Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Arithmetic rough statistical convergence for triple sequences

A. Esi, N. Subramanian, A. Esi

Received 7 January 2019; Revised 29 January 2019; Accepted 19 February 2019

Abstract. In this paper, using the concept of natural density, we introduce the notion of arithmetic rough statistical convergence of triple sequences. We define the set of arithmetic rough statistical limit points of a triple sequence and obtain arithmetic rough statistical convergence criteria associated with this set. Later, we prove this set is closed and convex and also examine the relations between the set of arithmetic rough statistical cluster points and the set of arithmetic rough statistical limit points of a triple sequence.

2010 AMS Classification: 40F05, 40J05, 40G05

Keywords: Arithmetic rough statistical convergence, Summability, natural density, Triple sequences, Chi sequence.

Corresponding Author: A. Esi (aesi23@hotmail.com)

1. Introduction

A triple sequence $x = (x_{mnk})$ defined on $\mathbb{N}$ and $(u, v, w) \in \mathbb{N}$, the notation $\sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk}$ means the finite sum of all the numbers x_{mnk} as (m, n, k) ranges over the integers that divide m, n, k including 1 and (m, n, k). In general for integers (m, n, k) and (u, v, w) we write $m \mid u, n \mid v$ and $k \mid w$ to mean m divides u, n divides v and k divides w or $"'m$ is a multiple of u, n is a multiple of v and k is a multiple of w. We use the symbol $\langle a, u \rangle$ to denote the greatest common divisor of two integers a and u.

Now we introduced the notions arithmetic rough summability and arithmetic rough convergence as follows:

Definition 1.1. A triple sequence $x = (x_{mnk})$ defined on $\mathbb{N}$ is called arithmetically rough summable, if for each $\epsilon > 0$, and r be a positive real number there is an integer

(u, v, w) such that for every integer (a, b, c) we have

$$\left| \sum_{m|u} \sum_{n|v} \sum_{k|w} x_{mnk} - \sum_{m|\langle a,u\rangle} \sum_{n|\langle b,v\rangle} \sum_{k|\langle c,w\rangle} x_{mnk} \right| < r + \epsilon.$$

Definition 1.2. A triple sequence $y = (y_{mnk})$ is called arithmetically rough convergent, if for each $\epsilon > 0$ and r be a positive number there is an integer u, v and w such that for every integer a, b and c we have

$$\left| y_{abc} - y_{\langle a,u\rangle \langle b,v\rangle \langle c,w\rangle} \right| < r + \epsilon.$$

The idea of statistical convergence was introduced by Steinhaus [13] and also independently by Fast [8] for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}\,(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [11, 12], Esi et al. [4, 5, 6, 7], Dutta et al. [2], Subramanian et al. [14], Debnath et al. [3], Tripathy and Goswami [15, 16, 17, 18] and many others.

Let K be a subset of the set $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and let us denote the set $\{(m, n, k) \in K : m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta\,(K) = \lim_{uvw\to\infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta\,(K^c) = 1 - \delta\,(K)$ where $K^c = \mathbb{N}\backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta\,(K_1) \leq \delta\,(K_2)$.

Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$, $m, n, k \in \mathbb{N}$.

A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - \lim x = 0$, provided that the set

$$\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - 0| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the ϵ-neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m, n, k except a set of natural density zero, then we say that the triple sequence x satisfies P for "almost all (m, n, k)" and we abbreviate this by "a.a. (m, n, k)".

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set $K = \left\{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\right\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non thin sub sequence of a triple sequence x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\}$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [10], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [9] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence. Throughout the paper r be a nonnegative real number.

In this paper, we introduce the notion of arithmetic rough statistical convergence of triple sequences. Defining the set of arithmetic rough statistical limit points of a triple sequence, we obtain arithmetic rough statistical convergence criteria associated with this set.

2. Arithmetic rough convergent sequence space Arc

In this section we study certain algebraic and toplogical properties of arithmetic rough convergent sequence space Arc defined as follows:

Definition 2.1. A arithmetic triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ is said to be rough convergent (r-convergent) to l (Pringsheim's sense), denoted as $\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \to^r l$, provided that

$$\begin{aligned} Arc = \{&\forall\epsilon > 0, \exists\, i_\epsilon, (u,v,w) \in \mathbb{N} : m,n,k \geq i_\epsilon \Longrightarrow \\ &\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| < r + \epsilon, \forall m,n,k\right\} \end{aligned} \tag{2.1}$$

or equivalently, if

$$\limsup \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \leq r. \tag{2.2}$$

Here r is called the roughness of degree. If we take $r = 0$, then we obtain the ordinary arithmetic convergence of a triple sequence.

Definition 2.2. It is obvious that the r-limit set of a arithmetic triple sequence is not unique. The r-limit set of the arithmetic triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ is defined as

$$LIM^r x_{mnk} := \left\{l \in \mathbb{R} : \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \to^r l\right\}.$$

Definition 2.3. A arithmetic triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ is said to be r-convergent, if $LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi$. In this case, r is called the convergence degree of the arithmetic triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$. For $r = 0$, we get the ordinary convergence.

Definition 2.4. A arithmetic triple sequence $\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ is said to be r-statistically convergent to l, denoted by $\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \to^{rst} l$, provided that the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq r + \epsilon\right\}$$

has natural density zero for every $\epsilon > 0$, or equivalently, if the condition

$$st - \limsup \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \leq r$$

is satisfied.

In addition, we can write $\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \to^{rst} l$ if and only if the inequality

$$\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| < r + \epsilon$$

holds for every $\epsilon > 0$ and almost all (m,n,k). Here r is called the roughness of degree. If we take $r = 0$, then we obtain the statistical convergence of arithmetic triple sequences.

In a similar fashion to the idea of classical rough convergence, the idea of rough statistical convergence of a arithmetic triple sequence can be interpreted as follows:

Assume that a arithmetic triple sequence $y = \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ is statistically convergent and cannot be measured or calculated exactly; one has to do with an approximated (or statistically approximated)arithmetic triple sequence

$$x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$$

satisfying $\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right| \leq r$, for all m,n,k (or for almost all (m,n,k), i.e.,

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right| > r\right\}\right) = 0.$$

Then the arithmetic triple sequence x is not statistically convergent any more, but as the incluson

$$\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq \epsilon\right\}$$
$$\supseteq \left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq r + \epsilon\right\}$$

holds and we have

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq \epsilon\right\}\right) = 0,$$

i.e., we get

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq r + \epsilon\right\}\right) = 0,$$

i.e., the arithmetic triple sequence spaces x is r-statistically convergent in the sense of Definition 2.3.

In general, the rough statistical limit of a arithmetic triple sequence may not unique for the roughness degree $r > 0$. So we have to consider the so called r-statistical limit set of a arithmetic triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, which is defined by

$$st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = \left\{L \in \mathbb{R} : \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \to^{rst} l\right\}.$$

The arithmetic triple sequence x is said to be r-statistically convergent, provided that

$$st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi.$$

It is clear that if $st–LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi$ for a arithmetic triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ of real numbers, then we have

$$\begin{aligned} & st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \\ & = \left[st - \limsup \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - r, st - \liminf \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) + r\right]. \end{aligned}$$

We know that $LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = \phi$ for an unbounded arithmetic triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$. But such a arithmetic triple sequence might be rough statistically convergent. For instance, define

$$\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = \begin{cases} (-1)^{mnk}, & \text{if } (m,n,k) \neq (i,j,\ell)^2 \,(i,j,\ell \in \mathbb{N}), \\ (mnk), & \text{otherwise} \end{cases}$$

in $\mathbb{R}$. Because the set $\{1, 64, 739, \ldots\}$ has natural density zero, we have

$$st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = \begin{cases} \phi, & \text{if } r < 1, \\ [1-r, r-1], & \text{otherwise} \end{cases}$$

and $LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = \phi$ for all $r \geq 0$.

As can be seen by the example above, the fact that

$$st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi$$

does not imply $LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi$. Because a finite set of natural numbers has natural density zero, $LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi$ implies $st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi$. Therefore, we get

$$LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \subseteq st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right).$$

This obvious fact means

$$\left\{r \geq 0 : LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi\right\} \subseteq \left\{r \geq 0 : st - LIM^r x \neq \phi\right\}$$

in this language of sets and yields immediately

$$\begin{aligned} & \inf \left\{r \geq 0 : LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi\right\} \\ & \qquad \geq \inf \left\{r \geq 0 : st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi\right\}. \end{aligned}$$

Moreover, it also yields directly

$$\operatorname{diam}\left(LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right) \leq \operatorname{diam}\left(st - LIM^r \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right).$$

3. Main Results

Theorem 3.1. *For a arithmetic triple sequence spaces* $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, *we have*

$$\text{diam}\left(st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right) \leq 2r.$$

In general

$$\text{diam}\left(st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right)$$

has an upper bound.

Proof. Assume that $\text{diam}\left(st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right) > 2r$. Then there exist $w, y \in st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ such that $|w-y| > 2r$. Take $\epsilon \in \left(0, \frac{|w-y|}{2} - r\right)$. Because $w, y \in st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, we have $\delta(K_1) = 0$ and $\delta(K_2) = 0$ for every $\epsilon > 0$ where

$$K_1 = \left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - w\right| \geq r + \epsilon\right\}$$

and

$$K_2 = \left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - y\right| \geq r + \epsilon\right\}.$$

Using the properties of natural density, we get $\delta\left(K_1^c \bigcap K_2^c\right) = 1$. Thus we can write

$$\begin{aligned} |w-y| &\leq \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - w\right| + \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - y\right| \\ &< 2(r+\epsilon) = 2\left(\frac{|w-y|}{2}\right) = |w-y| \end{aligned}$$

for all $(m,n,k) \in K_1^c \bigcap K_2^c$, which is a contradiction.

Now let us prove the second part of the theorem. Consider a arithmetic triple sequence $x = (x_{mnk})$ such that $st - \lim\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = l$. Let $\epsilon > 0$. Then we can write

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq \epsilon\right\}\right) = 0.$$

We have

$$\begin{aligned} \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - y\right| &\leq \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| + |l-y| \\ &\leq \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| + r \end{aligned}$$

for each $y \in \bar{B}_r(l) = \left\{y \in \mathbb{R}^3 : |y-l| \leq r\right\}$.

Thus we get $|l-y| < r + \epsilon$ for each $(m,n,k) \in \{(m,n,k) \in \mathbb{N}^3 : |(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}) - l| < \epsilon\}$. Because the arithmetic triple sequence spaces x is statistically convergent to l, we have

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| < \epsilon\right\}\right) = 1.$$

So we get $y \in st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$. Hence, we can write

$$st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = \bar{B}_r(l).$$

Because $\text{diam}\left(\bar{B}_r(l)\right) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ is not an lower bound. □

Theorem 3.2. *Let $r > 0$. Then a arithmetic triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ is r-statistically convergent to l if and only if there exists a triple sequence $y = \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ such that $st - \lim\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = l$ and*

$$\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right| \leq r$$

for each $(m,n,k) \in \mathbb{N}^3$.

Proof. **Necessity:** Assume that $\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \rightarrow^{rst} l$. Then we have

$$st - \limsup \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \leq r. \tag{3.1}$$

Now, define

$$\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = \begin{cases} l, \text{ if } \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \leq r \\ x_{mnk} + r\left(\frac{l - \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}{\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right|}\right), \text{ otherwise} \end{cases}$$

Then, we write

$$\left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| =$$
$$\begin{cases} |l - l|, \text{ if } \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \leq r, \\ \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| + r\left(\frac{|l-l| - \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right|}{\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right|}\right), \text{ otherwise} \end{cases}$$

(i.e)

$$\left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| =$$
$$\begin{cases} 0, \text{ if } \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \leq r \\ \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| - r\left(\frac{\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right|}{\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right|}\right), \text{ otherwise} \end{cases}$$

(i.e)

$$\left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| = \begin{cases} 0, \text{ if } \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \leq r \\ \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| - r, \text{ otherwise} \end{cases}.$$

We have

$$\left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| - r$$
$$\Longrightarrow \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l - \left(\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right) + l\right| \leq r$$

$$\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right| \leq r \tag{3.2}$$

for all $m, n, k \in \mathbb{N}$. By equation (3.1) and by definition of $\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, we get

$$st - \limsup \left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| = 0$$
$$\Longrightarrow st - \lim\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \rightarrow^{r} l.$$

Sufficiency: Because $st - \lim\ \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) = l$, we have

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq \epsilon\right\}\right) = 0$$

for each $\epsilon > 0$. It is easy to see that the inclusion

$$\begin{aligned}&\left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq \epsilon\right\}\\ &\supseteq \left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq r+\epsilon\right\}\end{aligned}$$

holds. Because $\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq \epsilon\right\}\right) = 0$, we get

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq r+\epsilon\right\}\right) = 0.$$

□

Remark 3.3. If we replace the condition

$$\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right| \leq r$$

for all $m,n,k\in\mathbb{N}$ in the hypothesis of the Theorem 3.2 with the condition

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right| > r\right\}\right) = 0$$

is valid.

Theorem 3.4. *For an arbitrary* $c \in \Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}$ *of arithmetic triple sequence* $x = \left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ *we have* $|l-c|\leq r$ *for all* $l\in st\text{–}LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$.

Proof. Assume on the contrary that there exist a point $c \in \Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}$ and $l \in st\text{–}LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ such that $|l-c|>r$. Define $\epsilon := \frac{|l-c|-r}{3}$. Then

$$\begin{aligned}&\left\{(m,n,k)\in\mathbb{N}^3 : |l-c| < \epsilon\right\}\\ &\subseteq \left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq r+\epsilon\right\}.\end{aligned} \tag{3.3}$$

Since $c\in\Gamma_x$, we have

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - c\right| < \epsilon\right\}\right) \neq 0.$$

Thus by (3.3), we get

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq r+\epsilon\right\}\right) \neq 0,$$

which contradicts the fact $l \in st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$. □

Proposition 3.5. *If a arithmetic triple sequence* $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ *is analytic, then there exists a non-negative real number* r *such that* $st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi$.

Proof. If we take the arithmetic triple sequence is to be statistically analytic, then the of proposition holds. Thus we have the following theorem. □

Theorem 3.6. *A arithmetic triple sequence* $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ *is statistically analytic if and only if there exists a non-negative real number* r *such that* $st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \neq \phi$.

Proof. Since the arithmetic triple sequence x is statistically analytic, there exists a positive real number M such that

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3:\left|\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right|^{1/m+n+k}\geq M\right\}\right)=0.$$

Define

$$r^{'}=\sup\left\{\left|\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right|^{1/m+n+k}:(m,n,k)\in K^c\right\},$$

where

$$K=\left\{(m,n,k)\in\mathbb{N}^3:\left|\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right|^{1/m+n+k}\geq M\right\}.$$

Then the set $st-LIM^{r^{'}}\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ contains the origin of $\mathbb{R}$. Thus we have $st-LIM^{r}\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\neq\phi$.

If $st-LIM^{r}\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\neq\phi$ for some $r\geq 0$, then there exists l such that $l\in st-LIM^{r}\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, i.e.,

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3:\left|\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)-l\right|^{1/m+n+k}\geq r+\epsilon\right\}\right)=0$$

for each $\epsilon>0$. Thus we say that almost all $\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ are contained in some ball with any radius greater than r. So the arithmetic triple sequence x is statistically analytic. □

Remark 3.7. If $x^{'}=\left(x_{m_in_jk_\ell}-x_{\langle m_i,u\rangle\langle n_j,v\rangle\langle k_\ell,w\rangle}\right)$ is a sub sequence of $x=\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, then $LIM^r x\subseteq LIM^r x^{'}$. But it is not valid for statistical convergence. For example, define

$$\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)=\begin{cases}(mnk), & \text{if } (m,n,k)=(i,j,\ell)^2\,(i,j,\ell\in\mathbb{N}),\\ 0, & \text{otherwise}\end{cases}$$

of real numbers. Then the arithmetic triple sequence $x^{'}=(1,64,739,\ldots)$ is a subsequence of x. We have $st-LIM^r x=[-r,r]$ and $st-LIM^r x^{'}=\phi$.

Theorem 3.8. *Let* $x^{'}=\left(x_{m_in_jk_\ell}-x_{\langle m_i,u\rangle\langle n_j,v\rangle\langle k_\ell,w\rangle}\right)$ *is a non-thin subsequence of arithmetic triple sequence* $x=\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, *then* $st-LIM^r x\subseteq st-LIM^r x^{'}$.

Proof. Easy, so omitted. □

Theorem 3.9. *The* r*-statistical limit set of a arithmetic triple sequence* $x=\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ *is closed.*

Proof. If $st-LIM^{r}\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\neq\phi$, then it is true. Assume that $st-LIM^{r}\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\neq\phi$. Then we can choose a arithmetic triple sequence spaces $\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\subseteq st-LIM^{r}\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ such that
$\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\to^r l$ as $m,n,k\to\infty$. If we prove that $l\in st-LIM^{r}\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, then the proof will be complete.

Let $\epsilon>0$ be given. Because $\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\to^r l$,

$\forall \epsilon > 0, \exists\, i_\epsilon \in \mathbb{N} : m, n, k \geq i_\epsilon$ such that

$$\left|\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| < \frac{\epsilon}{2} \text{ for all } m, n, k \geq i_\epsilon.$$

Now choose an $(m_0, n_0, k_0) \in \mathbb{N}$ such that $m_0, n_0, k_0 \geq i_\epsilon$. Then we can write

$$\left|\left(y_{m_0 n_0 k_0} - y_{\langle m_0,u\rangle\langle n_0,v\rangle\langle k_0,w\rangle}\right) - l\right| < \frac{\epsilon}{2}.$$

On the other hand, because

$$\left(y_{mnk} - y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \subseteq st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right),$$

we have $\left(y_{m_0 n_0 k_0} - y_{\langle m_0,u\rangle\langle n_0,v\rangle\langle k_0,w\rangle}\right) \in st{-}LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, namely,

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{m_0 n_0 k_0} - y_{\langle m_0,u\rangle\langle n_0,v\rangle\langle k_0,w\rangle}\right)\right| \geq r + \frac{\epsilon}{2}\right\}\right) = 0. \tag{3.4}$$

Now let us show that the inclusion

$$\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| < r + \epsilon\right\} \supseteq \left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{m_0 n_0 k_0} - y_{\langle m_0,u\rangle\langle n_0,v\rangle\langle k_0,w\rangle}\right)\right| < r + \frac{\epsilon}{2}\right\} \tag{3.5}$$

holds. Take

$$(i,j,\ell) \in \left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{m_0 n_0 k_0} - y_{\langle m_0,u\rangle\langle n_0,v\rangle\langle k_0,w\rangle}\right)\right| < r + \frac{\epsilon}{2}\right\}.$$

Then we have

$$\left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - \left(y_{m_0 n_0 k_0} - y_{\langle m_0,u\rangle\langle n_0,v\rangle\langle k_0,w\rangle}\right)\right| < r + \frac{\epsilon}{2}$$

and thus

$$|x_{ij\ell} - l| \leq \left|\left(x_{ij\ell} - x_{\langle i,u\rangle\langle j,v\rangle\langle \ell,w\rangle}\right) - y_{m_0 n_0 k_0}\right| + |y_{m_0 n_0 k_0} - l| < r + \frac{\epsilon}{2} + \frac{\epsilon}{2} < r + \epsilon$$

i.e.,

$$(i,j,\ell) \in \left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| < r + \epsilon\right\}$$

which proves the equation (3.5). So the natural density of the set on the LHS of equation (3.5) is equal to 1. Hence we get

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - l\right| \geq r + \epsilon\right\}\right) = 0.$$

□

Theorem 3.10. *The r-statistical limit set of a arithmetic triple sequence is convex.*

Proof. Let $y_1, y_2 \in st - LIM^r\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ for the triple sequence $x = \left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ and let $\epsilon > 0$ be given. Define

$$K_1 = \left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - y_1\right| \geq r + \epsilon\right\}$$

and

$$K_2 = \left\{(m,n,k) \in \mathbb{N}^3 : \left|\left(x_{mnk} - x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) - y_2\right| \geq r + \epsilon\right\}.$$

Because $y_1, y_2 \in st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, we have $\delta\left(K_1\right)=\delta\left(K_2\right)=0$. Thus we have

$$\begin{aligned}&\left|\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)-\left[\left(1-\lambda\right)y_1+\lambda y_2\right]\right|\\ &=\left|\left(1-\lambda\right)\left(\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)-y_1\right)+\lambda\left(\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)-y_2\right)\right|\\ &<r+\epsilon,\end{aligned}$$

for each $(m,n,k)\in\left(K_1^c\bigcap K_2^c\right)$ and each $\lambda\in[0,1]$. Because $\delta\left(K_1^c\bigcap K_2^c\right)=1$, we get

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3:\left|\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)-\left[\left(1-\lambda\right)y_1+\lambda y_2\right]\right|\geq r+\epsilon\right\}\right)=0,$$

i.e., $\left[\left(1-\lambda\right)y_1+\lambda y_2\right]\in st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$, which proves the convexity of the set $st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$. □

Theorem 3.11. *An arithmetic triple sequence* $x=\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ *statistically converges to* l *if and only if* $st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)=\bar{B}_r\left(l\right)$.

Proof. For the necessity part of this theorem is in proof of the Theorem 3.1.

Sufficiency: Because $st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)=\bar{B}_r\left(l\right)\neq\phi$, then by Theorem 3.6 we can say that the arithmetic triple sequence spaces x is statistically analytic. Assume on the contrary that the arithmetic triple sequence spaces x has another statistical cluster point $l^{'}$ different from l. Then the point

$$\bar{l}=l+\frac{r}{|l-l^{'}|}\left(l-l^{'}\right)$$

satisfies

$$\bar{l}-l^{'}=l-l^{'}+\frac{r}{|l-l^{'}|}\left(l-l^{'}\right)$$

$$\left|\bar{l}-l^{'}\right|=\left|l-l^{'}\right|+\frac{r}{|l-l^{'}|}\left(l-l^{'}\right)$$

$$\left|\bar{l}-l^{'}\right|=\left|l-l^{'}\right|+r>r.$$

Because $l^{'}$ is a statistical cluster point of the arithmetic triple sequence spaces x, by Theorem 3.4 this inequality implies that $\bar{l}\notin st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$. This contradicts the fact $\left|\bar{l}-l\right|=r$ and $st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)=\bar{B}_r\left(l\right)$. Thus l is the unique statistical cluster point of the arithmetic triple sequence spaces x. So the statistical cluster point of a statistically analytic arithmetic triple sequence spaces is unique. Hence the arithmetic triple sequence spaces x is statistically convergent to l. □

Theorem 3.12. (1) *If* $c\in\Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}$, *then*

$$st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\subseteq\bar{B}_r\left(c\right) \tag{3.6}$$

(2)

$$\begin{aligned}st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)&=\bigcap_{c\in\Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}}\bar{B}_r\left(c\right)\\ &=\left\{l\in\mathbb{R}^3:\Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}\subseteq\bar{B}_r\left(l\right)\right\}\end{aligned} \tag{3.7}$$

Proof. (1) Assume that $l \in st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ and $c \in \Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}$. Then by Theorem 3.4, we have

$$|l-c| \leq r;$$

otherwise, we get

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : \left|\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)-l\right| \geq r+\epsilon\right\}\right) \neq 0$$

for $\epsilon = \frac{|l-c|-r}{3}$. This contradicts the fact $l \in st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$.

(2) By the equation (3.6), we can write

$$st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \subseteq \bigcap_{c\in\Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}} \bar{B}_r(c). \tag{3.8}$$

Now assume that $y \in \bigcap_{c\in\Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}} \bar{B}_r(c)$. Then we have

$$\left|\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)-c\right| \leq r$$

for all $c \in \Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}$, which is equivalent to

$$\Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)} \subseteq \bar{B}_r\left(\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right),$$

i.e.,

$$\bigcap_{c\in\Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)}} \bar{B}_r(c) \subseteq \left\{l\in\mathbb{R} : \Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)} \subseteq \bar{B}_r(l)\right\}. \tag{3.9}$$

Now let $\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \notin st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$. Then there exists an $\epsilon > 0$ such that

$$\delta\left(\left\{(m,n,k)\in\mathbb{N}^3 : \left|l-\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right| \geq r+\epsilon\right\}\right) \neq 0.$$

the existence of a statistical cluster point c of the arithmetic triple sequence spaces x with

$$\left|\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)-c\right| \geq r+\epsilon,$$

i.e.,

$$\Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)} \nsubseteq \bar{B}_r\left(\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)\right)$$

and

$$\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \notin \left\{l\in\mathbb{R} : \Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)} \subseteq \bar{B}_r(l)\right\}.$$

Thus $y \in st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)$ follows from

$$\left(y_{mnk}-y_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right) \in \left\{l\in\mathbb{R} : \Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)} \subseteq \bar{B}_r(l)\right\},$$

i.e.,

$$\left\{l\in\mathbb{R} : \Gamma_{\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right)} \subseteq \bar{B}_r(l)\right\} \subseteq st-LIM^r\left(x_{mnk}-x_{\langle m,u\rangle\langle n,v\rangle\langle k,w\rangle}\right). \tag{3.10}$$

So the inclusions (3.6)-(3.9) ensure that (3.10) holds. ☐

Competing Interests: The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

References

[1] S. Aytar, Rough statistical Convergence, Numer. Funct. Anal. Optimi. 29 (3) (2008) 291–303.

[2] A. J. Dutta, A. Esi and B. C. Tripathy, Statistically convergent triple sequence spaces defined by Orlicz function, J. Math. Anal. 4 (2) (2013) 16–22.

[3] S. Debnath, B. Sarma and B. C. Das, Some generalized triple sequence spaces of real numbers, J. Nonlinear Anal. Optimi. 6 (1) (2015) 71–79.

[4] A. Esi, On some triple almost lacunary sequence spaces defined by Orlicz functions, Research and Reviews: Discrete Mathematical Structures 1 (2) (2014) 16–25.

[5] A. Esi and M. Necdet Catalbas, Almost convergence of triple sequences, Global J. Math. Anal. 2 (1) (2014) 6–10.

[6] A. Esi and E. Savas, On lacunary statistically convergent triple sequences in probabilistic normed space, Appl. Math. Inf. Sci. 9 (5) (2015) 2529–2534.

[7] A. Esi, N. Subramanian and A. Esi, On triple sequence space of Bernstein-Stancu operator of rough $I_\lambda-$statistical convergence of weighted g (A), Ann. Fuzzy Math. Inform. 16 (3) (2018) 337–361.

[8] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[9] S. K. Pal, D. Chandra and S. Dutta, Rough ideal Convergence, Hacee. J. Math. and Stat. 42 (6) (2013) 633–640.

[10] H. X. Phu, Rough convergence in normed linear spaces, Numer. Funct. Anal. Optimi. 22 (2001) 201–224.

[11] A. Sahiner, M. Gurdal and F. K. Duden, Triple sequences and their statistical convergence, Selcuk J. Appl. Math. 8 (2) (2007) 49–55.

[12] A. Sahiner, B. C. Tripathy, Some *I* related properties of triple sequences, Selcuk J. Appl. Math. 9 (2) (2008) 9–18.

[13] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73–74.

[14] N. Subramanian and A. Esi, The generalized tripled difference of χ^3 sequence spaces, Global Journal of Mathematical Analysis 3 (2) (2015) 54–60.

[15] B. C. Tripathy and R. Goswami, Vector valued multiple sequences defined by Orlicz functions, Boletim da Sociedade Paranaense de Matematica 33 (1) (2015) 67–79.

[16] B. C. Tripathy and R. Goswami, Multiple sequences in probabilistic normed spaces, Afrika Matematika 26 (5-6) (2015) 753–760.

[17] B. C. Tripathy and R. Goswami, Fuzzy real valued p-absolutely summable multiple sequences in probabilistic normed space, Afrika Matematika 26 (7-8) (2015) 1281–1289.

[18] B. C. Tripathy and R. Goswami, Statistically convergent multiple sequences in probabilistic normed spaces, U.P.B. Sci. Bull., Ser.A 78 (4) (2016) 83–94.

Ayhan Esi (aesi23@hotmail.com)
Department of Mathematics, Adiyaman University, 02040, Adiyaman, Turkey

Nagarajan Subramanian (nsmaths@yahoo.com)
School of Humanities and Sciences, SASTRA Deemed University, Thanjavur-613 401, India

Ayten Esi (aytenesi@yahoo.com)
Department of Mathematics, Adiyaman University, 02040, Adiyaman, Turkey