = (f(A_X^+), f(A_X^-))$, is a bipolar fuzzy set in Y defined as follows: for each $y \in Y$,

$$[(f(A_X^+)](y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A_X^+(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

and

$$[(f(A_X^-)](y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} A_X^-(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

(ii) The preimage of A_Y under f, denoted by $f^{-1}(A_Y) = (f^{-1}(A_Y^+), f^{-1}(A_Y^-))$, is a bipolar fuzzy set in Y defined as follows: for each $x \in X$,

$$[f^{-1}(A_Y^+)](x) = A_Y^+ \circ f(x) \text{ and } [f^{-1}(A_Y^-)](x) = A_Y^- \circ f(x).$$

Result 2.9 ([9], Proposition 3.10)**.** *Let $f : X \to Y$ be a mapping, and let $A,\ A_1,\ A_2 \in BPF(X)$, $(A_{j \in J}) \subset BPF(X)$, $B,\ B_1,\ B_2 \in BPF(Y)$ and $(B_{j \in J}) \subset BPF(Y)$. Then*

(1) *if $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$,*

(2) *$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$, $f(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} f(A_j)$,*

(3) *$f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$, $f(\bigcap_{j \in J} A_j) \subset \bigcap_{j \in J} f(A_j)$,*

$(3)'$ *if f is injective, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$, $f(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} f(A_j)$,*

(4) *$f(A) = \mathbf{0}_{bp}$ if and only if $A = \mathbf{0}_{bp}$,*

(5) *if $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$,*

(6) *$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$, $f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$,*

(7) *$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$, $f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$,*

(8) *$f^{-1}(B) = \mathbf{0}_{bp}$ if and only if $B \cap f(\mathbf{1}_{bp}) = \mathbf{0}_{bp}$,*

(9) *$A \subset f^{-1} \circ f(A)$; in particular, $A = f^{-1} \circ f(A)$, if f is injective,*

(10) *$f \circ f^{-1}(B) \subset B$; in particular, $f \circ f^{-1}(B) = B$, if f is sujective.*

(11) *$f^{-1}(B^c) = f^{-1}(B)^c$,*

(12) *if $f : X \to Y,\ g : Y \to Z$ are mappings, then $(g \circ f)(A) = g(f(A))$, for each $A \in BPF(X)$,*

(13) *if $f : X \to Y,\ g : Y \to Z$ are mappings, then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for each $C \in BPF(Z)$.*

3. Bases and subbases in a bipolar fuzzy topological space

In this section, we introduce the concepts of bipolar fuzzy point, bipolar fuzzy topology, base and subbase, and study some of their properties. Also we give some examples.

Definition 3.1. Let $x \in X$, $(\alpha, \beta) \in (0,1] \times [-1,0)$ and let $A \in BPF(X)$. Then

(i) $x_{(\alpha,\beta)}$ is called a bipolar fuzzy point in X with the value (α, β) and the support x, if for each $y \in X$,

$$[x_{(\alpha,\beta)}](y) = \begin{cases} (\alpha, \beta) & \text{if } y = x \\ (0,0) & \text{otherwise} \end{cases},$$

(ii) $x_{(\alpha,\beta)}$ is said to belong to A, denoted by $x_{(\alpha,\beta)} \in A$, if

$$A^+(x) \geq \alpha \text{ and } A^-(x) \leq \beta.$$

We will denote the set of all bipolar fuzzy points in X as $BPF_P(X)$.

It is clear that $A = \bigcup\{x_{(\alpha,\beta)} \in BPF_P(X) : x_{(\alpha,\beta)} \in A\}$, for each $A \in BPF(X)$.

Theorem 3.2 (The characterization of Definition 2.2 (i)). *Let $A,\ B \in BPF(X)$. Then $A \subset B$ if and only if $x_{(\alpha,\beta)} \in B$, $\forall x_{(\alpha,\beta)} \in A$.*

Proof. Suppose $A \subset B$ and let $x_{(\alpha,\beta)} \in A$. Then $A^+(x) \geq \alpha$ and $A^-(x) \leq \beta$. Since $A \subset B$, $A^+(x) \leq B^+(x)$ and $A^-(x) \geq B^-(x)$. Thus $B^+(x) \geq \alpha$ and $B^-(x) \leq \beta$. So $x_{(\alpha,\beta)} \in B$.

Suppose the necessary condition holds and assume that $A \not\subset B$. Then there is an $x \in X$ such that either $A^+(x) > B^+(x)$ or $A^-(x) < B^-(x)$, say $A^+(x) = \alpha \in [0,1]$ and $A^-(x) = \beta \in [-1,0]$. Thus $x_{(\alpha,\beta)} \in A$. By the hypothesis, $x_{(\alpha,\beta)} \in B$, i.e., $B^+(x) \geq \alpha$ and $B^-(x) \leq \beta$. So $\alpha > \alpha$ or $\beta < \beta$. This is a contradiction. Hence $A \subset B$. □

Theorem 3.3 (The characterization of Definition 2.2 (iii) and (iv)). *Let $A,\ B \in BPF(X)$ and let $x_{(\alpha,\beta)} \in BPF_P(X)$. Then*

(1) $x_{(\alpha,\beta)} \in A$ and $x_{(\alpha,\beta)} \in B$ if and only if $x_{(\alpha,\beta)} \in A \cap B$,

(2) If $x_{(\alpha,\beta)} \in A$ or $x_{(\alpha,\beta)} \in B$, then $x_{(\alpha,\beta)} \in A \cup B$.

Proof. (1) Suppose $x_{(\alpha,\beta)} \in A \cap B$. Then $(A \cap B)^+(x) = A^+(x) \wedge B^+(x) \geq \alpha$ and $(A \cap B)^-(x) = A^-(x) \vee B^-(x) \leq \beta$. Thus $A^+(x) \geq \alpha$, $A^-(x) \leq \beta$ and $B^+(x) \geq \alpha$, $B^-(x) \leq \beta$. So $x_{(\alpha,\beta)} \in A$ and $x_{(\alpha,\beta)} \in B$.

The converse is proved similarly.

(2) Suppose $x_{(\alpha,\beta)} \in A$ or $x_{(\alpha,\beta)} \in B$. Then $[A^+(x) \geq \alpha$ and $A^-(x) \leq \beta]$ or $[B^+(x) \geq \alpha$ and $B^-(x) \leq \beta]$. This implies that

$$[A^+(x) \geq \alpha \text{ or } B^+(x) \geq \alpha] \text{ and } [A^-(x) \leq \beta \text{ or } B^-(x) \leq \beta].$$

So $A^+(x) \vee B^+(x) \geq \alpha$ and $A^-(x) \wedge B^-(x) \leq \beta$. Thus $x_{(\alpha,\beta)} \in A \cup B$. □

Theorem 3.4 (The characterization of Definition 2.6). *Let $(A_j)_{j \in J} \subset BPF(X)$ and let $x_{(\alpha,\beta)} \in BPF_P(X)$. Then*

(1) $x_{(\alpha,\beta)} \in \bigcap_{j \in J} A_j$ if and only if $x_{(\alpha,\beta)} \in A_j$ for each $j \in J$,

(2) if there is a $j \in J$ such that $x_{(\alpha,\beta)} \in A_j$, then $x_{(\alpha,\beta)} \in \bigcup_{j \in J} A_j$.

Proof. (1) Suppose $x_{(\alpha,\beta)} \in \bigcap_{j \in J} A_j$. Then $(\bigcap_{j \in J} A_j)^+(x) = \bigwedge_{j \in J} A_j^+(x) \geq \alpha$ and $(\bigcap_{j \in J} A_j)^-(x) = \bigvee_{j \in J} A_j^-(x) \leq \beta$. Thus $A_j^+(x) \geq \alpha$, $A_j^-(x) \leq \beta$ for each $j \in J$. So $x_{(\alpha,\beta)} \in A_j$ for each $j \in J$.

The converse is proved similarly.

(2) It is immediate from $A_j \subset \bigcup_{j \in J} A_j$. □

The converses of Theorem 3.3 (2) and 3.4 (2) do not hold.

Example 3.5. Let X be the closed unit interval $[0,1]$ and let $A, B \in BPF(X)$ such that for each $x \in X$,

$$A^+(x) = x^2, \qquad A^-(x) = -x^2$$
$$B^+(x) = \frac{1}{x+1}, \qquad B^-(x) = -0.5x^3.$$

Then

$$A^+(0.5) = 0.25 < 0.3, \qquad A^-(0.5) = -0.25 \leq -0.2$$
$$B^+(0.5) = 2/3 \geq 0.3, \qquad B^-(0.5) = -0.0625 \geq -0.2$$

and

$$(A \cup B)^+(0.5) = A^+(0.5) \vee B^+(0.5) = 2/3 \geq 0.3$$
$$(A \cup B)^-(0.5) = A^-(0.5) \wedge B^-(0.5) = -0.25 \leq -0.2.$$

Therefore we have $0.5_{(0.3,-0.2)} \in A \cup B$, $0.5_{(0.3,-0.2)} \notin A$, and $0.5_{(0.3,-0.2)} \notin B$.

Definition 3.6 ([3])**.** Let X be a nonempty set and let $\tau \subset BPF(X)$. Then τ is called a bipolar fuzzy topology on X, if it satisfies the following axioms:
(BPFO1) $\mathbf{0}_{bp},\ \mathbf{1}_{bp} \in \tau$,
(BPFO2) $A \cap B \in \tau$ for any $A,\ B \in \tau$,
(BPFO3) $\bigcup_{j \in J} A_j \in \tau$ for any $(A_j)_{j \in J} \subset \tau$.
In this case, the pair (X, τ) is called a bipolar fuzzy topological space and each member of τ is called a bipolar fuzzy open set (in short, BPFOS) in X. $A \in BPF(X)$ is said to be closed in X, if $A^c \in \tau$.
We will denote the set of all bipolar fuzzy topologies on X as $BPFT(X)$.

It is obvious that for any bipolar fuzzy topological space (X, τ), the following families

$$\tau^+ = \{U^+ \in [0,1]^X : U \in \tau\},\ \tau^- = \{-U^- \in [0,1]^X : U \in \tau\}$$

are fuzzy topological spaces in Chang's sense (See [4]). Then (X, τ^+, τ^-) is a bifuzzy topological space.

Example 3.7. (1) Let X be a nonempty set and let $\tau^0 = \{\mathbf{0}_{bp}, \mathbf{1}_{bp}\}$. Then clearly, τ^0 is a bipolar fuzzy topology on X. In this case, τ^0 will be called the indiscrete bipolar fuzzy topology on X and the pair $((X, \tau^0)$ will be called the indiscrete bipolar fuzzy space.

(2) Let X be a nonempty set and let $\tau^1 = BPF(X)$. Then clearly, τ^1 is a bipolar fuzzy topology on X. In this case, τ^1 will be called the discrete bipolar fuzzy topology on X and the pair (X, τ^1) will be called the discrete bipolar fuzzy space.

We will denote the set of all bipolar fuzzy topologies on X as $BPFT(X)$.

(3) Let X be a nonempty set. Then for any $A \in BPF(X)$, the set

$$S(A) = \{x \in X : A^+(x) > 0,\ A^-(x) < 0\}$$

is called the support of A. If $S(A)$ is finite [resp. countable], then A is said to be finite [resp. countable].

Let $\tau = \{A \in BPF(X) : A = \mathbf{0}_{bp}$ or A^c is finite$\}$. Then clearly, τ is a bipolar fuzzy topology on X. In this case, τ is called the bipolar fuzzy cofinite topology on X and will be denoted by $Cof_{bp}(X)$.

(4) Let $\tau = \{A \in BPF(X) : A = \mathbf{0}_{bp}$ or A^c is countable$\}$. Then clearly, τ is a bipolar fuzzy topology on X. In this case, τ is called the bipolar fuzzy cocountable topology on X and will be denoted by $Coc_{bp}(X)$.

(5) Let (X, τ) be a fuzzy topological space in Chang's sense, where $\tau = \{0, 1\} \cup \{A_j : j \in J\}$. Then we can construct two bipolar fuzzy topologies on X as follows:

$$\tau_1 = \{\mathbf{0}_{bp}, \mathbf{1}_{bp}\} \cup \{(A_j, \mathbf{0}^-_{bp}) : A_j \in \tau\} \text{ and } \tau_2 = \{\mathbf{0}_{bp}, \mathbf{1}_{bp}\} \cup \{(\mathbf{0}^+_{bp}, -A_j) : A_j \in \tau\}.$$

(6) Let (X, τ) be a bipolar fuzzy topological space. Then we can also construct two bipolar fuzzy topologies on X as follows:

$$\tau_{0,1} = \{(A^+, -A^+) : A \in \tau\} \text{ and } \tau_{0,2} = \{(-A^-, A^-) : A \in \tau\}.$$

Let $\tau_1, \ \tau_2 \in BPFT(X)$. Then we say that τ_1 is weaker (or coarser) than τ_2, denoted by $\tau_1 \preceq \tau_2$, if $\tau_1 \subset \tau_2$. In this case, we say that τ_2 is stronger (or finer) then τ_1, denoted by $\tau_2 \succeq \tau_1$. In particular, if $\tau_1 \neq \tau_2$, then we say that τ_1 is strictly coarser than τ_2 or τ_2 is strictly finer than τ_1.

We can easily see that $\tau^0 \preceq \tau \preceq \tau^1$, for each $\tau \in BPFT(X)$.

Proposition 3.8. *Let $(\tau_j)_{j \in J} \subset BPFT(X)$. Then $\bigcap_{j \in J} \tau_j \in BPFT(X)$ and thus $(BPFT(X), \preceq)$ is a meet complete lattice with the smallest τ^0 and the largest τ^1, where 'meet' denotes the 'intersection'.*

Proof. It is straightforward. □

Remark 3.9. The following does not hold, in general: $\tau_1 \cup \tau_2 \in BPFT(X)$, for any $\tau_1, \ \tau_2 \in BPFT(X)$.

Example 3.10. Let $X = \{a, b, c\}$ and let $\tau_1 = \{\mathbf{0}_{bp}, \mathbf{1}_{bp}, A\}$, $\tau_1 = \{\mathbf{0}_{bp}, \mathbf{1}_{bp}, B\}$, where $A(a) = (0.7, -0.6), \ A(b) = (0.4, -0.7), \ A(c) = (0.6, -0.3)$,
$B(a) = (0.8, -0.5), \ B(b) = (0.3, -0.8), \ B(c) = (0.7, -0.5)$.
Then $\tau_1 \cup \tau_1 = \{\mathbf{0}_{bp}, \mathbf{1}_{bp}, A, B\}$ and $(A \cup B)(a) = (0.8, -0.6)$. Thus $A \cup B \notin \tau_1 \cup \tau_1$. So $\tau_1 \cup \tau_1 \notin BPFT(X)$.

Proposition 3.11. *Let (X, τ) be a bipolar fuzzy topological space and let $\mathcal{F}$ be the collection of all bipolar fuzzy closed sets in X. Then*

(1) $\mathbf{0}_{bp}, \ \mathbf{1}_{bp} \in \mathcal{F}$,
(2) $F_1 \cup F_2 \in \mathcal{F}$, *for any* $F_1, \ F_2 \in \mathcal{F}$,
(3) $\bigcap_{j \in J} F_j$, *for any* $(F_j)_{j \in J} \subset \mathcal{F}$.

Proof. It is straightforward. □

Proposition 3.12. *Let (X, τ) be a bipolar fuzzy topological space and let $A \in BPF(X)$. If for each $x_{(\alpha,\beta)} \in A$, there is $U_{(\alpha,\beta)} \in \tau$ such that $x_{(\alpha,\beta)} \in U_{(\alpha,\beta)} \subset A$, then $A \in \tau$.*

Proof. Suppose the sufficient condition holds. Then clearly, $A = \bigcup_{x_{(\alpha,\beta)} \in A} U_{(\alpha,\beta)}$. Thus by the condition (BPFO3), $A \in \tau$. □

Definition 3.13. Let (X, τ) be a bipolar fuzzy topological space.

(i) $\mathcal{B} \subset \tau$ is called a base for τ, if for each $U \in \tau$, $U = \mathbf{0}_{bp}$ or there is $\mathcal{B}' \subset \mathcal{B}$ such that $U = \bigcup \mathcal{B}'$.

(ii) $\mathcal{S} \subset \tau$ is called a subbase for τ, if the family of all finite intersections of members of $\mathcal{S}$ forms a base for τ.

Theorem 3.14. *Suppose that $\mathcal{B} \subset BPF(X)$ satisfies the following:*

(i) $\mathbf{1}_{bp} = \bigcup \mathcal{B}$,

(ii) *if $B_1,\ B_2 \in \mathcal{B}$ and $x_{(\alpha,\beta)} \in B_1 \cap B_2$, then there is $B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subset B_1 \cap B_2$.*

Then $\mathcal{B}$ is a base for a bipolar fuzzy topology τ on X.

Proof. Let

$$\tau = \{\mathbf{0}_{bp}\} \cup \{U \in BPF(X) : U \text{ is a union of members of } \mathcal{B}\}.$$

(BPFO1) From the definition of τ and the hypothesis, it is clear that $\mathbf{0}_{bp},\ \mathbf{1}_{bp} \in \tau$.

(BPFO3) Let $(U_j)_{j\in J} \subset \tau$. Then by the definition of τ, for each $j \in J$, there is $\mathcal{B}_j \subset \mathcal{B}$ such that $U_j = \bigcup \mathcal{B}_j$. Thus $\bigcup_{j\in J} U_j = \bigcup_{j\in J} \bigcup \mathcal{B}_j$. So $\bigcup_{j\in J} U_j$ is a union of member of $\mathcal{B}$. Hence $\bigcup_{j\in J} U_j \in \tau$.

(BPFO2) Let $U_1,\ U_1 \in \tau$. Then there are $\mathcal{B}_1 \subset \mathcal{B}$ and $\mathcal{B}_2 \subset \mathcal{B}$ such that $U_1 = \bigcup \mathcal{B}_1$ and $U_2 = \bigcup \mathcal{B}_2$. Thus $U_1 \cap U_2 = \bigcup(\mathcal{B}_1 \cap \mathcal{B}_2)$. Let $x_{(\beta,\alpha)} \in U_1 \cap U_2$. Then there are $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that $x_{(\alpha,\beta)} \in B_1 \cap B_2 \subset U_1 \cap U_2$. By the condition (i), there is $B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subset B_1 \cap B_2$. Thus there is $B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subset U_1 \cap U_2$. So $U_1 \cap U_2$ is a union of members of $\mathcal{B}$. Hence $U_1 \cap U_1 \in \tau$. □

But the converse of Theorem 3.14 does not hold as we see the following example.

Example 3.15. For a singleton $X = \{x\}$, let

$$\tau = \{\mathbf{0}_{bp}\} \cup \{A_{i,j},\ B_{i,j},\ C_{i,j},\ D_{i,j} : i, j \in \mathbb{N}\}$$

where for each positive integers i, j,

$$A_{i,j}(x) = \left(\frac{1}{2i}, -\frac{1}{2j}\right), \qquad B_{i,j}(x) = \left(\frac{1}{2i}, -\frac{1}{2j-1}\right),$$
$$C_{i,j}(x) = \left(\frac{1}{2i-1}, -\frac{1}{2j}\right), \qquad D_{i,j}(x) = \left(\frac{1}{2i-1}, -\frac{1}{2j-1}\right).$$

One can show easily that $D_{1,1} = \mathbf{1}_{bp}$ and $\tau \in BPFTS(X)$. Denote

$$\mathcal{B} = \{B_{i,j}, C_{i,j} : i, j \in \mathbb{N}\}.$$

Since $A_{i,j} = B_{i,k} \cup C_{k,j}$ if $2k-1 > \max\{2i, 2j\}$ and since $D_{i,j} = B_{k,j} \cup C_{i,k}$ if $2k > \max\{2i-1, 2j-1\}$, we obtain that $\mathcal{B}$ is a base for τ.

But $\mathcal{B}$ does not satisfy Theorem 3.14 (ii). For, $(B_{1,2} \cap C_{2,1})(x) = \left(\frac{1}{3}, -\frac{1}{3}\right)$ and $x_{(0.3,-0.3)} \in B_{1,2} \cap C_{2,1}$. But for all $B_{i,j} \in \mathcal{B}$ such that $B_{i,j} \subset B_{1,2} \cap C_{2,1}$, $x_{(0.3,-0.3)} \notin B_{i,j}$ because $\frac{1}{3} > 0.3 > \frac{1}{4} > \frac{1}{5} \cdots$. For all $C_{i,j} \in \mathcal{B}$, we have the same argument.

Proposition 3.16. *Let (X, τ) be a bipolar fuzzy topological space. Suppose $\mathcal{B} \subset \tau$ satisfies the following condition: for each $x_{(\alpha,\beta)} \in BPF_P(X)$ and each $U \in \tau$ such that $x_{(\alpha,\beta)} \in U$, there is $B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subset U$. Then $\mathcal{B}$ is a base for τ.*

Proof. Let $x_{(\alpha,\beta)} \in BPF_P(X)$. Since $\mathbf{1}_{bp} \in \tau$, by the hypothesis, there is $B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subset \mathbf{1}_{bp}$. Then $\mathbf{1}_{bp} = \bigcup \mathcal{B}$. Thus the condition (i) of Theorem 3.14 holds.

Suppose $B_1,\ B_2 \in \mathcal{B}$ and $x_{(\alpha,\beta)} \in B_1 \cap B_2$. Since $\mathcal{B} \subset \tau$, $B_1,\ B_2 \in \tau$. Then $B_1 \cap B_2 \in \tau$. Thus by the hypothesis, there is $B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subset B_1 \cap B_2$. So the condition (ii) of Theorem 3.14 holds. Hence by Theorem 3.14, $\mathcal{B}$ is a base for τ. □

Proposition 3.17. *Let $\mathcal{S} \subset BPF(X)$ such that $\mathbf{1}_{bp} = \bigcup \mathcal{S}$. Then there is a unique $\tau \in PFT(X)$ such that $\mathcal{S}$ is a subbase for τ.*

In this case, τ is called the bipolar fuzzy topology on X induced by $\mathcal{S}$.

Proof. The proof is similar to Theorem 3.14. □

Example 3.18. Let $X = \{a, b, c, d, e\}$ and let $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$, where

$S_1(a) = (1, -1),\ S_1(b) = S_1(c) = S_1(d) = S_1(e) = (0, 0),$
$S_2(a) = S_2(b) = S_2(c) = (1, -1),\ S_2(d) = S_2(e) = (0, 0),$
$S_3(b) = S_3(c) = S_3(d) = (1, -1),\ S_3(a) = S_3(e) = (0, 0),$
$S_4(c) = S_4(e) = (1, -1),\ S_4(a) = S_4(b) = S_4(d) = (0, 0).$

Then clearly, $\bigcup \mathcal{S} = S_1 \cup S_2 \cup S_3 \cup S_4 = \mathbf{1}_{bp}$. Moreover, $\mathcal{B} = \{\mathbf{0}_{bp}, S_1, S_2, S_3, S_4, B_1, B_2\}$, where $B_1(b) = B_1(c) = (1, -1),\ B_1(a) = B_1(d) = B_1(e) = (0, 0),$
$B_2(c) = (1, -1),\ B_2(a) = B_2(b) = B_2(d) = B_2(e) = (0, 0).$

Thus $\tau = \{\mathbf{0}_{bp}, \mathbf{1}_{bp}, S_1, S_2, S_3, S_4, B_1, B_2, U_1, U_2, U_3, U_4, U_5, U_6\}$, where

$U_1(a) = U_1(c) = (1, -1),\ U_1(b) = U_1(d) = U_1(e) = (0, 0),$
$U_2(a) = U_2(c) = U_2(e) = (1, -1),\ U_2(b) = U_2(d) = (0, 0),$
$U_3(b) = U_3(c) = U_3(e) = (1, -1),\ U_3(a) = U_3(d) = (0, 0),$
$U_4(a) = U_4(b) = U_4(c) = U_4(d) = (1, -1),\ U_4(e) = (0, 0),$
$U_5(a) = U_5(b) = U_5(c) = U_5(e) = (1, -1),\ U_5(d) = (0, 0),$
$U_6(b) = U_5(c) = U_5(d) = U_5(e) = (1, -1),\ U_5(a) = (0, 0).$

Definition 3.19. Let (X, τ) be a bipolar fuzzy topological space and let $a_{(\alpha,\beta)} \in BPF_P(X)$. Then $\mathcal{B}_{a_{(alpha,\beta)}} \subset \tau$ is called a local base at $a_{(\alpha,\beta)}$, if the following conditions hold:

(i) if $B \in \mathcal{B}_{a_{(\alpha,\beta)}}$, then $a_{(\alpha,\beta)} \in B$,
(ii) if $U \in \tau$ and $a_{(\alpha,\beta)} \in U$, then there is $B \in \mathcal{B}_{a_{(\alpha,\beta)}}$ such that $B \subset U$.

Definition 3.20. Let $BPF_{b^-}(X) = \{A \in BPF(X) : A^+ \leq -A^-\}$ (See [9], Proposition 3.15). Then $\tau_b \subset BPF_{b^-}(X)$ is called a bipolar fuzzy bounded topology on X, if it satisfies the following axioms:

(i) $\mathbf{0}_{bp},\ \mathbf{1}_{bp} \in \tau_b$,
(ii) if $A,\ B \in \tau_b$, then $A \cap B \in \tau_b$,
(iii) if $(A_j)_{j \in J} A_j \subset \tau_b$, then $\bigcup_{j \in J} A_j \in \tau_b$.

In this case, the pair (X, τ_b) is called a bipolar fuzzy bounded topological space.

Remark 3.21. Let (X, τ_b) be a bipolar fuzzy bounded topological space. Let

$$\tau_{b,1} = \{(U^+, 1 + U^-) : U \in \tau_b\}$$

and

$$\tau_{b,2} = \{[U^+, -U^-] : U \in \tau_b\}.$$

Then $\tau_{b,1}$ is an intuitionistic fuzzy topology on X in Coker's sense (See [5]) and $\tau_{b,2} = \{[U^+, -U^-] : U \in \tau_b\}$ is an interval-valued fuzzy topology on X in the sense of Mondal and Samanta (See [15]).

4. Continuities in a bipolar fuzzy topological space

We define a bipolar fuzzy neighborhood and continuity and study bipolar fuzzy analogues of many results concerning to classical neighborhood and continuity.

Definition 4.1. Let (X, τ) be a bipolar fuzzy topological space and let $A \in BPF(X)$. Then A is called a neighborhood (in short, *nbd*) of $x_{(\alpha,\beta)} \in BPF_P(X)$, if there is $U \in \tau$ such that $x_{(\alpha,\beta)} \in U \subset A$.

We will denote the set of all bipolar fuzzy *nbds* at $x_{(\alpha,\beta)}$ as $\mathcal{N}_{bp}(x_{(\alpha,\beta)})$.

Lemma 4.2. *Let (X, τ) be a bipolar fuzzy topological space and let $A \in BPF(X)$. Then $A \in \tau$ if and only if for each $x_{(\alpha,\beta)} \in A$, there is $U \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$ such that $U \subset A$.*

Proof. Suppose $A \in \tau$ and let $x_{(\alpha,\beta)} \in A$. Then clearly, $A \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$.

Conversely, suppose the necessary condition holds and let $x_{(\alpha,\beta)} \in A$. Then there is $U_{x_{(\alpha,\beta)}} \in \tau$ such that $x_{(\alpha,\beta)} \in U_{x_{(\alpha,\beta)}} \subset A$. Thus $A \in \tau$, by the Proposition 3.11. □

Theorem 4.3. *Let (X, τ) be a bipolar fuzzy topological space. Then*

(BPFN1) $\mathbf{1}_{bp} \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$, $\forall x_{(\alpha,\beta)} \in BPF_P(X)$ *and if* $A \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$, *then*

$$x_{(\alpha,\beta)} \in A,$$

(BPFN2) *if* $A,\ B \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$, *then* $A \cap B \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$,

(BPFN3) *if* $A \subset B$ *and* $A \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$, *then* $B \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$,

(BPFN4) *if* $A \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$, *then there is* $B \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$ *such that*

$$B \subset A \text{ and } B \in \mathcal{N}_{bp}(y_{(\delta,\eta)}),\ \forall y_{(\delta,\eta)} \in B.$$

Conversely, let X be a set and for each $x_{(\alpha,\beta)} \in BPF_P(X)$, let $\mathcal{U}_{(\alpha,\beta)}$ be a nonempty family of bipolar fuzzy sets in X satisfying (BPFN1)–(BPFN4). Then there is a unique $\tau \in BPFT(X)$ such that $\mathcal{U}(x_{(\alpha,\beta)})$ is precisely the system of all τ-neighborhoods of $x_{(\alpha,\beta)}$, for each $x_{(\alpha,\beta)} \in BPF_P(X)$.

Proof. The proofs of (BPFN1) and (BPFN3) are straightforward.

(BPFN2) Suppose $A,\ B \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$. Then there are $U,\ V \in \tau$ such that $x_{(\alpha,\beta)} \in U \subset A$ and $x_{(\alpha,\beta)} \in V \subset B$. Thus $x_{(\alpha,\beta)} \in U \cap V \subset A \cap B$ and $U \cap V \in \tau$. So $A \cap B \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$.

(BPFN4) Suppose $A \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})$. Then there is $U \in \tau$ such that $x_{(\beta,\alpha)} \in U \subset A$. Thus by Lemma 4.2, $U \in \mathcal{N}_{bp}(y_{(\delta,\eta)})$, for each $y_{(\delta,\eta)} \in U$. So the result holds.

Conversely, let X be a set and for each $x_{(\alpha,\beta)} \in BPF_P(X)$, let $\mathcal{U}_{(\alpha,\beta)}$ be a nonempty family of bipolar fuzzy sets in X satisfying (BPFN1)–(BPFN4). Let

$$\tau = \{\mathbf{0}_{bp}\} \cup \{U \in BPF(X) : U \in \mathcal{U}(x_{(\alpha,\beta)}),\ \forall\ x_{(\alpha,\beta)} \in U\}.$$

Then clearly, $\mathbf{0}_{bp} \in \tau$. By (BPFN1), $\mathbf{1}_{bp} \in \mathcal{U}(x_{(\alpha,\beta)})$, for each $x_{(\alpha,\beta)} \in X$. Thus $\mathbf{1}_{bp} \in \tau$. So the condition (BPFO1) holds.

Suppose $U,\ V \in \tau$ such that $U,\ V \in \mathcal{U}(x_{(\alpha,\beta)})$, for each $x_{(\alpha,\beta)} \in U \cap V$. Then by (BPFN2), $U \cap V \in \mathcal{U}(x_{(\alpha,\beta)})$. Thus $U \cap V \in \tau$. So the condition (BPFO2) holds.

Now, suppose $(U_j)_{j\in J} \subset \tau$. Then $U_j \in \mathcal{U}(x_{(\alpha,\beta)})$, for each $j \in J$ and each $x_{(\alpha,\beta)} \in U_j$. Thus $U_j \subset \bigcup_{j\in J} U_j$, for each $j \in J$. By (BPFN3), $\bigcup_{j\in J} U_j \in \mathcal{U}(x_{(\alpha,\beta)})$. So $\bigcup_{j\in J} U_j \in \tau$. Hence the condition (BPFO3) holds.

Finally, by (BPFN4), it is obvious that $\mathcal{U}(x_{(\alpha,\beta)})$ is precisely the system of all τ-neighborhoods of $x_{(\alpha,\beta)}$, for each $x_{(\alpha,\beta)} \in BPF_P(X)$. This completes the proof. □

Definition 4.4. Let (X, τ) be a bipolar fuzzy topological space and let $\mathcal{S} \subset \mathcal{N}(x_{(\alpha,\beta)})$. Then $\mathcal{S}$ is called a fundamental system of bipolar fuzzy neighborhoods of $x_{(\alpha,\beta)} \in BPF^P(X)$, if for each $U \in \mathcal{N}(x_{(\alpha,\beta)})$, there is $V \in \mathcal{S}$ such that $V \subset U$.

Theorem 4.5. *Let (X, τ) be a bipolar fuzzy topological space and let $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a bipolar fuzzy base for τ if and only if for each $x_{(\alpha,\beta)} \in BPF_P(X)$, the set $\{B \in \mathcal{B} : x_{(\alpha,\beta)} \in B\}$ is a fundamental system of bipolar fuzzy neighborhoods of $x_{(\alpha,\beta)}$.*

Proof. Suppose $\mathcal{B} \subset \tau$ is a bipolar fuzzy base for τ and for each $x_{(\alpha,\beta)} \in BPF_P(X)$, let us consider the set $\mathcal{S} = \{B \in \mathcal{B} : x_{(\alpha,\beta)} \in B\}$. Let $U \in \mathcal{N}(x_{(\alpha,\beta)})$. Then there is $O \in \tau$ such that $x_{(\alpha,\beta)} \in O \subset U$. Since $\mathcal{B} \subset \tau$ is a bipolar fuzzy base for τ, there is $\mathcal{B}' \subset \mathcal{B}$ such that $O = \bigcup \mathcal{B}'$. Since $x_{(\alpha,\beta)} \in O$, $x_{(\alpha,\beta)} \in \bigcup \mathcal{B}'$. Thus there is $B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subset O$. So by the definition of $\mathcal{S}$, $B \in \mathcal{S}$ and $B \subset O \subset U$. Hence the necessary condition holds.

Conversely, suppose the necessary condition holds. Let $U \in \tau$ and let $x_{(\alpha,\beta)} \in U$. Then by the hypothesis, there is $B_{x_{(\alpha,\beta)}} \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B_{x_{(\alpha,\beta)}} \subset U$. Thus $U = \bigcup_{x_{(\alpha,\beta)}\in U} B_{x_{(\alpha,\beta)}}$. So $\mathcal{B}$ is a bipolar fuzzy base for τ. □

Definition 4.6 ([3])**.** Let (X, τ) be a bipolar fuzzy topological space and let $A \in BPF(X)$. Then the bipolar fuzzy closure of A, denoted by $bcl_\tau(A)$ (in short, $bcl(A)$), is a bipolar fuzzy set in X defined by

$$bcl_\tau(A) = \bigcap\{F \in BPF(X) : F^c \in \tau,\ A \subset F\}.$$

Result 4.7 ([3], Theorem 1.8)**.** *Let (X, τ) be a bipolar fuzzy topological space and let $A,\ B \in BPF(X)$. Then*

(1) *$bcl(A)$ is the smallest bipolar fuzzy closed set containing A,*
(2) *A is bipolar fuzzy closed in X if and only if $A = bcl(A)$,*
(3) *$bcl(\mathbf{0}_{bp}) = \mathbf{0}_{bp}$,*
(4) *$bcl(bcl((A)) = bcl((A)$,*
(5) *$bcl(A \cup B) = bcl(A) \cup bcl(B)$,*
(6) *$bcl(A \cap B) \subset bcl(A) \cap bcl(B)$.*

A mapping $bcl : BPF(X) \to BPF(X)$ satisfying (1), (3), (4) and (5) of Result 4.7 will be called a bipolar fuzzy closure operator on a set X. Then we have the similar one to the property induced by an ordinary closure operator on X.

Proposition 4.8. *Let bcl^* be a bipolar fuzzy closure operator on a set X. Let $\mathcal{F} = \{F \in BPF(X) : bcl^*(F) = F\}$ and let $\tau = \{U \in BPF(X) : U^c \in \mathcal{F}\}$. Then $\tau \in BPFT(X)$. Moreover, if bcl is the bipolar fuzzy closure operator defined by τ, then $bcl^*(A) = bcl(A)$, for each $A \in BPF(X)$.*

Proof. The proof is similar to ordinary closure operator. □

Definition 4.9 ([3]). Let (X, τ) be a bipolar fuzzy topological space and let $A \in BPF(X)$. Then the bipolar fuzzy interior of A, denoted by $bint_\tau(A)$ (in short, $bint(A)$), is a bipolar fuzzy set in X defined by

$$bint_\tau(A) = \bigcup\{U \in BPF(X) : U \in \tau,\ U \subset A\}.$$

Result 4.10 ([3], Theorem 1.10). *Let (X, τ) be a bipolar fuzzy topological space and let $A,\ B \in BPF(X)$. Then*
(1) *$bint(A)$ is the largest bipolar fuzzy open set contained in A,*
(2) *A is bipolar fuzzy open in X if and only if $A = bint(A)$,*
(3) *if $A \subset B$, then $bint(A) \subset bint(B)$,*
(4) *$bint(bint((A)) = bint((A)$,*
(5) *$bint(A \cap B) = bint(A) \cap bint(B)$,*
(6) *$bint(A \cup B) \supset bint(A) \cup bint(B)$,*
(7) *$bint(A^c) = (bcl(A))^c$,*
(8) *$bcl(A^c) = (bint(A))^c$.*

The following is the immediate result of Definition 4.9.

Proposition 4.11. *Let (X, τ) be a bipolar fuzzy topological space. Then $bint(\mathbf{1}_{bp}) = \mathbf{1}_{bp}$.*

Remark 4.12. A mapping $bint : BPF(X) \to BPF(X)$ satisfying (1), (4) and (5) of Result 4.10 and Proposition 4.11 will be called a bipolar fuzzy interior operator on a set X. Then as one might expect, a proposition analogous to Proposition 4.8 holds for the bipolar fuzzy interior. Thus a bipolar fuzzy interior completely determines a bipolar fuzzy topology (a bipolar fuzzy set is open iff it equals its own interior) and in that topology, the operator is the bipolar fuzzy interior.

Definition 4.13. Let (X, τ_1), (Y, τ_2) be two bipolar fuzzy topological spaces. Then a mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be continuous, if $f^{-1}(V) \in \tau_1$, for each $V \in \tau_1$

The following is the immediate result of the above definition.

Proposition 4.14. *The identity mapping $1_X : (X, \tau) \to (X, \tau)$ is continuous.*

Proposition 4.15. *If mappings $f : (X, \tau_1) \to (Y, \tau_2)$ and $g : (Y, \tau_2) \to (Z, \tau_3)$ are continuous, then $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is continuous.*

Proof. Let $W \in \tau_3$ and let $x \in X$. Then by Result 2.9 (13),

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)).$$

Since $g : (Y, \tau_2) \to (Z, \tau_3)$, $g^{-1}(W) \in \tau_2$. Since $f : (X, \tau_1) \to (Y, \tau_2)$, $f^{-1}(g^{-1}(W)) \in \tau_1$. So $(g \circ f)^{-1}(W) \in \tau_1$. Hence $g \circ f$ is continuous. □

Remark 4.16. Let **BPFTop** be the collection of all bipolar fuzzy topological spaces and continuous mappings. Then from Propositions 4.14 and 4.15, we can easily see that **BPFTop** forms a concrete category.

From Definition 2.8, it is obvious that for a mapping $f : X \to Y$ and each $x_{(\alpha,\beta)} \in BPF_P(X)$, $f(x_{(\alpha,\beta)}) = f(x)_{(\alpha,\beta)} \in BPF_P(Y)$.

Definition 4.17. Let (X, τ_1), (Y, τ_2) be two bipolar fuzzy topological spaces. Then a mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous at $a_{(\alpha,\beta)} \in BPF_P(X)$, if for each $V \in \mathcal{N}_{bp}(f(a_{(\alpha,\beta)})) = \mathcal{N}_{bp}(f(a)_{(\alpha,\beta)})$, $f^{-1}(V) \in \mathcal{N}_{bp}(a_{(\alpha,\beta)})$.

Lemma 4.18. *A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous if and only if f is continuous at each $a_{(\alpha,\beta)} \in BPF_P(X)$.*

Proof. It is simple to verify. □

Theorem 4.19. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping. Then the followings are equivalent:*
(1) *f is continuous,*
(2) *$f^{-1}(F)$ is closed in X, for each bipolar fuzzy closed set F in Y,*
(3) *$f^{-1}(S) \in \tau_1$, for each member S of the subbase $\mathcal{S}$ for τ_2,*
(4) *f is continuous at each $a_{(\alpha,\beta)} \in BPF_P(X)$,*
(5) *for each $a_{(\alpha,\beta)} \in BPF_P(X)$ and each $V \in \mathcal{N}_{bp}(f(a)_{(\alpha,\beta)})$, there is $U \in \mathcal{N}_{bp}(a_{(\alpha,\beta)})$ such that $f(U) \subset V$,*
(6) *$f(bcl(A)) \subset bcl(f(A))$, for each $A \in BPF(X)$,*
(7) *$bcl(f^{-1}(B)) \subset f^{-1}(bcl(B))$, for each $B \in BPF(Y)$.*

Proof. (1)⇔ (2): Suppose f is continuous and let F be any bipolar fuzzy closed set in Y. Then clearly, $F^c \in \tau_2$. By Result 2.9 (11), $f^{-1}(F^c) = f^{-1}(F)^c$. Since f is continuous, $f^{-1}(F)^c \in \tau_1$. Thus $f^{-1}(F)$ is closed in X.

The prof of the converse is similar.

(1)⇒ (3): It is obvious.

(3)⇒ (4): Let $a_{(\alpha,\beta)} \in BPF_P(X)$ and let $V \in \mathcal{N}_{bp}(f(a)_{(\alpha,\beta)})$. Then there is a S of the subbase $\mathcal{S}$ for τ_2 such that $f(a)_{(\alpha,\beta)} \in S \subset V$. Thus by (3), $f^{-1}(S) \in \tau_1$. Since $f(a)_{(\alpha,\beta)} \in S \subset V$, $a_{(\alpha,\beta)} \in f^{-1}(S) \subset f^{-1}(V)$. So $f^{-1}(V) \in \mathcal{N}_{bp}(a_{(\alpha,\beta)})$.

(4)⇒ (5): It is obvious.

(5)⇒ (1): It is obvious.

(2)⇒ (6): Let $A \in BPF(X)$. Then clearly, $bcl(f(A))$ is closed in Y. By (2), $f^{-1}(bcl(f(A)))$ is closed in X. Since $f(A) \subset bcl(f(A))$, $A \subset f^{-1}(bcl(f(A)))$. Thus

$$bcl(A) \subset bcl[f^{-1}(bcl(f(A)))] = f^{-1}(bcl(f(A))).$$

So $f(bcl(A)) \subset bcl(f(A))$.

(6)⇒ (7): For any $B \in BPF(Y)$, let $A = f^{-1}(B)$. Then by (6), $f(bcl(A)) \subset bcl(f(A))$. Thus $f(bcl(f^{-1}(B))) \subset bcl(f(f^{-1}(B)))$, i.e., $f(bcl(f^{-1}(B))) \subset bcl(B)$. So $bcl(f^{-1}(B)) \subset f^{-1}(bcl(B))$.

(7)⇒ (2): Let F be any bipolar fuzzy closed set in Y. Then clearly, $F = bcl(F)$. Thus by (7), $bcl(f^{-1}(F)) \subset f^{-1}(bcl(F))$, i.e., $bcl(f^{-1}(F)) \subset f^{-1}(F)$. So $f^{-1}(F) = bcl(f^{-1}(F))$. Hence $f^{-1}(F)$ is closed in X. □

The following is the immediate result of Result 4.10 and Theorem 4.19.

Corollary 4.20. *A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is continuous if and only if $f^{-1}(bint(B)) \subset bint(f^{-1}(B))$.*

From Definition 4.13, we have the following lemma.

Lemma 4.21. *Let* $\tau_1,\ \tau_2 \in BPFT(X)$*. Then* $\tau_1 \preceq \tau_2$ *if and only if the identity mapping* $1_X : (X, \tau_2) \to (X, \tau_1)$ *is continuous.*

The following is the immediate result of Definition 4.13, Theorem 4.19 and Lemma 4.21.

Theorem 4.22. *Let* $\tau_1,\ \tau_2 \in BPFT(X)$*. Then the followings are equivalent:*
(1) τ_1 *is coarser than* τ_2*,*
(2) *for each* $x_{(\alpha,\beta)} \in BPF_P(X)$*,* $\mathcal{N}_{bp,\tau_1}(x_{(\alpha,\beta)}) \subset \mathcal{N}_{bp,\tau_2}(x_{(\alpha,\beta)})$*,*
(3) *for each* $A \in BPF(X)$*,* $bcl_{\tau_2}(A) \subset bcl_{\tau_1}(A)$*,*
(4) *if* F *is closed in* (X, τ_1)*, then* F *is closed in* (X, τ_2)*,*
(5) *if* F *is open in* (X, τ_1)*, then* F *is open in* (X, τ_2)*.*

Remark 4.23. (1) The *finer* the bipolar fuzzy topology, the *more* bipolar fuzzy open sets, bipolar fuzzy closed sets and bipolar fuzzy neighborhoods; the *finer* the bipolar fuzzy topology, the *smaller* (resp. the *larger*) the bipolar fuzzy closure (resp. the bipolar fuzzy interior) of a bipolar fuzzy set.
(2) If $f : X \to Y$ is a continuous mapping, then it remains continuous if the bipolar fuzzy topology of X is replaced by a *finer* bipolar fuzzy topology and the bipolar fuzzy topology of Y is replaced by a *coarser*. In other words, the *finer* the bipolar fuzzy topology of X and the *coarser* the bipolar fuzzy topology of Y, the *more* continuous mappings there are of X to Y.

Definition 4.24. Let $(X, \tau_1),\ (Y, \tau_2)$ be two bipolar fuzzy topological spaces. Then a mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be:
(i) open, if $f(U) \in \tau_2$, for each $U \in \tau_1$,
(ii) closed, if $f(F)$ is closed in Y, for each bipolar fuzzy closed set F in Y.

Proposition 4.25. *Let* $f : (X, \tau_1) \to (Y, \tau_2)$ *and* $g : (Y, \tau_2) \to (Z, \tau_3)$ *be mappings. If* $f,\ g$ *are open [resp. closed], then so is* $g \circ f$*.*

Proof. Suppose $f,\ g$ are open and let $U \in \tau_1$. Then by Result 2.9 (12),

$$(g \circ f)(U) = g(f(U)).$$

Since f is open, $f(U) \in \tau_2$. Since g is open, $(g(f(U)) \in \tau_3$. Thus $g \circ f$ is open.
The prof of the second part is similar. □

Theorem 4.26. *Let* $f : (X, \tau_1) \to (Y, \tau_2)$ *be a mapping. Then* f *is open if and only if* $f(bint(A)) \subset bint(f(A))$*, for each* $A \in BPF(X)$*.*

Proof. Suppose f is open and let $A \in BPF(X)$. Since $bint(A) \in \tau_1$, $f(bint(A)) \in \tau_2$ and $f(bint(A)) \subset f(A)$. Moreover, $bint(f(A))$ is the largest bipolar fuzzy open set contained in $f(A)$. Thus $f(bint(A)) \subset bint(f(A))$.

Conversely, suppose the necessary condition holds and let $U \in \tau_1$. Then clearly, $U = bint(U)$. Thus by the hypothesis, $f(U) = f(bint(U)) \subset bint(f(U))$. Since $bint(f(U)) \subset f(U)$. So $f(U) = bint(f(U))$. Hence f is open. □

Proposition 4.27. *Let* $f : (X, \tau_1) \to (Y, \tau_2)$ *be a injective mapping. If* f *is continuous, then* $bint(f(A)) \subset f(bint(A))$*, for each* $A \in BPF(X)$*.*

Proof. Suppose f is continuous and let $A \in BPF(X)$. Then clearly, $bint(f(A)) \in \tau_2$. Since f is continuous, $f^{-1}(bint(f(A))) \in \tau_1$. Since $bint(f(A)) \subset f(A)$ and f is injective, by Result 2.9 (5) and (9),

$$f^{-1}(bint(f(A))) \subset f^{-1}(f(A)) = A.$$

Since $bint(A)$ is the largest bipolar fuzzy open set contained in A,

$$f^{-1}(bint(f(A))) \subset bint(A).$$

So $bint(f(A)) \subset f(bint(A))$. □

The following is the immediate result of Theorem 4.26 and Proposition 4.27.

Corollary 4.28. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a continuous, open and injective mapping. Then $f(bint(A)) = bint(f(A))$, for each $A \in BPF(X)$.*

Theorem 4.29. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping. Then f is closed if and only if $bcl(f(A)) \subset f(bcl(A))$, for each $A \in BPF(X)$.*

Proof. Suppose f is closed and let $A \in BPF(X)$. Then clearly, $A \subset bcl(A)$ and $bcl(A)$ is closed in Y. Thus $f(A) \subset f(bcl(A))$. Since f is closed, $f(bcl(A))$ is closed in Y. So $bcl(f(A)) \subset f(bcl(A))$.

Conversely, suppose the necessary condition holds and let F be any bipolar fuzzy closed set in X. Then clearly, $F = bcl(F)$. Thus by the hypothesis,

$$bcl(f(F)) \subset f(bcl(F)) = f(F) \subset bcl(f(F)).$$

So $bcl(f(F)) = f(F)$, i.e., $f(F)$ is a bipolar fuzzy closed set in Y. Hence f is closed. □

The following is the immediate result of Theorems 4.19 and 4.29.

Corollary 4.30. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping. Then f is continuous and closed if and only if $bcl(f(A)) = f(bcl(A))$, for each $A \in BPF(X)$.*

Definition 4.31. A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is called a homeomorphism, if it is bijective, continuous and open.

A few useful facts about continuity are collected in the following.

Proposition 4.32. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping.*

(1) *If (X, τ_1) is a discrete bipolar fuzzy space, i.e., $\tau_1 = \tau^1$, then f is continuous.*

(2) *If (Y, τ_2) is an indiscrete bipolar fuzzy space, i.e., $\tau_2 = \tau^0$, then f is continuous.*

(3) *If both (X, τ_1) and (Y, τ_2) are discrete bipolar fuzzy spaces, then f is continuous and open.*

(4) *For discrete bipolar fuzzy spaces (X, τ_1) and (Y, τ_2), f is a homeomorphism if and only if f is bijective.*

5. Bipolar fuzzy subspaces and bipolar fuzzy quotient spaces

In this section, we define the concept of a bipolar fuzzy subspace and a bipolar fuzzy quotient space, and investigate some properties of each concept.

Definition 5.1. Let (X, τ) be a bipolar fuzzy topological space, let $A \in BPF(X)$ be fixed and let $\delta \subset BPF(X)$. Then δ is called a bipolar fuzzy topology on A, if it satisfies the following axioms:
(i) if $B \in \delta$, then $B \subset A$,
(ii) $\mathbf{0}_{bp},\ A \in \delta$,
(iii) if $B,\ C \in \delta$, then $B \cap C \in \delta$,
(iv) if $(B_j)_{j \in J} \subset \delta$, then $\bigcup_{j \in J} B_j \in \delta$.

It is clear that the set $\tau_A = \{U \cap A : U \in \tau\}$ is a bipolar fuzzy topology on A. In this case, τ_A will be called the bipolar fuzzy subspace topology or the bipolar fuzzy relative topology induced by A and a pair (A, τ_A) will be called a bipolar fuzzy subspace.

Example 5.2. (1) Let X be a discrete bipolar fuzzy space and let $A \in BPF(X)$. Then τ^1_A is a discrete bipolar fuzzy topology on A, i.e., $\tau^1_A = \{B \cap A : B \in BPF(X)\}$.
(2) Let X be an indiscrete bipolar fuzzy space and let $A \in BPF(X)$. Then τ^0_A is an indiscrete bipolar fuzzy topology on A, i.e., $\tau^0_A = \{\mathbf{0}_{bp}, A\}$.

Proposition 5.3. *Let (X, τ) be a bipolar fuzzy topological space and let $A,\ B \in BPF(X)$ such $A \subset B$. Then $\tau_A = (\tau_B)_A$.*

Proof. Let $C \in \tau_A$. Then there is $U \in \tau$ such that $C = U \cap A$. Since $A \subset B$, $A = A \cap B$. Thus $C = U \cap (A \cap B) = (U \cap B) \cap A$ and $U \cap B \in \tau_B$. So $C \in (\tau_B)_A$. Hence $\tau_A \subset (\tau_B)_A$.
Now let $D \in (\tau_B)_A$. Then there is $V \in \tau_B$ such that $D = V \cap A$. Since $V \in \tau_B$, there is $U \in \tau$ such that $V = U \cap B$. Thus $D = (U \cap B) \cap A = U \cap (B \cap A) = U \cap A$. So $D \in \tau_A$. Hence $(\tau_B)_A \subset \tau_A$. Therefore $\tau_A = (\tau_B)_A$. □

Proposition 5.4. *Let (X, τ) be a bipolar fuzzy topological space and let $A \in \tau$. If $U \in \tau_A$, then $U \in \tau$.*

Proof. Suppose $U \in \tau_A$. Then there is $V \in \tau$ such that $U = V \cap A$. Since $A \in \tau$, $V \cap A \in \tau$. Thus $U \in \tau$. □

Proposition 5.5. *Let (X, τ) be a bipolar fuzzy topological space, let $A \in BPF(X)$ and let $\mathcal{B}$ be a base for τ. Then $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$ is a base for τ_A.*

Proof. Let $U \in \tau_A$ and let $x_{(\alpha,\beta)} \in U$. Then there is $V \in \tau$ such that $U = V \cap A$, $x_{(\alpha,\beta)} \in V$ and $x_{(\alpha,\beta)} \in A$. Since $V \in \tau$, there is $B \in \mathcal{B}$ such that $x_{(\alpha,\beta)} \in B \subset V$. Thus $x_{(\alpha,\beta)} \in B \cap A \subset V \cap A$. So by Proposition 3.14, $\mathcal{B}_A$ is a base for τ_A. □

Remark 5.6. Let (X, τ) be a bipolar fuzzy topological space, let $A \in BPF(X)$ and let $x_{(\alpha,\beta)} \in A$. Then $\mathcal{N}_{bp,A}(x_{(\alpha,\beta)}) = \{U \cap A : U \in \mathcal{N}_{bp}(x_{(\alpha,\beta)})\}$ is the set of all neighborhoods of $x_{(\alpha,\beta)}$ in (A, τ_A).

For any $A \in BPF(X)$, the set $\{x \in X : A^+(x) > 0,\ A^-(x) < 0\}$ will be called the support of A and denoted by $S(A)$.

Definition 5.7. Let X, Y be nonempty sets, let $A \in BPF(X)$, $B \in BPF(Y)$ and let $f : S(A) \to S(B)$ be a mapping. Then f is called a mapping from A to B, denoted by $f : A \to B$, if for each $x \in S(A)$,

$$B^+ \circ f(x) \geq A^+(x) \text{ and } B^- \circ f(x) \leq A^-(x).$$

Definition 5.8. Let X, Y be nonempty sets, let $A \in BPF(X)$, $B \in BPF(Y)$ and let $f : A \to B$. Let $P_{bp}(A)$ be the set of all bipolar fuzzy subsets of A and let $\lambda \in P_{bp}(A)$, $\mu \in P_{bp}(B)$.

(i) The image of λ under f, denoted by $f(\lambda) = (f(\lambda^+), f(\lambda^-))$, is a bipolar fuzzy subset of B defined as: for each $y \in S(B)$,

$$[(f(\lambda^+)](y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \lambda^+(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

and

$$[(f(\lambda^-)](y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \lambda^-(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

(ii) The preimage of μ under f, denoted by $f^{-1}(\mu) = (f^{-1}(\mu^+), f^{-1}(\mu^-))$, is a bipolar fuzzy set in X defined as: for each $x \in S(A)$,

$$[f^{-1}(\mu^+)](x) = A^+(x) \wedge (\mu^+ \circ f(x)) \text{ and } [f^{-1}(\mu^-)](x) = A^-(x) \vee (\mu^- \circ f(x)).$$

Proposition 5.9. *Let X, Y be nonempty sets, let $A \in BPF(X)$, $B \in BPF(Y)$ and let $f : A \to B$. Let λ, λ_1, $\lambda_2 \in P_{bp}(A)$, $(\lambda_j)_{j \in J} \subset P_{bp}(A)$ and μ, μ_1, $\mu_2 \in P_{bp}(B)$, $(\mu_k)_{k \in K} \subset P_{bp}(B)$. Then*
(1) $f(\lambda) \subset B$, $f^{-1}(\mu) \subset A$,
(2) $f(\bigcup_{j \in J} \lambda_j) = \bigcup_{j \in J} f(\lambda_j)$,
(3) *if* $\lambda_1 \subset \lambda_2$, *then* $f(\lambda_1) \subset f(\lambda_2)$,
(4) $f^{-1}(\bigcup_{k \in K} \mu_k) = \bigcup_{k \in K} f^{-‘}(\mu_k)$, $f^{-1}(\bigcap_{k \in K} \mu_k) = \bigcap_{k \in K} f^{-1}(\mu_k)$,
(5) $f \circ f^{-1}(\mu) \subset \mu$,
(6) $\lambda \subset f^{-1} \circ f(\lambda)$.

Proof. We will prove only (2) and (4).
(2) Let $y \in S(B)$ such that $f^{-1}(y) \neq \phi$. Then

$$\begin{aligned} [f(\textstyle\bigcup_{j \in J} \lambda_j)]^+(y) &= f((\textstyle\bigcup_{j \in J} \lambda_j)^+)(y) \\ &= \textstyle\bigvee_{x \in f^{-1}(y)} (\bigcup_{j \in J} \lambda_j)^+(x) \\ &= \textstyle\bigvee_{x \in f^{-1}(y)} \bigvee_{j \in J} \lambda_j^+(x) \\ &= \textstyle\bigvee_{j \in J} \bigvee_{x \in f^{-1}(y)} \lambda_j^+(x) \\ &= \textstyle\bigvee_{j \in J} f(\lambda_j^+)(y) \\ &= [\textstyle\bigcup_{j \in J} f(\lambda_j^+)](y). \end{aligned}$$

Similarly, we have $[f(\bigcup_{j \in J} \lambda_j)]^-(y) = [\bigcup_{j \in J} f(\lambda_j^-)](y)$. Thus the result holds.
(3) Let $x \in S(A)$. Then

$$\begin{aligned} [f^{-1}(\textstyle\bigcup_{k \in K} \mu_k)]^+(x) &= A^+(x) \wedge (\textstyle\bigcup_{k \in K} \mu_k)^+ \circ f(x) \\ &= A^+(x) \wedge (\textstyle\bigvee_{k \in K} \mu_k^+ \circ f(x)) \\ &= \textstyle\bigvee_{k \in K} [A^+(x) \wedge (\mu_k^+ \circ f(x))] \\ &= \textstyle\bigvee_{k \in K} [f^{-1}(\mu_k^+)](x) \end{aligned}$$

$$= [\bigcup_{k\in K} f^{-1}(\mu_k^+)](x)$$

and

$$\begin{aligned}[f^{-1}(\bigcup_{k\in K}\mu_k)]^-(x) &= A^-(x)\vee(\bigcup_{k\in K}\mu_k)^-\circ f(x)\\ &= A^-(x)\vee(\bigwedge_{k\in K}\mu_k^-\circ f(x))\\ &= \bigwedge_{k\in K}[A^-(x)\vee(\mu_k^-\circ f(x))]\\ &= \bigwedge_{k\in K}[f^{-1}(\mu_k^-)](x)\\ &= [\bigcup_{k\in K} f^{-1}(\mu_k^-)](x).\end{aligned}$$

Thus $f^{-1}(\bigcup_{k\in K}\mu_k) = \bigcup_{k\in K} f^{-1}(\mu_k)$.

The proof of the second part is similar. □

Let $f : X \to Y$ be a mapping, let $A \in BPF(X)$, $B = f(A) \in BPF(Y)$ and let $f_A = f \mid S(A)$. Then clearly, for each $x \in S(A)$,

$$B^+\circ f(x) \geq A^+(x) \text{ and } B^-\circ f(x) \leq A^-(x).$$

Thus f_A is a mapping from A to B, i.e., $f_A : A \to B$. So we have following.

Proposition 5.10. *Let (X, τ), (Y, δ) be two bipolar fuzzy topological spaces, $f : (X, \tau) \to (Y, \delta)$ be a mapping and let $A \in BPF(X)$, $B = f(A) \in BPF(Y)$. If f is continuous, then $f_A : (A, \tau_A) \to (B, \delta_B)$ is continuous.*

Proposition 5.11. *Let (X, τ) be a bipolar fuzzy topological space and let $f : X \to Y$ be a mapping. Let $\delta = \{U \in BPF(Y) : f^{-1}(U) \in \tau\}$. Then*

(1) δ is a bipolar fuzzy topology on Y,

(2) $f : (X, \tau) \to (Y, \delta)$ is continuous,

(3) if η is a bipolar fuzzy topology on Y such that $f : (X, \tau) \to (Y, \eta)$ is continuous, then δ is finer than η, i.e., $\eta \preceq \delta$.

Proof. (1) (BPFO1) Let $\mathbf{0}_{bp,X}$ [resp. $\mathbf{1}_{bp,X}$] denotes the bipolar fuzzy empty [whole] set in X. Then clearly, $f^{-1}(\mathbf{0}_{bp,Y}) = \mathbf{0}_{bp,X}$ and $f^{-1}(\mathbf{1}_{bp,Y}) = \mathbf{1}_{bp,X}$. Thus

$$\mathbf{0}_{bp,Y} \in \delta \text{ and } \mathbf{1}_{bp,Y} \in \delta.$$

(BPFO2) Let $U,\ V \in \delta$. Then $f^{-1}(U),\ f^{-1}(V) \in \tau$. Thus

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \in \tau.$$

So $U \cap V \in \delta$.

(BPFO3) Let $(U_j)_{j\in J} \subset \delta$. Then clearly, $(f^{-1}(U_j))_{j\in J} \subset \tau$. Thus

$$f^{-1}(\bigcup_{j\in J} U_j) = \bigcup_{j\in J} f^{-1}(U_j) \in \tau.$$

So $\bigcup_{j\in J} U_j \in \delta$. Hence δ is a bipolar fuzzy topology on Y

(2), (3) The proofs are straightforward. □

Definition 5.12. Let (X, τ) be a bipolar fuzzy topological space, let Y be a set and let $f : X \to Y$ be a sujective mapping. Then

$$\delta = \{U \in BPF(Y) : f^{-1}(U) \in \tau\}$$

is a bipolar fuzzy topology on Y(See Proposition 5.11). In this case, δ is called the bipolar fuzzy quotient topology on Y induced by f, (Y, δ) is called a bipolar fuzzy quotient space of X and f is called a quotient mapping.

Proposition 5.13. *Let (X, τ), (Y, η) be two bipolar fuzzy topological spaces and let $f : (X, \tau) \to (Y, \eta)$ be sujective and continuous. Let δ be the bipolar fuzzy quotient topology on Y induced by f. If either f is open or closed, then $\delta = \eta$.*

Proof. Suppose f is open. Since $f : (X, \tau) \to (Y, \eta)$ is continuous, by Proposition 5.11, $\eta \preceq \delta$. Let $U \in \delta$. Then by the definition of δ, $f^{-1}(U) \in \tau$. Since $f : (X, \tau) \to (Y, \eta)$ is surjective and open, $U = f \circ f^{-1}(U) \in \eta$. Thus $\delta \preceq \eta$. So $\delta = \eta$. □

Remark 5.14. From Definition 5.12 and Proposition 5.13, we can easily see that if $f : (X, \tau) \to (Y, \eta)$ is open (or closed), surjective and continuous, then f is a quotient mapping.

Proposition 5.15. *If $f : (X, \tau) \to (Y, \eta)$ and $g : (Y, \eta) \to (Z, \delta)$ are quotient mappings, then $g \circ f$ is a quotient mapping*

Proof. It is straightforward. □

Theorem 5.16. *Let (X, τ), (Z, η) be bipolar fuzzy topological spaces, let Y be a set, $f : X \to Y$ be a sujective mapping and let δ be the bipolar fuzzy quotient topology on Y induced by f. Then $g : (Y, \delta) \to (Z, \eta)$ is continuous if and only if $g \circ f : (X, \tau) \to (Z, \eta)$ is continuous.*

Proof. Suppose $g : (Y, \delta) \to (Z, \eta)$ is continuous. Since δ be the bipolar fuzzy quotient topology on Y induced by f, $f : (X, \tau) \to (Y, \delta)$ is continuous. Then $g \circ f$ is continuous.

Conversely, suppose $g \circ f$ is continuous and let $V \in \eta$. Then $(g \circ f)^{-1}(V) \in \tau$ and $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$. Thus by the definition of the bipolar fuzzy quotient topology, $g^{-1}(V) \in \delta$. So g is continuous. □

6. Bipolar fuzzy initial topologies

Definition 6.1. Let X be a set, let $(Y_j, \tau_j)_{j \in J}$ be a family of bipolar fuzzy topological space and let let $(f_j : X \to (Y_j, \tau_j))_{j \in J}$ be a family of mappings. Let

$$\mathcal{S} = \{f_j^{-1}(U_j) \in BPF(X) : U^j \in \tau_j,\ j \in J\}.$$

Then the topology τ generated by the subbase $\mathcal{S}$ is called the bipolar fuzzy initial topology on X induced by $(f_j)_{j \in J}$ and $((Y_j, \tau_j))_{j \in J}$.

In fact, let $\mathcal{B}$ be the set of all finite intersections of members of $\mathcal{S}$. Then $\mathcal{B}$ is a base for τ and τ is the set of all unions of members of $\mathcal{B}$.

Proposition 6.2. *Let τ be the bipolar fuzzy initial topology on X induced by $(f_j)_{j \in J}$ and $(Y_j, \tau_j))_{j \in J}$. Then τ is the coarsest topology on X for which*

$$f_j : (X, \tau) \to (Y_j, \tau_j) \text{ is continuous, for each } j \in J.$$

Furthermore, for any bipolar fuzzy topological space (Z, δ), $g : (Z, \delta) \to (X, \tau)$ is continuous if and only if $f_j \circ g$ is continuous.

Proof. It is straightforward. □

Remark 6.3. From Proposition 6.2, we can see that the category **BPFTop** forms a topological category.

Example 6.4. (1) (*Inverse image of a bipolar fuzzy topology*) Let X be a set, let (Y, δ) be a bipolar fuzzy topological space and let $f : X \to (Y, \delta)$ be a mapping. Then by Proposition 6.2, there is the bipolar fuzzy initial topology τ on X for which $f : (X, \tau) \to (Y, \delta)$ is continuous.

In fact, the set $\mathcal{S} = \{f^{-1}(U) \in BPF(X) : U \in \delta\}$ is a subbase for τ.

In this case, τ is called the inverse image under f of δ.

In particular, $f : (X, \tau) \to (X', \delta)$ is continuous if and only if τ is finer than the inverse image τ_X under f of δ.

(2) (*bipolar fuzzy product topology*) Let $(X_j, \tau_j)_{j \in J}$ be a family of bipolar fuzzy topological space and let $X = \Pi_{j \in J} X_j$ be the product set of $(X_j)_{j \in J}$. Then by Proposition 6.2, there is the bipolar fuzzy initial topology τ on X for which the projections $pr_j : (X, \tau) \to (X_j, \tau_j)$ is continuous, for each $j \in J$.

In fact, the set $\mathcal{S} = \{pr_j^{-1}(U) \in BPF(X) : U \in \tau_j\}$ is a subbase for τ.

In this case, τ is called the bipolar fuzzy product topology on X and denoted by $\tau = \Pi_{j \in J} \tau_j$ and the pair (X, τ) is called a bipolar fuzzy product space.

Proposition 6.5. *Let $(\tau_j)_{j \in J} \subset BPFT(X)$. Then there is a bipolar fuzzy topology on X for which is coarsest among all the bipolar fuzzy topologies on X which are finer than each τ_j, i.e., $(\tau_j)_{j \in J}$ has a least upper bound in $(BPFT(X), \preceq)$.*

Proof. For each $j \in J$, let $(Y_j, \tau_j) = (X, \tau_j)$ and f_j be the identity mapping $X \to (Y_j, \tau_j)$. Then by by Proposition 6.2, there is a bipolar fuzzy initial topology τ on X for which all the mappings $f_j : (X, \tau) \to (Y_j, \tau_j)$ are continuous. Since each f_j is the identity mapping, by Lemma 4.21, $\tau_j \preceq \tau$. Thus τ is finer than each τ_j. So τ is a least upper bound in $(BPFT(X), \preceq)$. □

Remark 6.6. In Example 6.4 (2), the projection mappings are not necessarily open.

Example 6.7. Let $X_1 = \{a, b\} = X_2$ and let

$$\tau_1 = \{\mathbf{0}_{bp,X_1}, \mathbf{1}_{bp,X_1}, A\},\ \tau_2 = \{\mathbf{0}_{bp,X_2}, \mathbf{1}_{bp,X_2}, B\},$$

where

$$A(a) = (0.4, -0.7),\ A(b) = (0.6, -0.5)$$

and

$$B(a) = (0.8, -0.3),\ B(b) = (0.3, -0.6).$$

Let $\tau = \{\mathbf{0}_{bp,X_1 \times X_2}, \mathbf{1}_{bp,X_1 \times X_2}, U_1, U_2, U_1 \cap U_2, U_1 \cup U_2\}$, where $U_1 = pr_1^{-1}(A)$ and $U_2 = pr_2^{-1}(B)$. Then clearly, $\tau = \tau_1 \times \tau_2 \in BPFT(X_1 \times X_2)$ and $pr_2(U_1) \notin \tau_2$, $pr_1(U_2) \notin \tau_1$. Thus pr_1 and pr_2 are open.

Let the bipolar fuzzy set in X whose value is $(a, b) \in [0, 1] \times [-1, 0]$ for each $x \in X$, be denoted by $(a, b)_{bp}$. In particular, we will denote $(a, -a)_{bp}$ as a_{bp}.

Definition 6.8. Let (X, τ) be a bipolar fuzzy topological space. Then τ is said to be Lowen-type (See [13]), if $(a, b)_{bp} \in \tau$, for each $(a, b) \in [0, 1] \times [-1, 0]$.

Proposition 6.9. *Let $(X_j, \tau_j)_{j \in J}$ be a family of be Lowen-type bipolar fuzzy topological spaces and let $X = \Pi_{j \in J} X_j$, $\tau = \Pi_{j \in J} \tau_j$. Then each projection $pr_j : (X, \tau) \to (X_j, \tau_j)$ is open.*

Proof. Let $V = \bigcap_{i=1}^{n} pr_{j_i}^{-1}(U_{j_i})$, where $U_{j_i} \in \tau_{j_i}$, for $i = 1,\ 2, \cdots,\ n$.

Suppose $k(\neq j_i) \in J$, $i = 1, 2, \cdots, n$ and let $x_k \in X_k$. Then

$$\begin{aligned}[pr_k(V)]^+(x_k) &= \bigvee_{x \in pr_k^{-1}(x_k)} V^+(x) \\ &= \bigvee_{x \in pr_k^{-1}(x_k)} [\bigcap_{i=1}^n pr_{j_i}^{-1}(U_{j_i})]^+(x) \\ &= \bigvee_{x \in pr_k^{-1}(x_k)} [\bigwedge_{i=1}^n U_{j_i}^+ \circ pr_{j_i}(x)] \\ &= \bigwedge_{i=1}^n [\bigvee_{\xi \in X_{j_i}} U_{j_i}^+(\xi)] \\ &= a \in [0,1] \text{ (say)}.\end{aligned}$$

Similarly, we have $[pr_k(V)]^-(x_k) = \bigvee_{i=1}^n [\bigwedge_{\xi \in X_{j_i}} U_{j_i}^-(\xi)] = b \in [-1, 0]$ (say). Thus

$$pr_k(V) = (a, b)_{bp}.$$

Suppose $k = j_{i_0} \in J$, $i_0 = 1, 2, \cdots, n$ and let $x_k \in X_k$. Then

$$\begin{aligned}[pr_k(V)]^+(x_k) &= \bigvee_{x \in pr_k^{-1}(x_k)} V^+(x) \\ &= \bigvee_{x \in pr_k^{-1}(x_k)} [\bigwedge_{i=1}^n U_{j_i}^+ \circ pr_{j_i}(x)] \\ &= \bigwedge_{1 \leq i(\neq i_0) \leq n} [\bigvee_{\xi \in X_{j_i}} U_{j_i}^+(\xi))] \wedge U_{i_0}^+(x_k) \\ &= a^{'} \wedge U_{i_0}^+(x_k),\end{aligned}$$

where $a^{'} = \bigwedge_{1 \leq i(\neq i_0) \leq n} [\bigvee_{\xi \in X_{j_i}} U_{j_i}^+(\xi))]$.
Similarly, we have $[pr_k(V)]^+(x_k) = b^{'} \vee U_{i_0}^-(x_k)$,
where $b^{'} = \bigvee_{1 \leq i(\neq i_0) \leq n} [\bigwedge_{\xi \in X_{j_i}} U_{j_i}^-(\xi))]$.
Thus $pr_k(V) = (a^{'} \wedge U_{i_0}^+(x_k), b^{'} \vee U_{i_0}^-(x_k)_{bp}$. So either cases, $pr_k(V) \in \tau_k$. Hence pr_k is open. □

7. Compactness in a bipolar fuzzy topological space

Definition 7.1. Let (X, τ) be a bipolar fuzzy topological space and let $\mathcal{C} \subset BPF(X)$.
(i) $\mathcal{C}$ is called a cover of X, if $\mathbf{1}_{bp} \subset \bigcup \mathcal{C}$, i.e.,

$$(\bigcup \mathcal{C})^+(x) = 1 \text{ and } (\bigcup \mathcal{C})^-(x) = -1, \ \forall x \in X.$$

(ii) Let $\mathcal{C}$ be a cover of X. Then $\mathcal{D} \subset \mathcal{C}$ is a subcover of $\mathcal{C}$, if $\mathcal{D}$ is a cover of X.
(iii) $\mathcal{C}$ is called an open cover of X, if $\mathcal{C}$ is a cover of X and $\mathcal{C} \subset \tau$.

Definition 7.2. A bipolar fuzzy topological space (X, τ) is said to be compact, if each open cover of X has a finite subcover.

Proposition 7.3. *Let (X, τ) be a bipolar fuzzy compact space, (Y, δ) be a a bipolar fuzzy topological space and let $f : X, \tau) \to (Y, \delta)$ be surjective and continuous. Then (Y, δ) is compact.*

Proof. Let $\mathcal{C}$ be any open cover of Y. Then clearly, $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{C}\}$ is an open cover of X. Since (X, τ) is compact, there is a finite subcover

$$\mathcal{V} = \{f^{-1}(V_i) : V_i \in \mathcal{C}, \ 1 \leq i \leq n\}.$$

Thus for each $x \in X$,

$$(\bigcup \mathcal{V})^+(x) = \bigvee_{i=1}^n f^{-1}(V_i^+)(x) = \bigvee_{i=1}^n V_i^+ \circ f(x) \geq 1$$

and

$$(\bigcup \mathcal{U})^-(x) = \bigwedge_{i=1}^{n} f^{-1}(V_i^-)(x) = \bigwedge_{i=1}^{n} V_i^- \circ f(x) \leq -1.$$

Let $y \in Y$. Since f is surjctive, $f^{-1}(y) \neq \phi$, say $y = f(x)$. Then

$$\begin{aligned}(\bigcup_{i=1}^{n} V_i)^+(y) &= \bigvee_{i=1}^{n} V_i^+(y) \\ &= \bigvee_{i=1}^{n} V_i^+(f(x)) \\ &= \bigvee_{i=1}^{n} V_i^+ \circ f(x) \geq 1.\end{aligned}$$

Similarly, we have $(\bigcup_{i=1}^{n} V_i)^-(y) \leq 1$. Thus $\mathcal{A} = \{V_i \in \mathcal{C} : f^{-1}(V_i) \in \mathcal{V}\}$ is a finite subcover of $\mathcal{C}$. So (Y, δ) is compact. □

Definition 7.4 (See [8]). Let $\mathcal{C} \subset BPF(X)$. Then

(i) $\mathcal{C}$ is said to be inadequate, if it fails to cover X,

(ii) $\mathcal{C}$ is said to be finitely inadequate, if no finite subfamily of $\mathcal{C}$ covers X.

Definition 7.5 ([8]). A family $\mathcal{A}$ of ordinary sets is said to be of finite character, if each finite subset of a member of $\mathcal{A}$ is a member of $\mathcal{A}$.

Result 7.6 ([8], Turkey Lemma of Theorem 0.25). *There is a maximal member of nonempty family of finite character.*

Now we will prove the Alexander Theorem on compactness (See Theorem 5.6 in [8]) in bipolar fuzzy setting.

Proposition 7.7 (Alexander Subbase Theorem). *Let (X, τ) be a bipolar fuzzy topological space and let $\mathcal{S}$ be a subbase for τ. If every cover of X by members of $\mathcal{S}$ has a finite subcover, then (X, τ) is compact.*

Proof. By Definition 7.4, (X, τ) is compact iff each finitely inadequate family of bipolar fuzzy open sets in X is inadequate. Observe that the class of finitely inadequate families of bipolar fuzzy open sets in X is of finite character and thus each finitely inadequate family of bipolar fuzzy open sets in X is contained in a maximal finitely inadequate family $\mathcal{A}$ by Result 7.6. Then $\mathcal{A}$ has a special property which is established as follows: If $C \notin \mathcal{A}$ and $C \in \tau$, then by the maximality, there is finite subfamily $\{U_1, U_2, \cdots, U_n\}$ of $\mathcal{A}$ such that $\{U_1, U_2, \cdots, U_n, C\}$ is a cover of X, i.e., $\mathbf{1}_{bp} \subset C \cup U_1 \cup \cdots \cup U_n$. Thus no bipolar fuzzy open set containing C belongs to $\mathcal{A}$. Similarly, if $D \notin \mathcal{C}$ and $C \in \tau$, then there is finite subfamily $\{V_1, V_2, \cdots, V_m\}$ of $\mathcal{C}$ such that $\{V_1, V_2, \cdots, V_n, D\}$ is a cover of X. So $\{U_1, U_2, \cdots, U_n, V_1, V_2, \cdots, V_n, C \cap D\}$ is a cover of X. Hence $C \cap D \notin \mathcal{C}$. Therefore if no member of a finite family of bipolar fuzzy open sets belongs to $\mathcal{A}$, then no bipolar fuzzy open set containing the intersection belongs to $\mathcal{A}$, i.e., if a member of $\mathcal{A}$ contains a finite intersection $C_1 \cap C_2 \cdots \cap C_p$ of bipolar fuzzy open sets, then some $C_i \in \mathcal{A}$.

Suppose $\mathcal{S}$ is a subbase for τ such that every cover of X by members of $\mathcal{S}$ has a finite subcover (i.e., each finitely inadequate subfamily is inadequate). Let $\mathcal{B}$ be a finitely inadequate family of bipolar fuzzy open sets in X. Then there is a maximal finitely inadequate family $\mathcal{A}$ of bipolar fuzzy open sets such that $\mathcal{B} \subset \mathcal{A}$. If $\mathcal{A}$ is inadequate, then the family $\mathcal{S} \bigcap \mathcal{A}$ of all members $\mathcal{A}$ which belongs to $\mathcal{S}$ is finitely inadequate. Thus $\mathcal{S}\bigcap\mathcal{A}$ is inadequate, i.e., $\mathcal{S}\bigcap\mathcal{A}$ does not cover X. Now let us show that $\mathcal{A}$ is inadequate.

Assume that $\mathcal{A}$ is not inadequate. Then $\mathcal{A}$ covers X, i.e., $\mathbf{1}_{bp}\subset\bigcup\mathcal{A}$. Let $x\in X$. Then $1=\mathbf{1}^+_{bp}(x)\leq\bigvee_{A\in\mathcal{C}}A^+(x)$ and $-1=\mathbf{1}^-_{bp}(x)\geq\bigwedge_{A\in\mathcal{A}}A^-(x)$. Let $\varepsilon>0$. Then there is $A\in\mathcal{A}$ such that $A^+(x)>1-\frac{\varepsilon}{2}$. Since $A\in\tau$, there is $\{B_i\}_{i\in\triangle}$ such that $A=\bigcup_{i\in\triangle}B_i$, where $\{B_i\}_{i\in\triangle}$ is a set of basic open sets. Since each B_i is a basic open set, there is a finite index set J_i such that $B_i=\bigcap_{j\in J_i}S_j$, where $S_j\in\mathcal{S}$ and $B_i\subset A$. Since $A\in\mathcal{A}$, there is $j=j_{i\in J_i}$ such that $B_i=\bigcap_{j\in J_i}S_j\subset C_{j_i}\in\mathcal{A}\cap\mathcal{S}$. Since $A=\bigcup_{i\in\triangle}B_i$, there is B_{i_0} such that

$$B^+_{i_0}(x)>A^+(x)-\frac{\varepsilon}{2}.$$

Thus $C^+_{j_{i_0}}(x)\geq B^+_{i_0}(x)>1-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}$. Since $\varepsilon>0$ is arbitrary, $\bigvee_{C_j\in\mathcal{S}\cap\mathcal{C}}C^+_j(x)\geq 1$. Similarly $\bigwedge_{C_j\in\mathcal{S}\bigcap\mathcal{A}}C^-_j(x)\leq -1$. So $\mathbf{1}_{bp}\subset\bigcup(\mathcal{S}\cap\mathcal{A})$. Hence $\mathcal{S}\bigcap\mathcal{A}$ is not inadequate. This is a contradiction. This completes the proof. □

Proposition 7.8. *Let $(X_i,\tau_i)^n_{i=1}$ be a finite family of bipolar fuzzy compact space and let $X=\Pi^n_{i=1}X_i$, $\tau=\Pi^n_{i=1}\tau_i$. Then (X,τ) is compact.*

Proof. Let $\mathcal{C}$ be a finitely inadequate family of bipolar fuzzy subbasic open sets in X and for each $i\in\{1,2,\cdots,n\}$, let $\mathcal{B}_i=\{W\in\tau_i: pr_i^{-1}(W)\in\mathcal{C}\}$. Then clearly, each $\mathcal{B}_i$ is finitely inadequate in (X_i,τ_i). Since (X_i,τ_i) is compact, $\mathcal{B}_i$ is inadequate. Thus for each $i\in\{1,2,\cdots,n\}$, there is $x_i\in X_i$ such that

$$a_i=\bigvee_{W\in\mathcal{B}_i}W^+(x_i)<1 \text{ or } b_i=\bigwedge_{W\in\mathcal{B}_i}W^-(x_i)>-1.$$

Let $x=(x_1,x_2,\cdots,x_n)\in X$. Then

$$\begin{aligned}\textstyle\bigvee_{U\in\mathcal{C}}U^+(x)&=\textstyle\bigvee^n_{i=1}[\bigvee_{W\in\mathcal{B}_i}(pr_i^{-1}(W))^+(x)]\\&=\textstyle\bigvee^n_{i=1}[\bigvee_{W\in\mathcal{B}_i}W^+\circ pr_i(x)]\\&=\textstyle\bigvee^n_{i=1}[\bigvee_{W\in\mathcal{B}_i}W^+(x_i)]\\&=\textstyle\bigvee^n_{i=1}a_i<1\end{aligned}$$

or

$$\begin{aligned}\textstyle\bigwedge_{U\in\mathcal{C}}U^-(x)&=\textstyle\bigwedge^n_{i=1}[\bigwedge_{W\in\mathcal{B}_i}(pr_i^{-1}(W))^-(x)]\\&=\textstyle\bigwedge^n_{i=1}[\bigwedge_{W\in\mathcal{B}_i}W^-\circ pr_i(x)]\\&=\textstyle\bigwedge^n_{i=1}[\bigwedge_{W\in\mathcal{B}_i}W^-(x_i)]\\&=\textstyle\bigwedge^n_{i=1}b_i>-1.\end{aligned}$$

Thus $\mathcal{C}$ is inadequate. So by Proposition 7.7, (X,τ) is compact. □

Remark 7.9. The above Proposition does not hold when the number of spaces is infinite.

Example 7.10. For each positive integer i, let $X_i=[0,1]$ and let $\tau_i=\{\mathbf{0}_{bp},\mathbf{1}_{bp},\frac{i}{i+1}_{bp}\}$. Then clearly, each (X_i,τ_i) is compact. Let $X=\Pi^\infty_{i=1}X_i$, let $\tau=\Pi^\infty_{i=1}\tau_i$ and let $x=(x_1,x_2,\cdots)\in X$. Then

$$\begin{aligned}\textstyle[\bigcup^\infty_{i=1}pr_i^{-1}(\frac{i}{i+1}_{bp})]^+(x)&=\textstyle\bigvee^\infty_{i=1}[pr_i^{-1}(\frac{i}{i+1}_{bp})]^+(x)\\&=\textstyle\bigvee^\infty_{i=1}[\frac{i}{i+1}\circ pr_i(x)]\\&=\textstyle\bigvee^\infty_{i=1}[\frac{i}{i+1}(x_i)]\end{aligned}$$

$$= \bigvee_{i=1}^{\infty} \frac{i}{i+1} = 1$$

and

$$\begin{aligned}[\bigcup_{i=1}^{\infty} pr_i^{-1}(\tfrac{i}{i+1}{}_{bp})]^-(x) &= \bigwedge_{i=1}^{\infty}[pr_i^{-1}(\tfrac{i}{i+1}{}_{bp})]^-(x)\\ &= \bigwedge_{i=1}^{\infty}[\tfrac{-i}{i+1} \circ pr_i(x)]\\ &= \bigwedge_{i=1}^{\infty}[\tfrac{-i}{i+1}(x_i)]\\ &= \bigwedge_{i=1}^{\infty} \tfrac{-i}{i+1} = -1.\end{aligned}$$

Thus $(pr_i^{-1}(\frac{i}{i+1}{}_{bp}))_{i=1}^{\infty}$ is an open cover of X.

Now let J be a finite set of positive integers and let $x = (x_1, x_2, \cdots) \in X$. Then we can easily have $[\bigcup_{j\in J} pr_j(\frac{i}{i+1}{}_{bp})]^+(x) < 1$ and $[\bigcup_{j\in J} pr_j(\frac{i}{i+1}{}_{bp})]^-(x) > -1$. Thus $(pr_i^{-1}(\frac{i}{i+1}{}_{bp}))_{i=1}^{\infty}$ has no finite subcover. So (X, τ) is not compact.

Definition 7.11. Let $A \in BPF(X)$ and let $\mathcal{C} \subset BPF(X)$. Then $\mathcal{C}$ is said to be cover of A, if $A \subset \bigcup \mathcal{C}$.

Proposition 7.12. *Let (X, τ) be a bipolar fuzzy compact space and let $A \in BPF(X)$ be closed in (X, τ). Then A is compact in (X, τ).*

Proof. Let $\mathcal{C}$ be an open cover of A. Since A be closed in (X, τ), $A^c \in \tau$. Then $\mathcal{V} = \{A^c\} \bigcup \mathcal{C}$ is an open cover of X, i.e., $\mathbf{1}_{bp} \subset \bigcup \mathcal{V}$. Since (X, τ) is compact, there is a finite subcover $\mathcal{U} = \{A^c, V_1, V_2, \cdots, V_n\}$ of $\mathcal{V}$, i.e., $\mathbf{1}_{bp} \subset \bigcup \mathcal{U}$. Let $x \in X$. Then

$$(\bigcup \mathcal{U})^+(x) = (1 - A^+(x)) \bigvee_{i=1}^{n} V_i^+(x) \geq 1$$

and

$$(\bigcup \mathcal{U})^-(x) = (1 - A^-(x)) \bigwedge_{i=1}^{n} V_i^-(x) \leq -1.$$

Thus $\bigvee_{i=1}^{n} C_i^+(x) \geq 1 \geq A^+(x)$ and $\bigwedge_{i=1}^{n} V_i^-(x) \leq -1 \leq A^-(x)$. So $\mathcal{A} = \{V \in \mathcal{U} : V \in \mathcal{C}\}$ is a cover of A. Hence A is compact in (X, τ). □

Definition 7.13. Let (X, τ) be a bipolar fuzzy topological space, let $A \in BPF(X)$ and let $\mu \in P_{bp}(A)$. Then μ is said to be compact in (A, τ_A), if every open cover of μ by members of τ_A has a finite subcover.

Definition 7.14. Let (X, τ), (Y, δ) be two bipolar fuzzy topological spaces, let $A \in BPF(X)$, $B \in BPF(Y)$ and let $f : A \to B$ be a mapping. Then f is said to be continuous, if $f^{-1}(\mu) \in \tau_A$, for each $\mu \in \tau_B$.

Proposition 7.15. *Let (X, τ), (Y, δ) be two bipolar fuzzy topological spaces, let $A \in BPF(X)$, $B \in BPF(Y)$ and let $f : (A, \tau_A) \to (B, \tau_B)$ be continuous. If $\lambda \in P_{bp}(A)$ is compact in (A, τ_A), then $f(\lambda)$ is compact in (B, τ_B).*

Proof. Let $\mathcal{C}$ be any open cover of $f(\lambda)$ and let $x \in S(A)$, $y = f(x) \in S(B)$. Then clearly, $f(\lambda) \subset \bigcup \mathcal{C}$, i.e., $f(\lambda)^+(y) \leq (\bigcup \mathcal{C})^+(y)$ and $f(\lambda)^-(y) \geq (\bigcup \mathcal{C})^-(y)$, for each $y \in S(B)$.

Case (i): Suppose $f(\lambda)^+(y) = 0$. Then clearly, $f(\lambda)^+(y) \leq U^+(y)$, for each $U \in \mathcal{C}$. Thus $\bigvee_{x \in f^{-1}(y)} \lambda^+(x) \leq U^+ \circ f(x) = [f^{-1}(U)]^+(x)$. So $\lambda^+(x) \leq [f^{-1}(U)]^+(x)$.

Case (ii): Suppose $f(\lambda)^+(y) > 0$ and let $\varepsilon > 0$ such that $f(\lambda)^+(y) - \varepsilon > 0$. Then there is $\mu_\varepsilon \in \mathcal{C}$ such that $\mu_\varepsilon^+(y) > f(\lambda)^+(y) - \varepsilon$. On the other hand,

$$\begin{aligned}[f^{-1}(\mu_\varepsilon)]^+(x) &= A^+(x) \wedge \mu_\varepsilon^+ \circ f(x)\\ &= A^+(x) \wedge \mu_\varepsilon^+(y)\\ &\geq A^+(x) \wedge (f(\lambda)^+(y) - \varepsilon)\\ &= A^+(x) \wedge [(\bigvee_{z \in f^{-1}(y)} \lambda^+(z)) - \varepsilon]\\ &\geq A^+(x) \wedge (\lambda^+(x) - \varepsilon)\\ &= \lambda^+(x) - \varepsilon.\end{aligned}$$

Thus $[\bigcup_{\mu \in \mathcal{C}} f^{-1}(\mu)]^+(x) = \bigvee_{\mu \in \mathcal{C}} [f^{-1}(\mu)]^+(x) \geq \lambda^+(x) - \varepsilon$. Since ε is arbitrary,

$$[\bigcup_{\mu \in \mathcal{C}} f^{-1}(\mu)]^+(x) \geq \lambda^+(x).$$

Similarly, we have

$$[\bigcup_{\mu \in \mathcal{C}} f^{-1}(\mu)]^-(x) \leq \lambda^-(x).$$

So $f^{-1}(\mathcal{C}) = \{f^{-1}(\mu) : \mu \in \mathcal{C}\}$ is an open cover of λ. Since λ is compact in (A, τ_A), there is a finite subfamily $\{f^{-1}(\mu_1), f^{-1}(\mu_2), \cdots, f^{-1}(\mu_n)\}$ of $f^{-1}(\mathcal{C})$ which covers λ. Let $y \in S(B)$.

Case (1): Suppose $f^{-1}(y) = \phi$. Then $f(\lambda)^+(y) = 0 = f(\lambda)^-(y)$. Thus

$$[\bigcup_{i=1}^{n} ff^{-1}(\mu_i)]^+(y) \geq f(\lambda)^+(y).$$

Similarly, we have

$$[\bigcup_{i=1}^{n} ff^{-1}(\mu_i)]^-(y) \leq f(\lambda)^-(y).$$

Case (2): Suppose $f^{-1}(y) \neq \phi$. Then $f(\lambda) \subset f(\bigcup_{i=1}^n f^{-1}(\mu_i))$. Thus $f(\lambda) \subset \bigcup_{i=1}^n ff^{-1}(\mu_i)$. So $\bigcup_{i=1}^n \mu_i \supset \bigcup_{i=1}^n ff^{-1}(\mu_i) \supset f(\lambda)$. Hence $\{\mu_1, \mu_2, \cdots, \mu_n\}$ is a cover of $f(\lambda)$ which is a finite subfamily of $\mathcal{C}$. Therefore $f(\lambda)$ is compact in (B, τ_B). □

8. Conclusions

We dealt with some properties of bipolar fuzzy topology, neighborhood, continuity, base (subbase), subspace, quotient space and compactness. In particular, we proved that analogues to classical neighborhood system and Alexander Subbase Theorem hold in bipolar fuzzy topological spaces (See Theorem 4.3 and Proposition 7.7).

In the future, we try to study separation axioms and connectedness in bipolar fuzzy topological space. Also we will investigate the degree of openness of any ordinary set in the sense of bipolar fuzzy sets.

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