Some group's analogous results in multigroup setting

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Abstract. In furtherance of the exploration of the theory of multigroups drawn from multisets, this paper seeks to present some results in multigroup context which were hitherto established in group theory, by redefining some concepts in the light of multigroups. Also, the notion of commutator in multigroup setting is proposed, and it is proved that, if A and B are normal submultigroups of a multigroup C, then $[A, B] \subseteq A \cap B$.

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1. Introduction

The concept of multisets as noted in [17], was first suggested by N. G. de Bruijn (cf. [6]) in a private communication to D. E. Knuth, as an important generalisation of Cantorian set theory, by violating a basic property of Cantorian sets that an element can belong to a set only once. The notion of multisets is a boost to the concept of multigroups via multisets, which generalises group theory. In [18], the concept of multigroups in multisets framework was proposed and a number of results were obtained. The notion is parallel to other non-classical groups (e.g., fuzzy groups, intuitionistic fuzzy groups [5, 20], etc). A complete survey on the concept of multigroups was carried out in [14], and it was established that multigroup via multiset is a generalisation of group theory.

The concept of multigroups via multisets has been extensively researched upon since inception. A number of algebraic properties of order of an element in a multigroup were considered in [3] and some results on multigroups which cut across some homomorphic properties were explored in [4]. The notions of upper and lower cuts of multigroups were proposed and discussed in details with some results in [7], and the notions were extended to homomorphic sense and a number of results were explored [12]. The ideas of submultigroups of multigroups and abelian multigroups were explicated with some results in [13]. In continuation, the concepts of normal submultigroups and characteristic submultigroups of multigroups were proposed in [9, 16] with some results, and some homomorphic properties of multigroups were studied in [8]. Some group's analogous theoretic concepts were established in multigroup context like direct product of multigroups, comultisets, factor multigroups and multigroup actions on multiset with some number of results. See [9, 10, 11, 15] for details. The concept of multigroups has been extended to soft multigroups [1, 19] as a generalisation of soft groups [2].

The concept of multigroups which is an application of multisets to group theory has been elaborated in literature (cf. [4, 7, 8, 9, 10, 11, 12, 13, 15, 18]). Notwithstanding, some group's analogous results could be explored which are not hitherto studied in multigroup setting, hence the motivation of the work. In a nutshell, this present paper seeks to explicate some group theoretic results and concept (e.g., commutator) in the light of multigroup, which are fallowed in multigroup context. The paper is organised by presenting some definitions and existing results on multisets, multigroups, cuts of multigroups, comultisets, normal submultigroups, characteristic submultigroups, homomorphic properties and direct product of multigroups in Section 2. Section 3 discusses some further results of multigroup's concepts and proposes the notion of commutator in multigroup setting. Section 4 summarises and concludes the paper with future research direction.

2. Preliminaries

In this section, we present some existing definitions and results to be used in the sequel.

Definition 2.1 ([22])**.** Let X be a set. A multiset A over X is just a pair $\langle X, C_A\rangle$, where X is a set and $C_A : X \to \mathbb{N}$ is a function. Any ordinary set B is actually a multiset $\langle B, \chi_B\rangle$, where χ_B is its characteristic function.

The set X is called the ground or generic set of the class of all multisets (for short, msets) containing objects from X. We denote the set of all multisets over X by $MS(X)$.

A multiset $A = [a, a, b, b, c, c, c]$ over a set $X = \{a, b, c\}$ can be represented as $A = [a^2, b^2, c^3]$. Other forms of multiset representations can be found in literature.

Definition 2.2 ([22])**.** Let $A, B \in MS(X)$. Then A is called a submultiset of B written as $A \subseteq B$, if $C_A(x) \leq C_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 2.3 ([21])**.** Let $A, B \in MS(X)$. Then the intersection, union and sum of A and B, denoted by $A \cap B$, $A \cup B$ and $A + B$ respectively, are defined by the rules that for any object $x \in X$,

(i) $C_{A\cap B}(x) = C_A(x) \wedge C_B(x)$,
(ii) $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,
(iii) $C_{A+B}(x) = C_A(x) + C_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum, respectively.

Definition 2.4 ([21])**.** Let $A, B \in MS(X)$. Then A and B are comparable to each other if and only if $A \subseteq B$ or $B \subseteq A$, and $A = B$ if and only if

$$C_A(x) = C_B(x) \forall x \in X.$$

Definition 2.5. A multiset B of a set X is said to have sup-property, if for any subset $W \subset X$ $\exists$ $w_0 \in W$ such that

$$C_B(w_0) = \bigvee_{w\in W} \{C_B(w)\}.$$

Definition 2.6 ([13, 18]). Let X be a group. A multiset A over X is called a multigroupoid of X, if for all $x, y \in X$,

$$C_A(xy) \geq C_A(x) \wedge C_A(y),$$

where C_A denotes count function of A from X into a natural number $\mathbb{N}$.

Also, a multiset A over a group X is said to be a multigroup of X, if it satisfies the following two conditions:

(i) A is a multigroupoid of X,
(ii) $C_A(x^{-1}) = C_A(x), \forall x \in X$.

The set of all multigroups of X is denoted by $MG(X)$.

It can be easily verified that if A is a multigroup of X, then

$$C_A(e) = \bigvee_{x \in X} C_A(x) \ \forall x \in X,$$

that is, $C_A(e)$ is the tip of A, where e is the identity element of X. And a multigroup A is said to be regular, if $C_A(x) = C_A(y) \ \forall x, y \in X$.

Remark 2.7 ([18]). Let X be a group and G be a multiset over X. If

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y),$$

for all $x, y \in X$, then G is called a multigroup of X.

Remark 2.8 ([13]). Every multigroup is a multiset but the converse is not necessarily true.

Definition 2.9 ([13]). Let $A \in MG(X)$. A submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$, if B is a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Definition 2.10 ([11, 18]). Let X be a group. For any submultigroup A of a multigroup G of X, the submultiset yA of G for $y \in X$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \ \forall x \in X$$

is called the left comultiset of A. Similarly, the submultiset Ay of G for $y \in X$ defined by

$$C_{Ay}(x) = C_A(xy^{-1}) \ \forall x \in X$$

is called the right comultiset of A.

Proposition 2.11 ([18]). *Let $A \in MG(X)$. Then the sets A_* and A^* defined by*

$$A_* = \{x \in X \mid C_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid C_A(x) = C_A(e)\}$$

are subgroups of X.

Definition 2.12 ([7]). Let $A \in MG(X)$. Then the sets $A_{[n]}$ and $A_{(n)}$ defined by

$$A_{[n]} = \{x \in X \mid C_A(x) \geq n, n \in \mathbb{N}\}$$

and

$$A_{(n)} = \{x \in X \mid C_A(x) > n, n \in \mathbb{N}\}$$

are called the strong and weak upper cuts of A, respectively.

Similarly, The sets $A^{[n]}$ and $A^{(n)}$ defined by

$$A^{[n]} = \{x \in X \mid C_A(x) \leq n, n \in \mathbb{N}\}$$

and

$$A^{(n)} = \{x \in X \mid C_A(x) < n, n \in \mathbb{N}\}$$

are called the strong and weak lower cuts of A, respectively.

Theorem 2.13 ([7]). *Let $A \in MG(X)$. Then $A_{[n]}, n \in \mathbb{N}$ is a subgroup of X for $n \leq C_A(e)$ and $A^{[n]}, n \in \mathbb{N}$ is a subgroup of X, for $n \geq C_A(e)$.*

Definition 2.14 ([18]). Let $A \in MG(X)$. Then A^{-1} is defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}), \forall x \in X.$$

Thus we notice that $A \in MG(X) \Leftrightarrow A^{-1} \in MG(X)$.

Definition 2.15 ([9]). Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B, if for all $x, y \in X$,

$$C_A(xyx^{-1}) \geq C_A(y).$$

Proposition 2.16 ([9]). *Let A be a submultigroup of $B \in MG(X)$. Then the following statements are equivalent:*

(1) *A is a normal submultigroup of B,*
(2) *$C_A(xyx^{-1}) = C_A(y) \forall x, y \in X$,*
(3) *$C_A(xy) = C_A(yx) \forall x, y \in X$.*

Remark 2.17 ([9]). It follows that A is a normal submultigroup of $B \in MG(X)$ if and only if

$$C_A(xy) = C_A(yx) \text{ or } C_A(xyx^{-1}) = C_A(y) \; \forall x, y \in X,$$

rewritten as $C_{A^x}(y) = C_A(y)$. That is, $C_{A^x}(y) = C_A(xyx^{-1})$.

Definition 2.18 ([18]). Let $A \in MG(X)$. Then A is said to be commutative, if for all $x, y \in X$,

$$C_A(xy) = C_A(yx).$$

Definition 2.19 ([8]). Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively. Then f induces a homomorphism from A to B which satisfies

(i) $C_{f(A)}(y_1y_2) \geq C_{f(A)}(y_1) \wedge C_{f(A)}(y_2) \; \forall y_1, y_2 \in Y$,
(ii) $C_B(f(x_1x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2)) \; \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$ and

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \; \forall x \in X.$$

Definition 2.20 ([8]). Let f be a homomorphism of a group X into a group Y and $A \in MG(X)$. Then A is said to be f-invariant, if $\forall\ x, y \in X,\ f(x) = f(y)$ implies $C_A(x) = C_A(y)$.

Definition 2.21 ([8]). Let X be a group and let $A \in MG(X)$. Then a homomorphism θ from X onto X is called an automorphism of A onto A, if θ is both injective and surjective, that is, bijective.

Definition 2.22 ([16]). Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a characteristic submultigroup of B, if

$$C_{A^\theta}(x) = C_A(x)\ \forall x \in X$$

for every automorphism, θ of X. That is, $\theta(A) \subseteq A$, for every $\theta \in Aut(X)$.

Definition 2.23 ([16]). Let A be a multigroup of a group X and θ a function from X into itself. Define the multiset A^θ of X by

$$C_{A^\theta}(x) = C_A(x^\theta),\ \text{where}\ x^\theta = \theta(x) = x\ \forall x \in X.$$

Proposition 2.24 ([16]). *Let $B \in MG(X)$. Then every characteristic submultigroup of B is a normal submultigroup of B.*

Definition 2.25 ([10]). Let X and Y be groups, $A \in MG(X)$ and $B \in MG(Y)$, respectively. The direct product of A and B depicted by $A \times B$ is a function

$$C_{A\times B} : X \times Y \to \mathbb{N}$$

defined by

$$C_{A\times B}((x, y)) = C_A(x) \wedge C_B(y),\ \forall x \in X, \forall y \in Y.$$

Theorem 2.26 ([10]). *Let A and B be multigroups of groups X and Y, respectively. Then $A \times B$ is a multigroup of $X \times Y$.*

Definition 2.27 ([10]). Let $A_1, A_2, ..., A_k$ be multigroups of $X_1, X_2, ..., X_k$, respectively. Then the direct product of $A_1, A_2, ..., A_k$ is a function

$$C_{A_1\times A_2\times ...\times A_k} : X_1 \times X_2 \times ... \times X_k \to \mathbb{N}$$

defined by

$$C_{A_1\times A_2\times ...\times A_k}(x) = C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge ... \wedge C_{A_{k-1}}(x_{k-1}) \wedge C_{A_k}(x_k)$$

where $x = (x_1, x_2, ..., x_{k-1}, x_k),\quad \forall x_1 \in X_1, \forall x_2 \in X_2, ..., \forall x_k \in X_k$. If we denote $A_1, A_2, ..., A_k$ by $A_i, (i \in I)$, $X_1, X_2, ..., X_k$ by $X_i, (i \in I)$, $A_1 \times A_2 \times ... \times A_k$ by $\prod_{i=1}^k A_i$ and $X_1 \times X_2 \times ... \times X_k$ by $\prod_{i=1}^k X_i$. Then the direct product of A_i is a function

$$C_{\prod_{i=1}^k A_i} : \prod_{i=1}^k X_i \to \mathbb{N}$$

defined by

$$C_{\prod_{i=1}^k A_i}((x_i)_{i\in I}) = \wedge_{i\in I} C_{A_i}((x_i))\ \forall x_i \in X_i,\ I = 1, ..., k.$$

Unless otherwise specified, it is assumed that X_i is a group with identity e_i for all $i \in I$, $X = \prod_{i\in I}^k X_i$, and so $e = (e_i)_{i\in I}$.

3. Main results

In this section, we present some results on multigroups. The first three results are deduced from the concepts of cuts and comultisets of multigroups.

Proposition 3.1. *Let A be a multigroup of X and let $x \in X$. Then $C_A(x) = n_1$ if and only if $x \in A_{[n_1]}$ and $x \notin A_{[n_2]}$ such $n_2 > n_1$ for $n_1, n_2 \in \mathbb{N}$.*

Proof. Let $x \in X$ and $A \in MG(X)$. Suppose $C_A(x) = n_1$. It implies that $x \in A_{[n_1]}$, and if $\exists\ n_2 \in \mathbb{N}$ such that $n_2 > n_1$, then it follows that $x \notin A_{[n_2]}$.

Conversely, assume that $x \in A_{[n_1]}$ and $x \notin A_{[n_2]}$, for $n_2 > n_1$. Then we can see clearly by the hypothesis that $C_A(x) = n_1$. □

Theorem 3.2. *Let A be a submultigroup of $B \in MG(X)$. Then $gA = hA$, for $g, h \in X$ if and only if*

$$C_A(g^{-1}h) = C_A(h^{-1}g) = C_A(e).$$

Proof. Let $gA = hA$. Then $C_{gA}(g) = C_{hA}(g)$ and $C_{gA}(h) = C_{hA}(h)$ $\forall g, h \in X$. Thus

$$C_A(g^{-1}h) = C_A(h^{-1}g) = C_A(e).$$

Conversely, let $C_A(g^{-1}h) = C_A(h^{-1}g)$, $\forall g, h \in X$. For every $x \in X$, we have

$$\begin{aligned} C_{gA}(x) = C_A(g^{-1}x) &= C_A(g^{-1}hh^{-1}x) \\ &\geq C_A(g^{-1}h) \wedge C_A(h^{-1}x) \\ &= C_A(h^{-1}x) \\ &= C_{hA}(x). \end{aligned}$$

Similarly,

$$\begin{aligned} C_{hA}(x) = C_A(h^{-1}x) &= C_A(h^{-1}gg^{-1}x) \\ &\geq C_A(h^{-1}g) \wedge C_A(g^{-1}x) \\ &= C_A(g^{-1}x) \\ &= C_{gA}(x). \end{aligned}$$

Then $C_{gA}(x) = C_{hA}(x) \Rightarrow gA = hA$. □

Corollary 3.3. *Let A be a submultigroup of $B \in MG(X)$. Then $Ag = Ah$, for $g, h \in X$ if and only if*

$$C_A(gh^{-1}) = C_A(hg^{-1}) = C_A(e).$$

Proof. Straightforward from Theorem 3.2. □

Next, we consider some results that have to do with normal submultigroups of multigroups.

Proposition 3.4. *If $B \in MG(X)$ and A is a normal submultigroup of B. Then A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*.*

Proof. We know that A_* and A^* are subgroups of X, by Proposition 2.11. It is easy to prove that A_* and A^* are normal subgroups of X.

Let $x, y \in A_*$. Then by the definition of A_*, it follows that $C_A(x) > 0$ and $C_A(y) > 0$. That is,

$$C_A(xyx^{-1}) \geq C_A(y) > 0.$$

Thus $xyx^{-1} \in A_* \Rightarrow A_*$ is a normal subgroup of X.

Similarly, assume $x, y \in A^*$. Then by the definition of A^*, it follows that

$$C_A(x) = C_A(e) = C_A(y).$$

That is,

$$C_A(xyx^{-1}) \geq C_A(y) = C_A(e) \geq C_A(xyx^{-1}).$$

Thus $C_A(xyx^{-1}) = C_A(e)\ \forall x, y \in X$. So $xyx^{-1} \in A^*$. Hence the result follows.

Recall that, A is a normal submultigroup of B, and A_* and A^* are normal subgroups of X. Synthesizing these, it implies that A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*. □

Proposition 3.5. *Let A be a normal submultigroup of $B \in MG(X)$. Then $A_{[n]}$ is a normal subgroup of X $\forall\ n \leq C_A(e)$ and $A^{[n]}$ is a normal subgroup of X $\forall$ $n \geq C_A(e)$, where e is the identity element of X and $n \in \mathbb{N}$. Consequently, $A_{[n]}$ is a normal subgroup of $B_{[n]}$ and $A^{[n]}$ is a normal subgroup of $B^{[n]}$.*

Proof. It implies from Theorem 2.13 that, $A_{[n]}$ is a subgroup of X $\forall\ n \leq C_A(e)$ and $A^{[n]}$ is a subgroup of X $\forall\ n \geq C_A(e)$, where $n \in \mathbb{N}$. Now, we prove that $A_{[n]}$ and $A^{[n]}$ are normal subgroups of X.

Let $x, y \in A_{[n]}$. Then by the definition of $A_{[n]}$, we get

$$C_A(x) \geq n \text{ and } C_A(y) \geq n.$$

That is,

$$C_A(xyx^{-1}) = C_A(y) \geq n.$$

Thus, $xyx^{-1} \in A_{[n]}$. So $A_{[n]}$ is a normal subgroup of X. Similarly, it follows that $A^{[n]}$ is a normal subgroup of X.

But we know that, A is a normal submultigroup of B, and $A_{[n]}$ and $A^{[n]}$ are normal subgroups of X. Synthesizing these, it happens that $A_{[n]}$ is a normal subgroup of $B_{[n]}$ and $A^{[n]}$ is a normal subgroup of $B^{[n]}$. □

Theorem 3.6. *For a submultigroup A of $B \in MG(X)$, the following statements are equivalent:*

(1) *A is a normal submultigroup of B,*
(2) *$A_{[n]}$ (for $n \in \mathbb{N}$ and $n \leq C_A(e)$, where e is the identity element of X) is a normal subgroup of X. It also holds for $A^{[n]}$, whenever $n \geq C_A(e)$.*

Proof. (1)⇒(2): Let $x \in X$ and $y \in A_{[n]}$. Then by the hypothesis, we have

$$C_A(xyx^{-1}) = C_A(y) \geq n.$$

Thus $y = xyx^{-1} \in A_{[n]}$. So $A_{[n]}$ is a normal subgroup of X.

(2)⇒(1): Let $x, g \in X$. Take $n_1 = C_A(x)$ and $n_2 = C_B(g)$. then $x \in A_{[n_1]}$ and $g \in B_{[n_2]}$, for $n_1, n_2 \in \mathbb{N}$.

Case 1: Suppose $n_1 \geq n_2$. Then $n_0 \geq C_A(x) \geq n_2 = C_B(g)$, for $n_1 \in [0, n_0]$. Thus $x \in A_{[n_2]}$ and $g \in B_{[n_2]}$. So by the hypothesis, $A_{[n_2]}$ is a normal subgroup of $B_{[n_2]}$. Hence $gxg^{-1} \in A_{[n_2]}$. So,

$$C_A(gxg^{-1}) \geq n_2 = C_B(g) = C_A(x) \wedge C_B(g).$$

Case 2: Suppose $n_2 \geq n_1$. Thent

$$C_B(g) \geq n_1 = C_A(x).$$

Thus $x \in A_{[n_1]}$ and $g \in B_{[n_1]}$. So by the hypothesis, $A_{[n_1]}$ is a normal subgroup of $B_{[n_1]}$. Consequently, $gxg^{-1} \in A_{[n_1]}$. Hence

$$C_A(gxg^{-1}) \geq n_1 = C_A(x) = C_A(x) \wedge C_B(g).$$

Therefore (1) holds. □

Definition 3.7. Let A and B be submultigroups of $C \in MG(X)$. Then the commutator of A and B is the multiset (A, B) of X defined as follows:

$$C_{(A,B)}(x) = \begin{cases} \bigvee_{x=[a,b]}[C_A(a) \wedge C_B(b)], & \text{if } x \text{ is a commutator in } X \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$C_{(A,B)}(x) = \bigvee_{x=aba^{-1}b^{-1}} [C_A(a) \wedge C_B(b)].$$

Since the supremum of an empty set is zero, $C_{(A,B)}(x) = 0$ if x is not a commutator.

Definition 3.8. Let A and B be submultigroups of $C \in MG(X)$. Then the commutator multigroup of A and B is the multigroup generated by the commutator (A, B). It is denoted by $[A, B]$.

Definition 3.9. Let A be a submultigroup of $B \in MG(X)$. Then the submultigroup of B generated by A is the least submultigroup of B containing A. It is denoted by $< A >$. That is

$$< A >= \bigcap\{A_i \in MG(X) | C_A(x) \leq C_{A_i}(x)\}.$$

With the aid of Definitions 3.7 and 3.8, we obtain the result that follows.

Theorem 3.10. *Let A and B be normal submultigroups of $C \in MG(X)$. Then $[A, B] \subseteq A \cap B$.*

Proof. Let $x \in X$and assume that x is not a commutator. Then $C_{(A,B)}(x) = 0$. Thus there is nothing to prove. Suppose that $x = aba^{-1}b^{-1}$, for some $a, b \in X$. Then

$$\begin{aligned} C_{A\cap B}(x) &= C_A(x) \wedge C_B(x) \\ &= C_A(aba^{-1}b^{-1}) \wedge C_B(aba^{-1}b^{-1}) \\ &\geq (C_A(a) \wedge C_A(ba^{-1}b^{-1})) \wedge (C_B(aba^{-1}) \wedge C_B(b^{-1})) \\ &\geq (C_A(a) \wedge C_C(b)) \wedge (C_B(b) \wedge C_C(a)) \\ &= C_A(a) \wedge C_B(b). \end{aligned}$$

This implies that

$$\begin{array}{rcl} C_{A\cap B}(x) & \geq & \bigvee_{x=aba^{-1}b^{-1}} C_A(a)\wedge C_B(b) \\ & = & C_{(A,B)}(x). \end{array}$$

Thus $C_{A\cap B}(x)\geq C_{(A,B)}(x)$. So $[A,B]\subseteq A\cap B$. $\square$

Theorem 3.11. *Let $A,B\in MG(X)$ such that $C_A(e)=C_B(e)$, where e is the identity element of X. Then B is commutative if and only if A is a commutative multigroup of X.*

Proof. Let X be a group such that $x,y\in X$. Suppose B is commutative. Then it follows that

$$\begin{array}{rcl} C_B((xy)(xy)^{-1})=C_B(e) & = & C_B((xy)(yx)^{-1}) \\ & = & C_A((xy)(yx)^{-1}) \\ & = & C_A(e), \end{array}$$

since $C_A(e)=C_B(e)$. Thus $C_A(xy)=C_A(yx)\ \forall x,y\in X$.

Conversely, suppose A is a commutative multigroup of X. Then we have $C_B(xy)=C_B(yx)$, $\forall x,y\in X$ using the same logic in the necessity part. $\square$

In what follow are some results on homomorphic properties of multigroups and the notion of characteristic submultigroups.

Theorem 3.12. *Let $f:X\to Y$ be a homomorphism and $A\in MG(X)$. If A is f-invariant, then A is regular.*

Proof. Suppose A is f-invariant. Then $\forall x,y\in X$, it follows that $f(x)=f(y)$, by Definition 2.20. Thus

$$\begin{array}{rcl} C_{f(A)}(f(x))=C_{f(A)}(f(y)) & \Rightarrow & C_A(f^{-1}(f(x)))=C_A(f^{-1}(f(y))) \\ & \Rightarrow & C_A(x)=C_A(y). \end{array}$$

So A is regular. $\square$

Corollary 3.13. *With the same hypothesis as in Theorem 3.12, for all $x,y\in X$, $f(x)=f(y)$ if and only if A is f-invariant.*

Proof. Suppose $f(x)=f(y)\ \forall x,y\in X$. Then by Theorem 3.12, $C_A(x)=C_A(y)$. Thus A is f-invariant.

Conversely, suppose A is f-invariant. Then it follows that, A is regular by Theorem 3.12. Thus $C_A(x)=C_A(y)\ \forall x,y\in X$. So we get

$$C_A(f^{-1}(f(x)))=C_A(f^{-1}(f(y))) \quad\Rightarrow\quad C_{f(A)}(f(x))=C_{f(A)}(f(y)).$$

So $f(x)=f(y)$. $\square$

Proposition 3.14. *Let $A\in MG(X)$ and $g\in X$. If θ is an automorphism of X defined by $\theta(x)=gxg^{-1}\ \forall x\in X$, then $A^g=A^\theta$.*

Proof. Let $A \in MG(X)$ and $g \in X$. Suppose $\theta : X \to X$ is defined by $\theta(x) = gxg^{-1}$ $\forall x \in X$. Then by Remark 2.17, we get

$$C_{A^g}(x) = C_A(gxg^{-1}) = C_A(\theta(x)) = C_{A^\theta}(x).$$

Thus the result follows. □

Proposition 3.15. *Let $A \in MG(X)$. If θ is a homomorphism of X into itself, then A^θ is a multigroup of X.*

Proof. Let $x, y \in X$. Then $C_{A^\theta}(xy) = C_A((xy)^\theta) = C_A(x^\theta y^\theta)$, since θ is a homomorphism. Since A is a multigroup of X, we have

$$C_A(x^\theta y^\theta) \geq C_A(x^\theta) \wedge C_A(y^\theta) = C_{A^\theta}(x) \wedge C_{A^\theta}(y).$$

Thus

$$C_{A^\theta}(xy) \geq C_{A^\theta}(x) \wedge C_{A^\theta}(y).$$

Also,

$$C_{A^\theta}(x^{-1}) = C_A((x^{-1})^\theta) = C_A((x^\theta)^{-1}) = C_A(x^\theta) = C_{A^\theta}(x).$$

So A^θ is a multigroup of X. □

Theorem 3.16. *Let $\theta : X \to X$ be an automorphism and $A \in MS(X)$. Then $A^\theta \in MG(X)$ if and only if $A \in MG(X)$.*

Proof. Suppose $A \in MG(X)$. Then using the same logic in the proof of Proposition 3.15, it follows that $A^\theta \in MG(X)$.

Conversely, assume A^θ is a multigroup of X. Then

$$C_{A^\theta}(xy) \geq C_{A^\theta}(x) \wedge C_{A^\theta}(y) \text{ and } C_{A^\theta}(x^{-1}) = C_{A^\theta}(x)$$

, $\forall x, y \in X$ and for every $\theta \in Aut(X)$. Thus

$$\begin{aligned} C_{A^\theta}(xy) = C_A((xy)^\theta) &= C_A(\theta(xy)) \\ &= C_A(xy), \end{aligned}$$

$$\Rightarrow C_A(xy) \geq C_A(x) \wedge C_A(y) \; \forall x, y \in X.$$

Also,

$$\begin{aligned} C_{A^\theta}(x^{-1}) = C_A((x^{-1})^\theta) &= C_A((x^\theta)^{-1}) \\ &= C_A((\theta(x))^{-1}) = C_A(x^{-1}), \end{aligned}$$

implying that $C_A(x^{-1}) = C_A(x)$ $\forall x \in X$. So $A \in MG(X)$. □

Proposition 3.17. *If $B \in MG(X)$ and A is a characteristic submultigroup of B. Then A^* and A_* are characteristic subgroups of X. Also, A^* is a characteristic subgroup of B^* and A_* is a characteristic subgroup of B_*.*

Proof. We know that A_* and A^* are subgroups of X, by Proposition 2.11. Now, we prove that A_* and A^* are characteristic subgroups of X. It is sufficient to show that $\theta(A^*) \subseteq A^*$ $\forall \theta \in Aut(X)$.

Let $\theta \in Aut(X)$. Then $C_{A^\theta}(x) = C_A(x)$, since A is a characteristic submultigroup of B. Let $x \in A^*$. Then $C_A(x) = C_A(e)$. Thus

$$C_{A^\theta}(x) = C_A(\theta(x)) = C_A(x) = C_A(e).$$

So $\theta(x) \in A^*$. Hence $\theta(A^*) \subseteq A^*$. This completes the proof.

Similarly, the proof of the fact that A_* is a characteristic subgroup of X follows.

Recall that, A is a characteristic submultigroup of B, and A^* and A_* are characteristic subgroups of X. Synthesizing these, it implies that A^* is a characteristic subgroup of B^* and A_* is a characteristic subgroup of B_*. □

Theorem 3.18. *Let X be a group and A be a characteristic submultigroup of $B \in MG(X)$. Then $A_{[n]}$ is a characteristic subgroup of X $\forall\ n \leq C_A(e)$ and $A^{[n]}$ is a characteristic subgroup of X $\forall\ n \geq C_A(e)$, where e is the identity element of X and $n \in \mathbb{N}$.*

Proof. It implies from Theorem 2.13 that, $A_{[n]}$ is a subgroup of X $\forall\, n \leq C_A(e)$ and $A^{[n]}$ is a subgroup of X $\forall\, n \geq C_A(e)$. Now, we prove first that $A_{[n]}$ is a characteristic subgroups of X.

Whenever we show that $\theta(A_{[n]}) \subseteq A_{[n]}$ $\forall\theta \in Aut(X)$ we are done. Let $\theta \in Aut(X)$. Then $C_{A^\theta}(x) = C_A(x)$, since A is a characteristic submultigroup of B. Let $x \in A_{[n]}$. Then $C_A(x) \geq n$, which implies that

$$C_{A^\theta}(x) = C_A(\theta(x)) = C_A(x) \geq n.$$

Thus $\theta(x) \in A_{[n]}$. So $\theta(A_{[n]}) \subseteq A_{[n]}$. Hence $A_{[n]}$ is a characteristic subgroups of X. Similarly, $A^{[n]}$ is a characteristic subgroups of X. □

Remark 3.19. Since A is a characteristic submultigroup of B, and $A_{[n]}$ and $A^{[n]}$ are characteristic subgroups of X. Synthesizing these, it happens that $A_{[n]}$ is a characteristic subgroup of $B_{[n]}$ and $A^{[n]}$ is a characteristic subgroup of $B^{[n]}$.

Now, we give a statement of the converse of Theorem 3.18.

Theorem 3.20. *Let X be a group and A be a submultigroup of $B \in MG(X)$. If $A_{[n]}$, for $n \in \mathbb{N}$ (also $A^{[n]}$) is a characteristic subgroup of X, then A is a characteristic submultigroup of B.*

Proof. By hypothesis, it follows that

$$A_{[n]} = \{x \in X |\, C_A(x) \geq n_i\}$$

is a characteristic subgroup of X, $\forall i = 1, ..., k$. Let $\theta \in Aut(X)$. Also

$$C_{A^\theta}(x) = C_A(\theta(x)) = C_{\theta^{-1}(A)}(x) = C_A(x).$$

Moreover, $\forall i = 1, ..., k$, we get $(A^\theta)_{[n_i]} = A_{[n_i]}$, since $x \in (A^\theta)_{[n_i]} \Leftrightarrow C_{A^\theta}(x) \geq n_i \Leftrightarrow C_A(\theta(x)) \geq n_i \Leftrightarrow \theta(x) \in A_{[n_i]} \Leftrightarrow x \in \theta^{-1}(A_{[n_i]}) \Leftrightarrow x \in A_{[n_i]}$. Thus $A^\theta = A$. The result follows. □

In the next theorem, we combine Theorems 3.18 and 3.20 together and get:

Theorem 3.21. *For a submultigroup A of $B \in MG(X)$, the following statements are equivalent:*

(1) *A is a characteristics submultigroup of B,*
(2) *$A_{[n]}$ for $n \in \mathbb{N}$ is a characteristic subgroup of X $\forall\ n \leq C_A(e)$, where e is the identity element of X (also $A^{[n]}$ $\forall\ n \geq C_A(e)$).*

Proof. (1)⇒(2): Let $\theta \in Aut(X)$ and $x \in A_{[\alpha]}$. Then

$$C_A(\theta(x)) = C_A(x) \geq n,$$

since A is a characteristic submultigroup of B. It follows that $\theta(x) \in A_{[n]}$. Thus $\theta(A_{[n]}) \subseteq A_{[n]}$. We prove that $A_{[n]} \subseteq \theta(A_{[n]})$, by symmetry. Let $x \in A_{[n]}$ and let $y \in X$ such that $\theta(y) = x$. Then

$$C_A(y) = C_A(\theta(y)) = C_A(x) \geq n.$$

Thus $y \in A_{[n]}$. So $x \in \theta(A_{[n]})$. Hence $A_{[n]} \subseteq \theta(A_{[n]})$. Therefore $A_{[n]}$ is a characteristic subgroup of X.

(2)⇒(1): Let $x \in X$, $\theta \in Aut(X)$ and $C_A(x) = n_1$. Then $x \in A_{[n_1]}$ and $x \notin A_{[n_2]}$ $\forall n_2 > n_1$, by Proposition 3.1. Thus by hypothesis, $\theta(A_{[n_1]}) = A_{[n_1]}$. So $\theta(x) \in A_{[n_1]}$. Hence $C_A(x) = C_A(\theta(x)) \geq n_1$.

Let $n_2 = C_A(\theta(x))$. Assume $n_2 > n_1$. Then $\theta(x) \in A_{[n_2]} = \theta(A_{[n_2]})$. Since θ is one-to-one, it follows that $x \in A_{[n_2]}$, which is a contradiction. Thus

$$C_A(\theta(x)) = n_1 = C_A(x),$$

implying that A is a characteristic submultigroup of B. □

Finally, we obtain some results on the concept of direct product in multigroup setting.

Theorem 3.22. *Let $A \in MG(X)$ and $B \in MG(Y)$. Suppose C and D are two submultisets of A and B, respectively. Then $C \times D$ is a submultigroup of $A \times B$ if and only if both C and D are submultigroups of A and B, respectively.*

Proof. Suppose C and D are two submultigroups of A and B, respectively. Then it is clear that $C \in MG(X)$ and $D \in MG(Y)$. It follows that $C \times D$ is a multigroup of $X \times Y$, by Theorem 2.26. Since $A \times B$ is a multigroup of $X \times Y$ by the same reason, and $C \sqsubseteq A$ and $D \sqsubseteq B$, $C \times D$ is a submultigroup of $A \times B$.

Conversely, suppose $C \times D$ is a submultigroup of $A \times B$. Then it follows that $C \sqsubseteq A$ and $D \sqsubseteq B$. These complete the proof. □

Corollary 3.23. *Let $A \in MG(X)$ and $B \in MG(Y)$. Suppose C and D are two submultigroups of A and B, respectively. Then $C \times D$ is a normal submultigroup of $A \times B$ if and only if both C and D are normal submultigroups of A and B, respectively.*

Proof. Combining both Definitions 2.15 and 2.25, and Theorems 2.26 and 3.22, the proof follows. □

Corollary 3.24. *Let $A \in MG(X)$ and $B \in MG(Y)$. Suppose C and D are two submultigroups of A and B, respectively. Then $C \times D$ is a characteristic submultigroup of $A \times B$ if and only if both C and D are characteristic submultigroups of A and B, respectively.*

Proof. Combining both Definition 2.22, Theorems 2.26 and 3.22, the proof follows. □

Corollary 3.25. *With the same hypothesis as in Corollary 3.24, it follows that $C \times D$ is a normal submultigroup of $A \times B$, if both C and D are characteristic submultigroups of A and B, respectively.*

Proof. Straightforward from Proposition 2.24 and Theorem 3.22. □

Corollary 3.26. *Let $A \in MG(X)$ and C be a submultiset of A. Then $C \times C$ is a submultigroup of $A \times A$ if and only if C is a submultigroup of A.*

Proof. The proof is straightforward from Theorem 3.22. □

Remark 3.27. Let $A \in MG(X)$ and C be a submultigroup of A. Then

(1) $C \times C$ is a normal submultigroup of $A \times A$ if and only if C is a normal submultigroup of A,
(2) $C \times C$ is a characteristic submultigroup of $A \times A$ if and only if C is a characteristic submultigroup of A,
(3) $C \times C$ is a normal submultigroup of $A \times A$ if C is a characteristic submultigroup of A.

Corollary 3.28. *Let $B_1, ..., B_k$ be multigroups of groups $X_1, ..., X_k$. Suppose $A_1, ..., A_k$ are submultisets of $B_1, ..., B_k$, respectively. Then $A_1 \times ... \times A_k$ is a submultigroup of $B_1 \times ... \times B_k$ if and only if $A_1, ..., A_k$ are submultigroups of $B_1, ..., B_k$.*

Proof. Similar to Theorem 3.22. □

Remark 3.29. Let $B_1, ..., B_k$ be multigroups of groups $X_1, ..., X_k$. Suppose $A_1, ..., A_k$ are submultigroups of $B_1, ..., B_k$, respectively. Then

(1) $A_1 \times ... \times A_k$ is a normal submultigroup of $B_1 \times ... \times B_k$ if and only if $A_1, ..., A_k$ are normal submultigroups of $B_1, ..., B_k$,
(2) $A_1 \times ... \times A_k$ is a characteristic submultigroup of $B_1 \times ... \times B_k$ if and only if $A_1, ..., A_k$ are characteristic submultigroups of $B_1, ..., B_k$,
(3) $A_1 \times ... \times A_k$ is a normal submultigroup of $B_1 \times ... \times B_k$ if $A_1, ..., A_k$ are characteristic submultigroups of $B_1, ..., B_k$.

Remark 3.30. Let A and B be multisets over groups X and Y, respectively such that $A \times B$ is a multigroup of $X \times Y$. Then

(1) either A or B is a multigroup of either X or Y,
(2) both A and B are multigroups of X and Y, respectively.

Theorem 3.31. *Let A and B be multigroups of groups X and Y, respectively. Then A and B are commutative if and only if $A \times B$ is a commutative multigroup of $X \times Y$.*

Proof. Suppose A and B are commutative. We show that $A \times B$ is a commutative multigroup of $X \times Y$. It is a known fact that $A \times B \in MG(X \times Y)$, by Theorem

2.26. Let $(x, y) \in X_1 \times X_2$ such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we get

$$\begin{aligned} C_{A\times B}(xy) &= C_{A\times B}((x_1, x_2)(y_1, y_2)) \\ &= C_{A\times B}(x_1y_1, x_2y_2) \\ &= C_A(x_1y_1) \wedge C_B(x_2y_2) \\ &= C_A(y_1x_1) \wedge C_B(y_2x_2) \\ &= C_{A\times B}(y_1x_1, y_2x_2) \\ &= C_{A\times B}((y_1, y_2)(x_1, x_2)) \\ &= C_{A\times B}(yx). \end{aligned}$$

Thus $A \times B$ is a commutative multigroup of $X \times Y$, by Definition 2.18.

Conversely, suppose $A\times B$ is a commutative multigroup of $X\times Y$. Then it is clear that both A and B are commutative multigroups of groups X and Y, respectively. □

4. Conclusions

We have presented some results in the area of cuts of multigroups, comultisets, normal submultigroups, characteristic submultigroups, homomorphic properties and direct product of multigroups. The notion of commutator in multigroup setting was proposed. Further results on multigroup theory could still be exploited, especially, some properties of commutator in multigroup context.

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