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Fuzzy $r-$ ideals in $\Gamma-$incline

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Abstract. In this paper, we introduce the notion of fuzzy ideal, fuzzy $k-$ ideal and fuzzy $r-$ ideal in $\Gamma-$incline. We study the properties of fuzzy ideals,fuzzy $k-$ ideals and fuzzy $r-$ ideals in $\Gamma-$incline.

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1. Introduction

The non trivial example of semiring first appeared in the work of German mathematician Richard Dedikind in 1894 in connection with the algebra of ideals of a commutative ring. The notion of semiring was introduced by American mathematician Vandiver [35] in 1934. Semiring is a well known universal algebra. Semiring have been used for studying optimization theory, graph theory, matrices, determinants, theory of automata, coding theory, analysis of computer programmes, etc.

The concept of incline was first introduced by Cao et al. [3] in 1984. Inclines are additively idempotent semirings in which products are less than or equal to either factor. Products reduce the values of quantities and make them go down which is why the structures were named inclines. Idempotent semirings and Kleene algebras have recently been established as fundamental structures in computer sciences. An incline is a generalization of Boolean algebra, fuzzy algebra and distributive lattice and incline is a special type of semiring. An incline has both semiring structure and the poset structure. Every distributive lattice and every Boolean algebra is an incline but an incline need not be a distributive lattice. Set of all idempotent elements in an incline is a distributive lattice. Yao and Han studied the relations between ideals, filters and congruences in inclines and it is shown that there is a one to one correspondence between the set of ideals and the set of all regular congruences. Kim and Rowsh have studied matrices over an incline. Many research scholars have been researched the theory of incline matrices. Few research scholars studied the

algebraic structure of incline. Inclines and matrices over inclines are useful tools in diverse areas such as automata theory, design of switching circuits, graph theory, information systems, modeling, decision making, dynamical programming, control theory, classical and non classical path finding problems in graphs, fuzzy set theory, data analysis, medical diagnosis, nervous system, probable reasoning, physical measurement and so on.
The notion of Γ-ring was introduced by Nobusawa [29] as a generalization of ring in 1964. Sen [31] introduced the notion of $\Gamma-$semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [12] in 1932. Lister introduced ternary ring. Dutta and Kar [4] introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. In 1995, Murali Krishna Rao [15, 16, 17] introduced the notion of $\Gamma-$semiring which is a generalization of $\Gamma-$ring, ternary semiring and semiring. After the paper [15] was published, many mathematicians obtained interesting results on Γ-semirings. Murali Krishna Rao and Venkateswarlu [26] introduced the notion of regular $\Gamma-$incline and field $\Gamma-$semiring.

Ahn et al. [1, 2] studied ideals in incline and quotient incline. Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers and it was generalized by Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory and the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal. In 1952, the concept of bi-ideals was introduced by Good and Hughes [5] for semigroups. Henriksen [6] studied ideals in semigroups. The notion of bi-ideals in rings and semirings were introduced by Lajos and Szasz [11]. Quasi ideals are generalization of right ideals and left ideals whereas bi-ideals are generalization of quasi ideals. In 1956, Steinfeld [33] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [7] introduced the concept of quasi ideal for a semiring. Quasi ideals in $\Gamma-$semirings were studied by Jagtap and Pawar [8]. Murali Krishna Rao [24] introduced the concept of bi-interior ideal for a semigroup.

The fuzzy set theory was developed by Zadeh [36] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by Rosenfeld [30] with the notion of fuzzy subgroups in 1971. Swamy and Swamy [34] studied fuzzy prime ideals in rings in 1988. In 1982, Liu [13] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Applying the concept of fuzzy sets to the theory of $\Gamma-$ring, Jun and Lee [9] introduced the notion of fuzzy ideals in $\Gamma-$ring and studied the properties of fuzzy ideals of $\Gamma-$ring. Murali Krishna Rao and Venkateswarlu [27] studied $L-$fuzzy ideals in $\Gamma-$semirings. Mandal [14] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. Murali Krishna Rao [18, 19, 20, 21, 22, 23, 25, 28] studied fuzzy soft $\Gamma-$semiring, fuzzy soft $k-$ideal, $T-$fuzzy ideals,left bi-quasi ideals, bi interior ideals in semigroups and $\Gamma-$ semirings. Kuroki [10] studied fuzzy interior ideals in semigroups. Srinivas and Nagaiah [32] studied some results on $T-$Fuzzy ideals of $\Gamma-$ near-rings.

In this paper, we introduce the notion of ideal, $k-$ideal, $r-$ideal, fuzzy ideal, fuzzy $r-$ideal and $k-$fuzzy ideal in $\Gamma-$incline. We study the properties of fuzzy ideal, fuzzy $r-$ideal and fuzzy $k-$ideal in $\Gamma-$incline and relations between them.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. A commutative incline M with additive identity 0 and multiplicative identity 1 is a non-empty set M with operations addition $(+)$ and multiplication $(\cdot)$ defined on $M \times M \to M$ such that satisfying the following conditions: for all $x, y, z \in M$,

(i) $x + y = y + x$,
(ii) $x + x = x$,
(iii) $x + xy = x$,
(iv) $y + xy = y$,
(v) $x + (y + z) = (x + y) + z$,
(vi) $x(yz) = x(yz)$,
(vii) $x(y + z) = xy + xz$,
(viii) $(x + y)z = xz + yz$,
(ix) $x1 = 1x = x$,
(x) $x + 0 = 0 + x = x$,
(xi) $xy = yx$.

Definition 2.2. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M as a $\Gamma-$semiring, if there exists a mapping $M \times \Gamma \times M \to M$ is written (x, α, y) as $x\alpha y$ such that it satisfies the following axioms: for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$,
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a $\Gamma-$ semiring with $\Gamma = R$ and ternary operation $x\gamma y$ as the usual semiring multiplication.

Definition 2.3. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \to M\big((x, \alpha, y) \to x\alpha y\big)$ such that it satisfies the following axioms: for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$,
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$,
(v) $x + x = x$,
(vi) $x + x\alpha y = x$,
(vii) $y + x\alpha y = y$.

Then M is called a $\Gamma-$incline.

Every incline M is a $\Gamma-$incline with $\Gamma = M$ and ternary operation $x\gamma y$ as the usual incline multiplication.

In an $\Gamma-$incline define the order relation such that for all $x, y \in M, y \leq x$ if and only if $y + x = x$. Obviously $\leq$ is a partially order relation over M.

Definition 2.4. A $\Gamma-$incline M is said to have zero element, if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$.

Definition 2.5. A $\Gamma-$incline M is said to be commutative $\Gamma-$incline, if $x\alpha y = y\alpha x$, for all $x, y \in M$.

Definition 2.6. A $\Gamma-$ subincline I of a $\Gamma-$incline M is a nonempty subset of M which is closed under the $\Gamma-$incline operations addition and ternary operation.

Definition 2.7. An element $a \in M$ is said to be idempotent of M, if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be $\alpha-$idempotent.

Definition 2.8. If every element of M is an idempotent of M then M is said to be an idempotent $\Gamma-$incline M.

Definition 2.9. If every element of M is a regular element of M then M is said to be a regular $\Gamma-$incline M.

Example 2.10. If $M = [0,\ 1]$ and $\Gamma = N$, a binary operation $+$ is defined as $a + b = max\{a, b\}$ and ternary operation is defined as $xry = min\{x, r, y\}$, for all $x, y \in M, r \in \Gamma$, then M is a $\Gamma-$incline .

Example 2.11. If $M = [0, 1], \Gamma = \{0, 1\}$, binary operation $+$ is maximum, ternary operation $a\alpha b$ is usual multiplication, for all $a, b \in M, \alpha \in \Gamma$, then M is $\Gamma-$incline with unity 1.

Definition 2.12. Let M and N be $\Gamma-$inclines. A mapping $f : M \to N$ is called a homomorphism, if

(i) $f(a + b) = f(a) + f(b)$,
(ii) $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M, \alpha \in \Gamma$.

Definition 2.13. Let M be a non-empty set. A mapping $\mu : M \to [0, 1]$ is called a fuzzy subset of M.

Definition 2.14. If μ is a fuzzy subset of M and $t \in [0, 1]$, then the set $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is called a level subset of M with respect to the fuzzy subset μ.

Definition 2.15. A fuzzy subset $\mu : M \to [0, 1]$ is a non-empty fuzzy subset, if μ is not a constant function.

Definition 2.16. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$, for all $x \in M$.

Definition 2.17. Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M is defined by:

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.18. Let $\phi : M \to M`$ be a homomorphism of $\Gamma-$semirings and $f$ be a fuzzy subset of $M$. We define a fuzzy subset $\phi(f)$ of $M`$ by:

$$\phi(f)(x) = \begin{cases} \sup\limits_{y \in \phi^{-1}(x)} f(y), & \text{if } \phi^{-1}(x) \neq \varnothing, \\ 0, & \text{otherwise} \end{cases}.$$

Definition 2.19. Let $\phi : M \to N$ be a homomorphism of $\Gamma-$semirings M, N and μ be a fuzzy subset of M. Then μ is said to be $\phi-$homomorphism invariant, if $\phi(a) = \phi(b)$, then $\mu(a) = \mu(b)$, for all $a, b \in M$.

Definition 2.20. Let M be a $\Gamma-$semiring. A fuzzy subset μ of M is said to be fuzzy $\Gamma-$subsemiring of M, if it satisfying the following conditions:
(i) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.21. A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left (right) ideal of M, if it satisfying the following conditions:
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq \mu(y)\ (\mu(x))$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.22. A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy ideal of M, if it satisfying the following conditions:
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

3. Fuzzy ideals, fuzzy $r-$ ideals and fuzzy $k-$ideals in $\Gamma-$incline

In this section, we introduce the notion of ideal, $r-$ideal, $k-$ideal, fuzzy ideal, fuzzy $r-$ideal and fuzzy $k-$ideal in $\Gamma-$inclines. We study the properties of fuzzy ideals, fuzzy $k-$ ideals and fuzzy $r-$ ideals in $\Gamma-$incline.

Definition 3.1. A $\Gamma-$subincline I of a $\Gamma-$incline M is a non-empty subset of M which is closed under the $\Gamma-$incline operations addition and ternary operation.

Definition 3.2. A $\Gamma-$subincline I of a $\Gamma-$incline M is called an ideal, if it is a lower set, i.e., for any $x \in I, y \in M$ and $y \leq x \Rightarrow y \in I$.

Definition 3.3. A $\Gamma-$subincline I of a $\Gamma-$incline M is said to be left (right) $r-$ideal of M, if $M\Gamma I \subseteq I$ $(I\Gamma M \subseteq I)$.

Definition 3.4. If I is both left and right $r-$ideal then I is called a $r-$ideal of a $\Gamma-$incline M.

Definition 3.5. A $\Gamma-$subincline I of an $\Gamma-$incline M is said to be $k-$ideal, if $x \in M,\ x + y \in I, y \in I$, then $x \in I$.

Definition 3.6. A $\Gamma-$incline M is said to be $r-$simple, if it has no proper $r-$ideals of M.

Theorem 3.7. *Let I be a subincline of a $\Gamma-$incline M. Then I is an ideal of M if and only if I is a $k-$ideal of M.*

Proof. Let I be an ideal of the $\Gamma-$incline M, $x \in M$, $x+y \in I$ and $y \in I$. Then

$$\begin{aligned} x+y =&(x+x)+y \\ =&x+(x+y) \\ \Rightarrow x \leq& x+y. \end{aligned}$$

Thus by definition of an ideal, $x \in I$. So I is a $k-$ideal of M.

Conversely, suppose that I is a $k-$ideal of the $\Gamma-$incline M. Let $y \in M, x \in I$ and $y \leq x$. Then $y+x=x$. Thus $y+x \in I$. Since I is a $k-$ideal of the $\Gamma-$incline M, $y \in I$. So I is an ideal of the $\Gamma-$incline M. □

Theorem 3.8. *If I is a $r-$ideal of an idempotent $\Gamma-$incline M, then I is a $k-$ideal of M.*

Proof. Suppose $x+y \in I, y \in I$ and $x \in M$. Then $x+x\beta y=x$, for all $\beta \in \Gamma$ and there exists $\alpha \in \Gamma$ such that $x\alpha x=x$. We have $x+y\alpha x=x$.

$$\begin{aligned} &(x+y)\alpha x \in I \\ \Rightarrow &x\alpha x+y\alpha x \in I \\ \Rightarrow &x+y\alpha x \in I \\ \Rightarrow &x \in I. \end{aligned}$$

Suppose $x \leq y, y \in I$. Then $x+y=y$. Thus $x \in I$. So I is a $k-$ideal of the $\Gamma-$incline M. □

Theorem 3.9. *Let M be a $\Gamma-$incline. If I is an ideal of a $\Gamma-$incline M, then I is a $r-$ideal of M.*

Proof. Suppose I is an ideal of the $\Gamma-$incline M, $x \in I, y \in M$ and $\alpha \in \Gamma$. Then $x\alpha y \leq x$ and $y\alpha x \leq x$. Since I is an ideal, $x\alpha y,\ y\alpha x \in I$. Thus I is a $r-$ideal of the $\Gamma-$incline M. □

The following example shows that converse of the Theorem 3.9 need not be true.

Example 3.10. Let $I=[0,1]$ be a set of real numbers between 0 and 1 with $x+y=\max\{x,y\}$ and $x \cdot y = xy$, where $\cdot$ is a usual multiplication for all $x, y \in I$. Then I is a incline.

Let M be the set of all 2×2 matrices whose elements be in I and $\Gamma = M$. Now we define as

$$A+B=(a_{ij}+b_{ij}) \text{ and } A \times B = (a_{ij}b_{ij}),$$

where $A=(a_{ij}) and B=(b_{ij})$ are in M. Then M is a $\Gamma-$incline.
Let $B=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a \neq b, a, b \in I\right\}$. Suppose $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in B, \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M$. Then $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} apx & bqy \\ 0 & 0 \end{pmatrix} \in B$.

Suppose $A=(a_{ij})$ and $B=(b_{ij}) \in M$. We define $A \leq B$ if and only if $a_{ij} \leq b_{ij}$, for all i, j. Then we have $\begin{pmatrix} 0.5 & 0.5 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0.5 & 0.6 \\ 0 & 0 \end{pmatrix} \in B$ but $\begin{pmatrix} 0.5 & 0.5 \\ 0 & 0 \end{pmatrix} \notin B$. Thus B is a $r-$ideal but not an ideal of the $\Gamma-$incline M.

Definition 3.11. A fuzzy subset μ of a $\Gamma-$incline M is called a fuzzy $\Gamma-$subincline of M, if the fuzzy subset μ satisfies:

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 3.12. A fuzzy $\Gamma-$subincline μ of a $\Gamma-$incline M is called a fuzzy ideal of M, if it satisfies

$$\mu(x) \geq \mu(y), \text{ if } x \leq y, \text{ for all } x, y \in M.$$

Definition 3.13. Let M be a $\Gamma-$incline and μ be a fuzzy ideal of a $\Gamma-$incline M. Then μ is called a fuzzy $k-$ideal of a $\Gamma-$incline M, if it satisfies

$$\mu(x) \geq \min\{\mu(x+y), \mu(y)\}, \quad \text{for all } x, y \in M.$$

Definition 3.14. A fuzzy ideal μ of a $\Gamma-$incline M is called a fuzzy $r-$left (right) ideal, if $\mu(x\alpha y) \geq \mu(y)$ $(\mu(x))$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 3.15. A fuzzy ideal μ of a $\Gamma-$incline M is called a fuzzy$r-$ideal of M, if μ is a left and right fuzzy $r-$ideal of M.

Example 3.16. Let M be a $\Gamma-$incline. Then set $I = \{a \mid a \leq x\}$ is an ideal of M. Define a fuzzy subset μ of M by:

$$\mu(x) = \begin{cases} 0.8, & \text{if } x \in I, \\ 0.3, & \text{otherwise.} \end{cases}$$

Then it is obvious that μ is a fuzzy ideal of $\Gamma-$incline.

Definition 3.17. For any $r-$ideal I of a $\Gamma-$incline M, the fuzzy subset defined by

$$\mu(x) = \begin{cases} s, & \text{if } x \in I \\ t, & \text{otherwise,} \end{cases}$$

for all $s,\ t \in [0,1]$ with $s > t$.

Theorem 3.18. *Let M be a $\Gamma-$incline and μ be a fuzzy $\Gamma-$subincline of M. Then μ is a fuzzy ideal of M if and only if μ is a fuzzy $k-$ideal of M.*

Proof. Suppose μ is a fuzzy ideal of the $\Gamma-$incline M. Let $x, y \in M$ and $x + y = z$. Then $z = x + y = x + x + y = x + z$. Thus $x \leq z$. So $\mu(x) \geq \mu(z) = \mu(x + y)$. Hence $\mu(x) \geq \min\{\mu(y), \mu(x+y)\}$. Therefore μ is a fuzzy $k-$ideal of M.

Conversely, suppose that μ is a fuzzy $k-$ideal of M. Let $x, y \in M$ and $x \leq y$. Then $x + y = y$ and

$$\begin{aligned} \mu(x) &\geq \min\{\mu(y), \mu(x+y)\} \\ &= \min\{\mu(y), \mu(y)\} \\ &= \mu(y). \end{aligned}$$

Thus μ is a fuzzy ideal of M. □

Let μ be a fuzzy subset of $\Gamma-$ incline M. Then the set $\{x \in M \mid \mu(x) \geq t, t \in [0,1]\}$ is called a level subset of M and it is denoted by μ_t.

Theorem 3.19. *A fuzzy subset μ of a $\Gamma-$incline M is a fuzzy ideal of M if and only if μ_t is an ideal of M.*

Corollary 3.20. *A fuzzy subset μ of a $\Gamma-$incline M is a fuzzy $r-$ideal of M if and only if μ_t is a $r-$ideal of M.*

Theorem 3.21. *A fuzzy subset μ of M is a fuzzy $r-$left ideal of a $\Gamma-$incline M if and only if the non-empty level subset μ_t is a $r-$left ideal of M.*

Proof. Suppose μ is a fuzzy $r-$left ideal of the $\Gamma-$incline M. Let $x \in M, a \in \mu_t$ and $\alpha \in \Gamma$. Then $\mu(x\alpha a) \geq \mu(a) \geq t$. Thus $x\alpha a \in \mu_t$. So level subset μ_t is a $r-$left ideal of M.

Conversely, suppose that level subset $\mu_t \neq \varnothing$ is a $r-$left ideal of M, for all $t \in [0,1]$. Assume that there exists $x, y \in M, \alpha \in \Gamma$ such that $\mu(x\alpha y) < \mu(y)$. Then $\mu(x\alpha y) < t_0 < \mu(y)$, where $t_0 = \frac{1}{2}(\mu(x\alpha y) + \mu(y))$. Thus $y \in \mu_{t_0}$ but $x\alpha y \notin \mu_{t_0}$, which is a contradiction. So $\mu(x\alpha y) \geq \mu(y)$, for all $x, y \in M$. Hence μ is a fuzzy $r-$left ideal of M. □

Corollary 3.22. *A fuzzy subset μ of a $\Gamma-$incline M is a fuzzy $r-$right ideal of M if and only if the non-empty level subset μ_t is a $r-$right ideal of M.*

Corollary 3.23. *A fuzzy subset μ of $\Gamma-$incline M is a fuzzy $r-$ideal of the $\Gamma-$incline M if and only if the non-empty level subset μ_t is a $r-$ideal of M.*

Theorem 3.24. *Let I be any $r-$left ideal of M. Then there exists a fuzzy $r-$left ideal μ of M such that $\mu_t = I$, for some $t \in [0,1]$.*

Proof. Let μ be a fuzzy subset of M defined by:

$$\mu(x) = \begin{cases} t \in (0,1], & \text{if } x \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is clear that $\mu_t = I$.

Suppose $s \in [0,1]$. Then

$$\mu_s = \begin{cases} \mu_0 = M, & \text{if } s = 0, \\ \mu_t = I, & \text{if } 0 < s \leq t, \\ \varnothing, & \text{if } t < s \leq 1. \end{cases}$$

Thus μ_s is a $r-$left ideal. Therefore μ is a fuzzy $r-$left ideal of M, by Theorem 3.21. □

Theorem 3.25. *Let N be a non-empty subset of M. A fuzzy subset μ of a $\Gamma-$incline M is defined by:*

$$\mu(x) = \begin{cases} s, & \textit{if } x \textit{ satisfies } a\alpha(a\alpha x) = 0, \textit{ for all } a \in N \textit{ and } \alpha \in \Gamma, \\ t, & \textit{otherwise,} \end{cases}$$

where $s > t, s, t \in [0,1]$. Then μ is a fuzzy $r-$right ideal of M.

Proof. Let $x, y \in M$ and $a\alpha(a\alpha x) = 0,\ a\alpha(a\alpha y) = 0$, for all $a \in N$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} a\alpha(a\alpha(x+y)) &= a\alpha(a\alpha x + a\alpha y)) \\ &= a\alpha(a\alpha x) + a\alpha(a\alpha y) \\ &= 0. \\ a\alpha(a\alpha(x\alpha y)) &= a\alpha(a\alpha x)\alpha y \\ &= 0\alpha y \\ &= 0. \\ \mu(x+y) &= s,\ \mu(x\alpha y) = s \\ \mu(x+y) &= \min\{\mu(x), \mu(y)\}. \\ \mu(x\alpha y) &= \min\{\mu(x), \mu(y)\}. \end{aligned}$$

Thus μ is a fuzzy $\Gamma-$subincline of M.

Let $x, y \in M, x \leq y$ and $a \in N$ and $y \in Annl(N)$. Then $x + y = y$

$$\begin{aligned} 0 &= a\alpha(a\alpha y) \\ &= a\alpha(a\alpha(x+y)) \\ &= a\alpha(a\alpha x) + a\alpha(a\alpha y) \\ &= a\alpha(a\alpha x) \\ \Rightarrow x &\in Annl(N). \end{aligned}$$

Thus $\mu(x) = \mu(y) = s$.

Suppose $y \notin Annl(N)$. Then $\mu(y) = t \leq \mu(x)$. Thus μ is a fuzzy ideal of M.

Suppose $x, y \in M,\ \alpha \in \Gamma$ and $x \in Annl(N)$. Then $a\alpha(a\alpha x\alpha y) \leq a\alpha(a\alpha x) = 0$. Thus $x\alpha y \in Annl(N)$. So $\mu(x\alpha y) = s = \mu(x)$. Otherwise, $\mu(x\alpha y) = t \leq \mu(x)$. Hence μ is a fuzzy $r-$right ideal of M. □

Definition 3.26. A function $f : M \to N$ where M and N are $\Gamma-$inclines is called a homomorphism of $\Gamma-$inclines, if
$f(a+b) = f(a)+f(b), f(a\alpha b) = f(a)\alpha f(b)$ and $a \leq b \Rightarrow f(a) \leq f(b),$ for all $a, b \in M, \alpha \in \Gamma$.

Definition 3.27. Let $\phi : M \to M$‘ be a homomorphism of $\Gamma-$inclines and f be a fuzzy subset of M. We define a fuzzy subset $\phi(f)$ of M‘ by:

$$\phi(f)(x) = \begin{cases} \sup\limits_{y \in \phi^{-1}(x)} f(y), & \text{if } \ \phi^{-1}(x) \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3.28. Let a function $\phi : M \to N$ be a homomorphism of $\Gamma-$inclines M, N and μ be a fuzzy subset of M. Then μ is said to be $\phi-$homomorphism invariant, if $\phi(a) \leq \phi(b)$, then $\mu(a) \leq \mu(b)$, for all $a, b \in M$.

Theorem 3.29. *Let M and N be $\Gamma-$inclines and $\phi : M \to N$ be an onto homomorphism. If f is a homomorphism $\phi-$invariant fuzzy $k-$ideal of M, then $\phi(f)$ is a fuzzy $k-$ideal of N.*

Proof. Let M and N be $\Gamma-$inclines, $\phi : M \to N$ be an onto homomorphism, f be a homomorphism $\phi-$invariant fuzzy ideal of M and $a \in M$. Suppose $x \in N, t \in \phi^{-1}(x)$ and $x = \phi(a)$. Then $a \in \phi^{-1}(x) \;\Rightarrow\; \phi(t) = x = \phi(a)$,

$$\begin{aligned} \phi(a\alpha b) =& \phi(a)\alpha\phi(b) \\ =& x\alpha y \\ \phi(f)(x\alpha y) =& f(a\alpha b) \\ \leq& \min\{f(a), f(b)\} \\ =& \min\{\phi(f(x)), \phi(f(y))\}. \end{aligned}$$

Since f is a $\phi-$invariant, $f(t) = f(a)$. Thus $\phi(f)(x) = \sup\limits_{t\in\phi^{-1}(x)} \{f(t)\} = f(a)$. So $\phi(f)(x) = f(a)$.

Let $x, y \in N$. Then there exist $a, b \in M$ such that $\phi(a) = x$ and $\phi(b) = y$. Thus $\phi(a + b) = x + y$. So

$$\phi(f)(x + y) = f(a + b) \leq \max\{f(a), f(b)\} = \min\{\phi(f)(x), \phi(f)(y)\}.$$

Since f is a fuzzy $k-$ideal, we have

$$f(a) \leq \max\{f(a + b), f(b)\}.$$

Hence $\phi(f)(x) \leq \max\{\phi(f)(x + y), \phi(f)(y)\}, \quad$ for all $\; x, y \in M$.

Let $x, y \in N$ and $x \leq y$. Then there exist $a, b \in M$ such that $\phi(a) = x, \phi(b) = y$ and $\phi(f)(x) = f(a), \phi(f)(y) = f(b)$. Thus

$$\begin{aligned} x \leq y \Rightarrow& \phi(a) \leq \phi(b) \\ \Rightarrow& f(a) \leq f(b) \\ \Rightarrow& \phi(f)(x) \leq \phi(f)(y). \end{aligned}$$

So $\phi(f)$ is a fuzzy $k-$ideal of N. Hence the theorem holds. □

Definition 3.30. Let M and N be be $\Gamma-$inclines and f be a function from M into N. If μ is a fuzzy ideal of N then the pre-image of μ under f is the fuzzy subset of M, defined by $f^{-1}(\mu)(x) = \mu(f(x))$, for all $x \in M$.

Theorem 3.31. *Every homomorphic pre-image of a fuzzy $r-$ideal of $\Gamma-$incline is a fuzzy $r-$ideal.*

Proof. Let $f : M \to N$ be a homomorphism of $\Gamma-$inclines and μ be a fuzzy ideal of N. Suppose $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned} f^{-1}(\mu)(x+y) &= \mu[f(x+y)] \\ &= \mu(f(x)+f(y)] \\ &\geq \min\{\mu(f(x)), \mu(f(y))\} \\ &= \min\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\} \\ f^{-1}(\mu)(x\alpha y) &= \mu[f(x\alpha y)] \\ &= \mu(f(x)\alpha f(y)] \\ &\geq \min\{\mu(f(x)), \mu(f(y))\} \\ &= \min\{f^{-1}(\mu)(x), f^{-1}(\mu)(y)\}. \end{aligned}$$

Thus $f^{-1}(\mu)$ is a $\Gamma-$subincline of M.

Suppose $x, y \in M, x \leq y$ and $\alpha \in \Gamma$. Then $x + y = y$. Thus $f(y) = f(x + y) = f(x) + f(y)$. So $f(x) \leq f(y)$. Hence $f^{-1}(\mu)(x) = \mu(f(x)) \geq \mu(f(y)) = f^{-1}(\mu)(y)$ and so $f^{-1}(\mu)(x\alpha y) = \mu[f(x\alpha y)] = \mu(f(x)\alpha f(y)] \geq \mu(f(y)) = f^{-1}(\mu)(y)$.

Similarly, $f^{-1}(\mu)(x\alpha y) \geq f^{-1}(\mu)(x)$. Therefore $f^{-1}(\mu)$ is a fuzzy $r-$ideal of M. □

Definition 3.32. A fuzzy $r-$ ideal μ of a $\Gamma-$incline M is called a characteristic fuzzy ideal of M, if $\mu(f(x)) = \mu(x)$, for all $x \in M$ and for all $f \in Aut(M)$.

Definition 3.33. A $r-$ideal I of a $\Gamma-$incline M is said to be characteristic $r-$ideal, if $f(I) = I$, for all $f \in Aut(M)$.

Theorem 3.34. *Let I be a non-empty subset of a $\Gamma-$incline M. Then I is a characteristic $r-$ideal of M if and only if its characteristic function χ_I is a characteristic fuzzy $r-$ideal of M.*

Proof. Let I be a characteristic $r-$ideal of M. Then I is a $r-$ideal of M.

If $x \in I$, then $\chi_I(x) = 1$. Thus for all $f \in Aut(M), f(I) = I \Rightarrow \chi_I(f(x)) = 1$.

If $x \notin I$, then $\chi_I(x) = 0$. Thus $f(x) \notin f(I),\ \chi_I(f(x)) = 0$.

So χ_I is a characteristic fuzzy $r-$ideal of M..

Conversely, suppose that χ_I is a characteristic fuzzy $r-$ideal of M. Then I is a $r-$ideal of M.

Suppose $x \in I$ and $f \in Aut(M)$. Then $\chi_I(x) = 1$. Thus $\chi_I(f(x)) = \chi_I(x) = 1$. So $f(x) \in I$. Hence $f(I) \subseteq I$.

Suppose $a \in I,\ f \in Aut(M)$. Then there exists $b \in M$ such that $f(b) = a$.

Suppose $b \notin I$. Then $\chi_I(a) = 0$ and $\chi_I(f(b)) = \chi_I(b) = 0$. Thus $a \notin I$, which is a contradiction. So $b \in I$. Hence $a = f(b) \in f(I)$. Therefore $I \subseteq f(I)$. □

Definition 3.35. A fuzzy $r-$left (right) ideal of a $\Gamma-$incline M is called a normal, if $\mu(0) = 1$.

Theorem 3.36. *Let μ be a fuzzy $r-$left (right) ideal of M. If $\mu^+(x) = \mu(x)+1-\mu(0)$, for all $x \in M$, then μ^+ is a normal fuzzy $r-$left ideal and $\mu \subseteq \mu^+$.*

Theorem 3.37. *Let $\mu^+(x) = \mu(x)+1-\mu(0)$, where μ is a fuzzy $r-$left (right) ideal of a $\Gamma-$incline M. Then*

(1) ${\mu^{+}}^{+} = \mu$,
(2) *If* $\mu^{+}(x) = 0$ *then* $\mu(x) = 0$,
(3) *If* μ *is normal, then* $\mu^{+} = \mu$.

Theorem 3.38. *If* μ *is a fuzzy ideal of a* $\Gamma-$*incline* M*, then* $\mu(x+y) = \min\{\mu(x), \mu(y)\}$*, for all* $x, y \in M$.

Proof. It is obvious that

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\},\ for\ all\ x,\ y \in M. \tag{3.1}$$

Let $x,\ y \in M$. Then

$$x + (x + y) = (x + x) + y = x + y,$$

$$y + (x + y) = (y + x) + y = (x + y) + y = x + y + y = x + y.$$

Thus $x \leq (x + y),\ y \leq (x + y)$.

Since μ is a fuzzy ideal of M, we have

$$\begin{aligned} x \leq (x + y) \Rightarrow \mu(x) \geq \mu(x + y) \\ y \leq (x + y) \Rightarrow \mu(y) \geq \mu(x + y). \end{aligned}$$

So

$$\mu(x+y) \leq \min\{\mu(x), \mu(y)\}. \tag{3.2}$$

From (3.1) and (3.1), $\mu(x + y) = \min\{\mu(x), \mu(y)\}$. This completes the proof. ☐

Definition 3.39. If μ is a fuzzy $r-$ ideal of $\Gamma-$ incline M and δ is an endomorphism of M, then we define a mapping $\mu^{\delta} : M \to [0, 1]$ by $\mu^{\delta}(x) = \mu(\delta(x))$, for all $x \in M$.

Theorem 3.40. *If* μ *is a fuzzy* $r-$*ideal and* δ *is an endomorphism of* $\Gamma-$*incline* M*, then* μ^{δ} *is a fuzzy* $r-$*ideal of* M.

Proof. Suppose $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} \mu^{\delta}(x + y) &= \mu(\delta(x + y)) \\ &= \mu(\delta(x) + \delta(y)) \\ &\geq \min\{\mu(\delta(x)), \mu(\delta(y))\}, \\ \mu^{\delta}(x\alpha y) &= \mu(\delta(x\alpha y)) \\ &= \min\{\mu(\delta(x)), \mu(\delta(y))\} \\ &= \mu(\delta(x)\alpha\delta(y)) \\ &\geq \min\{\mu(\delta(x)), \mu(\delta(y))\}. \end{aligned}$$

Thus μ is a fuzzy $\Gamma-$subincline of M.

Let μ be a fuzzy $r-$ideal of M and $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}\mu^\delta(x+y) &= \mu(\delta(x+y))\\ &= \mu(\delta(x)+\delta(y))\\ &\geq \min\{\mu(\delta(x)), \mu(\delta(y))\}\\ &= \min\{\mu^\delta(x), \mu^\delta\delta(y)\},\\ \mu^\delta(x\alpha y) &= \mu(\delta(x\alpha y))\\ &= \mu(\delta(x)\alpha\delta(y))\\ &\geq \max\{\mu(\delta(x)), \mu(\delta(y))\}\\ &= \max\{\mu^\delta(x), \mu^\delta\delta(y)\}.\end{aligned}$$

Thus μ^δ is a fuzzy $r-$ideal of M. □

Theorem 3.41. *Let M be a $\Gamma-$incline. Then μ is a characteristic fuzzy $r-$ ideal of M if and only if each level subset μ_t of M is a characteristic $r-$ideal of M.*

Proof. Let μ be a characteristic fuzzy $r-$ ideal of M. Then $\mu_t,\ t \in Im(\mu)$ is a $r-$ideal of M.

Suppose δ is an automorphism of M and $x \in \mu_t$. Since μ is a fuzzy characteristic, we have $\mu(\delta(x)) = \mu(x) \geq t$. Thus $\delta(x) \in \mu_t$. So $\delta(\mu_t) \subseteq \mu_t$.

Let $y \in \mu_t$. Then there exists $x \in M$ such that $\delta(x) = y$. Thus

$$\mu(x) = \mu(\delta(x)) = \mu(y) \geq t.$$

So $x \in \mu_t$ and thus $y = \delta(x) \in \delta(\mu_t) \Rightarrow \mu_t \subseteq \delta(\mu_t)$. Hence $\delta(\mu_t) = \mu_t$.
Therefore μ is a characteristic $r-$ideal of M.

Conversely, suppose that the level subset μ_t of M is a characteristic $r-$ideal of $M, x \in M,\ \delta$ is an automorphism of M and $\mu(x) = t$. Then $x \notin \mu_s$, for all $s > t$. Thus we have

$$\begin{aligned}\delta(\mu_t) &= \mu_t\\ \Rightarrow \delta(x) &\in \delta(\mu_t) = \mu_t\\ \Rightarrow \mu(\delta(x)) &\geq t = \mu(x).\end{aligned}$$

Suppose $\mu(\delta(x)) = s$ and $s > t$. Then $\delta(x) \in \mu_s = \delta(\mu_s)$. Thus $x \in \mu_s$, since δ is an automorphism. This a contradiction. So $\mu(\delta(x)) = t = \mu(x)$. Hence μ is a characteristic fuzzy $r-$ideal of M. This completes the proof. □

The following theorem can be verified easily.

Theorem 3.42. *Let M, S be $\Gamma-$inclines. If we define*
(1) $(x, y) + (z, w) = (x + z, y + w)$,
(2) $(x, y)\alpha(z, w) = (x\alpha z, y\alpha w)$, *for all* $(x, y), (z, w) \in M \times S, \alpha \in \Gamma$,
then $M \times S$ is a $\Gamma-$incline M.

Definition 3.43. Let μ and γ be fuzzy subsets of X. The cartesian product of μ and γ is defined by $\mu \times \gamma(x, y) = \min\{\mu(x), \gamma(y)\}$, for all $(x, y) \in X \times X$.

Definition 3.44. Let μ and γ be fuzzy $k-$ideals of a $\Gamma-$incline M. Then $\mu \times \gamma$ is said to be fuzzy left (right) $k-$ideal of $M \times M$, if

(i) $\mu \times \gamma((x,z)+(y,w)) \geq \min\{\mu(x+y),\ \gamma(z+w)\}$,
(ii) $\mu \times \gamma((x,z)\alpha(y,w)) \geq \min\left\{\mu \times \gamma(y,w),\ \mu \times \gamma(x,z)\right\}$,
(iii) if $(x,z) \leq (y,w)$, then $\mu \times \gamma(x,z) \geq \mu \times \gamma(y,w)$,
(iv) $\mu \times \gamma(x,z) \geq \min\{\mu \times \gamma(x+y,z+y), \mu \times \gamma(y,y)\}$, for all $(x,z),(y,w) \in M \times M, \alpha \in \Gamma$.

Theorem 3.45. *Let μ be a fuzzy subset of a $\Gamma-$incline M. If μ is a fuzzy $k-$ideal, then $\mu \times \mu$ is a fuzzy $k-$ideal of a $\Gamma-$incline $M \times M$.*

Proof. Suppose μ is a fuzzy $k-$ideal of a $\Gamma-$incline M and $(x,x) \in M \times M$. It is obvious that $\mu \times \mu$ is a fuzzy $k-$ideal of $\Gamma-$incline $M \times M$. Then

$$\begin{aligned}\mu \times \mu(x,z) &= \min\{\mu(x),\mu(z)\}\\ &\geq \min\{\min\{\mu(x+y),\mu(y)\},\min\{\mu(z+y),\mu(y)\}\}\\ &= \min\{\min\{\mu(x+y),\mu(z+y)\},\min\{\mu(y),\mu(y)\}\}\\ &= \min\{\mu \times \mu(x+y,z+y),\mu \times \mu(y,y)\},\end{aligned}$$

for all $(x,z) \in M \times M, y \in M$. Thus $\mu \times \mu$ is a fuzzy $k-$ideal of M. □

Theorem 3.46. *Let μ and γ be fuzzy $k-$ideals of a $\Gamma-$incline M. Then $\mu \times \gamma$ is a fuzzy $k-$ideal of $\Gamma-$semiring $M \times M$.*

Proof. Let μ and γ be fuzzy $k-$ideals of the $\Gamma-$incline M and $(x_1,x_2),(y_1,y_2) \in M \times M, \alpha \in \Gamma$. Then

$$\begin{aligned}\mu \times \gamma((x_1,x_2)+(y_1,y_2)) &= \mu \times \gamma(x_1+y_1,x_2+y_2)\\ &= \min\{\mu(x_1+y_1),\gamma(x_2+y_2)\}\\ &\geq \min\{\min\{\mu(x_1),\mu(y_1)\},\min\{\gamma(x_2),\gamma(y_2)\}\}\\ &= \min\{\min\{\mu(x_1),\gamma(x_2)\},\min\{\mu(y_1),\gamma(y_2)\}\}\\ &= \min\{\mu \times \gamma(x_1,x_2),\mu \times \gamma(y_1,y_2)\},\\ \mu \times \gamma((x_1,x_2)\alpha(y_1,y_2)) &= \mu \times \gamma(x_1\alpha y_1,x_2\alpha y_2)\\ &= \min\{\mu(x_1\alpha y_1),\gamma(x_2\alpha y_2)\}\\ &\geq \min\{\mu(y_1),\gamma(y_2)\}\\ &= \mu \times \gamma(y_1,y_2),\end{aligned}$$

$$\begin{aligned}\mu \times \gamma(x,z) &= \min\{\mu(x),\gamma(z)\}\\ &\geq \min\{\min\{\mu(x+y),\mu(y)\},\min\{\gamma(z+y),\gamma(y)\}\}\\ &= \min\{\min\{\mu(x+y),\gamma(z+y)\},\min\{\mu(y),\gamma(y)\}\}\\ &= \min\{\mu \times \gamma(x+y,z+y),\mu \times \gamma(y,y)\},\end{aligned}$$

for all $(x,z) \in M \times M, y \in M$.

Suppose $(x_1,x_2) \leq (y_1,y_2)$. Then

$$\begin{aligned}\mu \times \gamma(x_1,x_2) &= \min\{\mu(x_1),\gamma(x_2)\}\\ &\geq \min\{\mu(y_1),\gamma(y_2)\}\\ &= \mu \times \gamma(y_1,y_2).\end{aligned}$$

Thus $\mu \times \gamma$ is a fuzzy $k-$ideal of the $\Gamma-$incline $M \times M$. □

Corollary 3.47. *Let M_1 and M_2 be $\Gamma-$inclines. If μ_1 and μ_2 are fuzzy left $k-$ideals of M_1 and M_2 respectively, then $\mu = \mu_1 \times \mu_2$ is a fuzzy left $k-$ideal of the $\Gamma-$incline $M_1 \times M_2$.*

Definition 3.48. A fuzzy relation on any set X is a fuzzy subset, if $\mu : X \times X \to [0,1]$.

Definition 3.49. Let γ be a fuzzy subset on a set M. Then the strongest fuzzy relation μ_γ on M is a fuzzy relation which is defined by:

$$\mu_\gamma(x,y) = min(\gamma(x), \gamma(y)), \text{ for all } (x,y) \in M \times M.$$

Theorem 3.50. *Let γ be a fuzzy subset of a $\Gamma-$incline M. Then γ is a fuzzy left $k-$ideal of M if and only if the strongest fuzzy relation μ_γ on M is a fuzzy left $k-$ideal of the $\Gamma-$incline $M \times M$.*

Proof. Suppose γ is a fuzzy $k-$ideal of the $\Gamma-$incline M. Then

$$\begin{aligned}
\mu_\gamma(x_1, x_2) &= min(\gamma(x_1), \gamma(x_2)) \\
&= min(min(\gamma(x_1+y_1), \gamma(y_1)), min(\gamma(x_2+y_2), \gamma(y_2))) \\
&= min(min(\gamma(x_1+y_1), \gamma(x_2+y_2)), min(\gamma(y_1), \gamma(y_2))) \\
&= min(\mu_\gamma(x_1+y_1, x_2+y_2), \mu_\gamma(y_1, y_2)), \\
min(\mu_\gamma(x_1,x_2), \mu_\gamma(x_1,x_2)) &= min(min(\gamma(x_1), \gamma(x_2)), min(\gamma(x_1), \gamma(x_2))) \\
&= min(min(\gamma(x_1), \gamma(x_1)), min(\gamma(x_2), \gamma(x_2))) \\
&= min(\gamma(x_1), \gamma(x_2)) \\
&= \mu_\gamma(x_1, x_2), \text{ for all } (x_1,x_2),(y_1,y_2) \in M \times M.
\end{aligned}$$

Suppose $(x_1,x_2),(y_1,y_2) \in M \times M$ and $(x_1,x_2) \le (y_1,y_2)$. Then $x_1 \le y_1, x_2 \le y_2$. Thus $\gamma(x_1) \ge \gamma(y_1), \gamma(x_2) \ge \gamma(y_2)$. So $\min\{\gamma(x_1), \gamma(x_2)\} \ge \min\{\gamma(y_1), \gamma(y_2)\}$. Hence $\mu_\gamma(x_1,x_2) \ge \mu_\gamma(y_1,y_2)$. Therefore μ_γ is a fuzzy left $k-$ ideal of the $\Gamma-$incline M.

Conversely, suppose that μ_γ is a fuzzy $k-$ideal of the $\Gamma-$incline M, $x,y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
\gamma(x+y) &= \min\{\gamma(x+y), \gamma(x+y)\} \\
&= \mu_\gamma(x+y, x+y) \\
&= \mu_\gamma((x,x)+(y,y)) \\
&\ge \min\{\mu_\gamma(x,x), \mu_\gamma(y,y)\} \\
&= \min\{\min\{\gamma(x), \gamma(x)\}, \min\{\gamma(y), \gamma(y)\}\} \\
&= \min\{\min\{\gamma(x), \gamma(y)\}, \min\{\gamma(x), \gamma(y)\}\} \\
&= \min\{\mu_\gamma(x,y), \mu_\gamma(x,y)\} \\
&= \min\{\mu_\gamma(x,y)\} \\
&= \min\{\gamma(x), \gamma(y)\}, \\
\gamma(x\alpha y) &= \min\{\gamma(x\alpha y), \gamma(x\alpha y)\} \\
&= \mu_\gamma(x\alpha y, x\alpha y) \\
&= \mu_\gamma((x,x)\alpha(y,y)) \\
&\ge \mu_\gamma(y,y) \\
&= \min\{\gamma(y), \gamma(y)\} = \gamma(y), \\
\gamma(x) &= \min\{\gamma(x), \gamma(x)\} \\
&= \mu_\gamma(x,x) \\
&\ge \min\{\mu_\gamma(x+y, x+y), \mu_\gamma(y,y)\} \\
&= \min\{\min\{\gamma(x+y), \gamma(x+y)\}, \min\{\gamma(y), \gamma(y)\}\} \\
&= \min\{\gamma(x+y), \gamma(y)\}.
\end{aligned}$$

Suppose $x,y \in M$ and $x \le y$. Then $(x,x) \le (y,y)$. Thus $\mu_\gamma(x,x) \ge \mu_\gamma(y,y)$. So $\min\{\gamma(x), \gamma(x)\} \ge min\{\gamma(y), \gamma(y)\}$. Hence $\gamma(x) \ge \gamma(y)$. Therefore γ is a fuzzy $k-$ideal of the $\Gamma-$incline M. □

Conclusion

In this paper, we introduced the notion of fuzzy $r-$ideal and fuzzy $k-$ideal in $\Gamma-$incline. We studied the properties of fuzzy $k-$ideals and fuzzy $r-$ideals. We proved that every homomorphic pre image of fuzzy $r-$ideal is fuzzy $r-$ideal.

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References

[1] S. S. Ahn, Y. B. Jun and H. S. Kim, Ideals and quotients of incline algebras, Comm. Korean Math. Soc. 16 (2001) 573–583.
[2] S. S. Ahn and H. S. Kim, On $r-$ideals in incline algebras, Comm. Korean Math. Soc. 17 (2) (2002) 229–235.
[3] Z. Q. Cao, K. H. Kim and F. W. Roush, Incline algebra and applications, John Wiley and Sons, New York 1984.
[4] T. K. Dutta and S. Kar, On regular ternary semirings, Advances in algebra, Proceedings of the ICM Satellite conference in algebra and related topics, World scientific (2003) 343–355.
[5] R. A. Good and D. R. Hughes, Associated groups for a semigroup, Bull. Amer. Math. Soc. 58 (1952) 624–625.
[6] M. Henriksen, Ideals in semirings with commutative addition, Amer. Math. Soc. Notices 6 (1958) 321.
[7] K. Iseki, Quasi-ideals in semirings without zero, Proc. Japan Acad. 34 (1958) 79–84.
[8] R. D. Jagatap and Y. S. Pawar, Quasi-ideals and minimal quasi-ideals in $\Gamma-$semirings, Novi Sad J. Math. 39 (2) (2009) 79–87.
[9] Y. B. Jun and C. Y. Lee, Fuzzy $\Gamma-$rings, Pusan kyongnan Math. J. 8 (1991) 163–170.
[10] N. Kuroki, On fuzzy semigroups, Inform. Sci. 53 (1991) 203–236.
[11] S. Lajos and F. A. Szasz, On the bi-ideals in associative ring, Proc. Japan Acad. 46 (1970) 505–507.
[12] H. Lehmer, A ternary analogue of abelian groups, Amer. J. of Math. 59 (1932) 329–338.
[13] W. J. Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy sets and Systems 8 (1982) 133–139.
[14] D. Mandal, Fuzzy ideals and fuzzy interior ideals in ordered semirings, Fuzzy info. and Engg. 6 (2014) 101–114.
[15] M. Murali Krishna Rao, $\Gamma-$semirings-I, Southeast Asian Bull. Math. 19 (1) (1995) 49–54.
[16] M. Murali Krishna Rao, $\Gamma-$semirings-II, Southeast Asian Bull. Math. 21 (5) (1997) 281–287.
[17] M. Murali Krishna Rao, The Jacobson Radical of a $\Gamma-$semiring, Southeast Asian Bull. of Math. 23 (1999) 127–134.
[18] M. Murali Krishna Rao, Fuzzy soft $\Gamma-$semiring and fuzzy soft $k-$ideal over $\Gamma-$semiring, Ann. Fuzzy Math. Infom. 9 (2) (2015) 12–25.
[19] M. Murali Krishna Rao, T-fuzzy ideals in ordered $\Gamma-$semirings, Ann. fuzzy Math. Inform. 13 (2) (2017) 253–276.
[20] Marapureddy Murali Krishna Rao, Bi-quasi-ideals and fuzzy bi-quasi ideals of $\Gamma-$ semigroups, Bull. Int. Math. Virtual Inst. 7 (2)(2017) 231–242.
[21] M. Murali Krishna Rao, $\Gamma-$ semiring with identity, Discuss. Math. Gen. Algebra Appl. 37 (2017) 189–207.
[22] M. Murali Krishna Rao, Left bi-quasi ideals of semirings, Bull. Int. Math. Virtual Inst. 8 (2018) 45–53.
[23] M. Murali Krishna Rao, Ideals in ordered $\Gamma-$semirings, Discuss. Math. Gen. Algebra Appl. 38 (2018) 47–68.

[24] M. Murali Krishna Rao, Bi-interior ideals in semigroups, Discuss. Math. Gen. Algebra Appl. 38 (1) (2018) 69–78.
[25] M. Murali Krishna Rao, Bi-interior ideals in $\Gamma-$semirings, Discuss. Math. Gen. Algebra Appl. 38 (2)(2018) 239–254.
[26] M. Murali Krishna Rao and B. Venkateswarlu, Regular $\Gamma-$incline and field $\Gamma-$semiring, Novi sad. J. Math. 45 (2) (2015) 155–171.
[27] M. Murali Krishna Rao and B. Venkateswarlu, $L-$ fuzzy ideals in $\Gamma-$semirings, Ann. fuzzy Math. Inform. 10 (1) (2015) 1–16.
[28] M. Murali Krishna Rao, B. Venkateswarlu and N. Rafi, Left bi-quasi-ideals of $\Gamma-$semirings, Asia Pacific J. of Math. 4 (2) (2017) 144–153.
[29] N. Nobusawa, On a generalization of the ring theory, Osaka. J. Math. 1 (1964) 81 – 89.
[30] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512 – 517.
[31] M. K. Sen, On $\Gamma-$semigroup, Proc. of Inter. Con. of Alg. and its Appl. Decker Publication, New York (1981) 301–308.
[32] T. Srinivas and T. Nagaiah, Some results on $T-$Fuzzy ideals of Gamma near-rings, Ann. fuzzy Math. Inform. 4 (2) (2012) 305–319.
[33] O. Steinfeld, Uher die quasi ideals, Von halbgruppend Publ. Math. Debrecen 4 (1956) 262 – 275.
[34] U. M. Swamy and K. L. N. Swamy, Fuzzy prime ideals of rings, J. Math. Anal. Appl. 134 (1988) 94– 103.
[35] H. S. Vandiver, Note on a simple type of algebra in which cancelation law of addition does not hold, Bull. Amer. Math. Soc. (N.S.) 40 (1934) 914–920.
[36] L. A. Zadeh, Fuzzy sets, Information and control 8 (1965) 338–353.

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