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Hesitant fuzzy subgroups and subrings

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Abstract. We define a hesitant fuzzy subgroupoid and obtain some of its properties. Next, we introduce the concepts of hesitant fuzzy subgroups, hesitant fuzzy ideals and hesitant fuzzy normal subgroups, and obtain some of its properties, respectively (In particular, see Theorems 4.5, 4.7 and 4.16, and Propositions 4.23 and 4.24). Finally, we define a hesitant fuzzy subring and investigate some of its properties. In particular, we give a characteristic of a (usual) field by a hesitant fuzzy ideal (See Proposition 5.9).

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1. Introduction

In 2010, Torra [13, 14] introduced the notion of a hesitant fuzzy set (Refer to [10, 12]) which further characterized an element by a set of membership values thereby decreasing the loss of information during fuzzufication. After then, Jun et al. [7] studied hesitant fuzzy bi-ideals in semigroups. Xia and Xu [15] applied hesitant fuzzy set to decision making. Furthermore, Deepark and John [2] investigated hesitant fuzzy rough sets through hesitant fuzzy relations. Also They [3, 4, 5] studied homomorphisms of hesitant fuzzy subgroups, and hesitant fuzzy subrings and ideals. Alshehri and Alshehri [1] applied Hesitant anti-fuzzy soft sets to BCK-algebras. Solariaju and Mahalakshmi [11] investigated hesitant intuitionistic fuzzy soft groups. Deepark and Mashinchi [6] studied hesitant L-fuzzy relations. Recently, Kim et al [8] introduced the category $\mathbf{HSet}(H)$ consisting of all hesitant H-fuzzy spaces and all morphisms between them and studied $\mathbf{HSet}(H)$ in the sense of a topological universe. Also they [9] studied hesitant fuzzy relations.

In this paper, we define a hesitant fuzzy subgroupoid and obtain some of its properties. Next, we introduce the concepts of hesitant fuzzy subgroups, hesitant fuzzy ideals and hesitant fuzzy normal subgroups, and obtain some of its properties, respectively (In particular, see Theorems 4.5, 4.7 and 4.16, and Propositions 4.23 and 4.24). Finally, we define a hesitant fuzzy subring and investigate some of its properties. In particular, we give a characteristic of a (usual) field by a hesitant fuzzy ideal (See Proposition 5.9).

2. Preliminaries

In this section, we list some basic definitions and some properties needed in the next sections.

Definition 2.1 ([8, 13])**.** Let X be a reference set and let $P[0,1]$ denote the power set of $[0,1]$. Then a mapping $h : X \to P[0,1]$ is called a hesitant fuzzy set in X.

The hesitant fuzzy empty [resp. whole] set, denoted by $h^{\mathbf{0}}$ [resp. $h^{\mathbf{1}}$], is a hesitant fuzzy set in X defined as: for each $x \in X$,

$$h^{\mathbf{0}}(x) = \phi \text{ [resp. } h^{\mathbf{1}}(x) = [0,1]].$$

In this case, we will denote the set of all hesitant fuzzy sets in X as $HS(X)$.

Definition 2.2 ([2])**.** Let $h_1,\ h_2 \in HS(X)$. Then

we say that h_1 is a subset of h_2, denoted by $h_1 \subset h_2$, if $h_1(x) \subset h_2(x)$, for each $x \in X$,

(ii) we say that h_1 is equal to h_2, denoted by $h_1 = h_2$, if $h_1 \subset h_2$ and $h_2 \subset h_1$.

Definition 2.3 ([8])**.** Let $h_1,\ h_2 \in HS(X)$ and let $(h_j)_{j\in J} \subset HS(X)$. Then

(i) the intersection of h_1 and h_2, denoted by $h_1 \widetilde{\cap} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(h_1 \widetilde{\cap} h_2)(x) = h_1(x) \cap h_2(x),$$

(ii) the intersection of $(h_j)_{j\in J}$, denoted by $\widetilde{\bigcap}_{j\in J} h_j$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(\widetilde{\bigcap}_{j\in J} h_j)(x) = \bigcap_{j\in J} h_j(x),$$

(iii) the union of h_1 and h_2, denoted by $h_1 \widetilde{\cup} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(h_1 \widetilde{\cup} h_2)(x) = h_1(x) \cup h_2(x),$$

(iv) the union of $(h_j)_{j\in J}$, denoted by $\widetilde{\bigcup}_{j\in J} h_j$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$,

$$(\widetilde{\bigcup}_{j\in J} h_j)(x) = \bigcup_{j\in J} h_j(x).$$

Definition 2.4 ([8])**.** Let X be a nonempty set and let $h \in HS(X)$. Then the complement of h, denoted by h^c, is a hesitant fuzzy set in X defined as: for each $x \in X$,

$$h^c(x) = h(x)^c = [0,1] \setminus h(x).$$

Result 2.5 ([8], Proposition 3.14)**.** *Let X be a nonempty set, let $h,\ h_1,\ h_2,\ h_3 \in HS(X)$ and let $(h_j)_{j\in J} \subset HS(X)$. Then*

(1) (Idempotent laws): $h\widetilde{\cup}h = h,\ h\widetilde{\cap}h = h$,
(2) (Commutative laws): $h_1\widetilde{\cup}h_2 = h_2\widetilde{\cup}h_1,\ h_1\widetilde{\cap}h_2 = h_2\widetilde{\cap}h_1$,
(3) (Associative laws): $h_1\widetilde{\cup}(h_2\widetilde{\cup}h_3) = (h_1\widetilde{\cup}h_2)\widetilde{\cup}h_3,\ h_1\widetilde{\cap}(h_2\widetilde{\cap}h_3) = (h_1\widetilde{\cap}h_2)\widetilde{\cap}h_3$,
(4) (Distributive laws): $h_1\widetilde{\cup}(h_2\widetilde{\cap}h_3) = (h_1\widetilde{\cup}h_2)\widetilde{\cap}(h_1\widetilde{\cup}h_3)$,
$h_1\widetilde{\cap}(h_2\widetilde{\cup}h_3) = (h_1\widetilde{\cap}h_2)\widetilde{\cup}(h_1\widetilde{\cap}h_3)$,
(4)$'$ (Generalized Distributive laws): $h\widetilde{\cup}(\widetilde{\bigcap}_{j\in J}h_j) = \widetilde{\bigcap}_{j\in J}(h\widetilde{\cup}h_j)$,
$h\widetilde{\cap}(\widetilde{\bigcup}_{j\in J}h_j) = \widetilde{\bigcup}_{j\in J}(h\widetilde{\cap}h_j)$,
(5) (Absorption laws): $h_1\widetilde{\cup}(h_1\widetilde{\cap}h_2) = h_1,\ h_1\widetilde{\cap}(h_1\widetilde{\cup}h_2) = h_1$.
(6) (DeMorgan's laws): $(h_1\widetilde{\cup}h_2)^c = h_1^c\widetilde{\cap}h_2^c,\ (h_1\widetilde{\cap}h_2)^c = h_1^c\widetilde{\cup}h_2^c$,
(6)$'$ (Generalized DeMorgan's laws): $(\widetilde{\bigcup}_{j\in J}h_j)^c = \widetilde{\bigcap}_{j\in J}h_j^c,\ (\widetilde{\bigcap}_{j\in J}h_j)^c = \widetilde{\bigcup}_{j\in J}h_j^c$,
(7) $(h^c)^c = h$,
(8) $h_1\widetilde{\cap}h_2 \subseteq h_1$ *and* $h_2\widetilde{\cap}h_1 \subseteq h_2$,
(9) $h_1 \subseteq h_2\widetilde{\cup}h_1$ *and* $h_1 \subseteq h_2\widetilde{\cup}h_2$,
(10) *if* $h_1 \subseteq h_2$ *and* $h_2 \subseteq h_3$, *then* $h_1 \subseteq h_3$,
(11) *if* $h_1 \subseteq h_2$, *then* $h_1\widetilde{\cap}h \subseteq h_2\widetilde{\cap}h$ *and* $h_1\widetilde{\cup}h \subseteq h_2\widetilde{\cup}h$,
(12) $h^{\mathbf{0}} \subseteq h \subseteq h^{\mathbf{1}}$,
(13) $h\widetilde{\cap}h^{\mathbf{0}} = h^{\mathbf{0}},\ h\widetilde{\cup}h^{\mathbf{0}} = h,\ h\widetilde{\cap}h^{\mathbf{1}} = h,\ h\widetilde{\cup}h^{\mathbf{1}} = h^{\mathbf{1}}$.

From the above proposition, we can easily see that $(HS(X), \widetilde{\cap}, \widetilde{\cup}, ^c)$ is a Boolean algebra with the least element $h^{\mathbf{0}}$ and the largest element $h^{\mathbf{1}}$.

Definition 2.6 ([8])**.** Let X and Y be a nonempty sets, let $h_X \in HS(X)$ and $h_Y \in HS(Y))$ and let $f : X \to Y$ be a mapping. Then the image of h_X under f, denoted by $f(h_X)$, is a hesitant fuzzy set in Y defined as follows: for each $y \in Y$,

$$f(h_X)(y) = \begin{cases} \widetilde{\bigcup}_{x\in f^{-1}(y)} h_X(x) & \text{if } f^{-1}(y) \neq \phi \\ \phi & \text{otherwise.} \end{cases}$$

Result 2.7 ([8], Proposition 3.16)**.** *Let $f : X \to Y$ be a mapping, and let $h_X,\ h_{X1},\ h_{X2} \in HS(X)$, $(h_{X_j})_{j\in J} \subset HS(X)$, $h_Y,\ h_{Y1},\ h_{Y2} \in HS(Y)$ and $(h_{Y_j})_{j\in J} \subset HS(Y)$. Then*

(1) *if* $h_{X1} \subseteq h_{X2}$, *then* $f(h_{X1}) \subseteq f(h_{X2})$,
(2) $f(h_{X1}\widetilde{\cup}h_{X2}) = f(h_{X1})\widetilde{\cup}f(h_{X2}),\ f(\widetilde{\bigcup}_{j\in J}h_{X_j}) = \widetilde{\bigcup}_{j\in J}f(h_{X_j})$,
(3) $f(h_{X1}\widetilde{\cap}h_{X2}) \subseteq f(h_{X1})\widetilde{\cap}f(h_{X2}),\ f(\widetilde{\bigcap}_{j\in J}h_{X_j}) \subseteq \widetilde{\bigcap}_{j\in J}f(h_{X_j})$,
(3)$'$ *if f is injective, then* $f(h_{X1}\widetilde{\cap}h_{X2}) = f(h_{X1})\widetilde{\cap}f(h_{X2}),\ f(\widetilde{\bigcap}_{j\in J}h_{X_j}) = \widetilde{\bigcap}_{j\in J}f(h_{X_j})$,
(4) $f(A) = h^{\mathbf{0}}$ *if and only if* $A = h^{\mathbf{0}}$,
(5) *if* $h_{Y1} \subseteq h_{Y2}$, *then* $f^{-1}(h_{Y1}) \subseteq f^{-1}(h_{Y2})$,
(6) $f^{-1}(h_{Y1}\widetilde{\cup}h_{Y2}) = f^{-1}(h_{Y1})\widetilde{\cup}f^{-1}(h_{Y2}),\ f^{-1}(\widetilde{\bigcup}_{j\in J}h_{Y_j}) = \widetilde{\bigcup}_{j\in J}f^{-1}(h_{Y_j})$,
(7) $f^{-1}(h_{Y1}\widetilde{\cap}h_{Y2}) \subseteq f^{-1}(h_{Y1})\widetilde{\cap}f^{-1}(h_{Y2}),\ f^{-1}(\widetilde{\bigcap}_{j\in J}h_{Y_j}) \subseteq \widetilde{\bigcap}_{j\in J}f^{-1}(h_{Y_j})$,
(8) $f^{-1}(h_Y) = h^{\mathbf{1}}$ *if and only if* $h_Y\widetilde{\cap}f(h^{\mathbf{1}}) = h^{\mathbf{1}}$,
(9) $h_X \subset f^{-1} \circ f(h_X)$; *in particular,* $h_X = f^{-1} \circ f(h_X)$, *if f is injective,*
(10) $f \circ f^{-1}(h_Y) \subset h_Y$; *in particular,* $f \circ f^{-1}(h_Y) = h_Y$, *if f is sujective.*

3. Hesitant fuzzy subgroupoids

Definition 3.1. Let $h \in HS(X)$. Then h is called a hesitant fuzzy point with the support $x \in X$ and the value λ, denoted by x_λ, if $x_\lambda : X \to P[0,1]$ is the mapping given by: for each $y \in X$,

$$x_\lambda(y) = \begin{cases} \lambda \subset [0,1] & \text{if } y \neq x \\ \phi & \text{otherwise.} \end{cases}$$

We will denote the set of all hesitant fuzzy points in X as $H_P(X)$.

Definition 3.2. Let $h \in HS(X)$ and let $x_\lambda \in H_P(X)$. Then x_λ is said to be belong to h, denoted by $x_\lambda \in h$, if $\lambda \subset h(x)$.

It is obvious that $h = \widetilde{\bigcap}_{x_\lambda \in A} x_\lambda$.

The following is the immediate result of Definitions 2.2, 2.3 and 3.1.

Theorem 3.3. *Let $h_1,\ h_2 \in HS(X)$ and let $(h_j)_{j \in J} \subset HS(X)$.*
(1) *$h_1 \subset h_2$ if and only if $x_\lambda \in h_2$, for each $x_\lambda \in h_1$.*
(2) *$x_\lambda \in h_1 \widetilde{\cap} h_2$ if and only if $x_\lambda \in h_1$ and $x_\lambda \in h_2$.*
(3) *If $x_\lambda \in h_1$ or $x_\lambda \in h_2$, then $x_\lambda \in h_1 \widetilde{\cup} h_2$.*
(4) *$x_\lambda \in \widetilde{\bigcap}_{j \in J} h_j$ if and only if $x_\lambda \in h_j$, for each $j \in J$.*
(5) *If $x_\lambda \in h_j$ for some $j \in J$, then $x_\lambda \in \widetilde{\bigcup}_{j \in J} h_j$.*

Remark 3.4. The converses of (3) and (5) of Theorem 3.3 need not to be true in general as shown the following example.

Example 3.5. Let $X = \{a, b, c\}$, let h_1 and h_2 be two hesitant fuzzy sets given by, respectively: $h_1(a) = \{0, 0.4, 0.7\},\ h_1(b) = [0, 0.6),\ h_1(c) = (0, 0.8]$
and

$$h_2(a) = \{0, 0.5, 0.7\},\ h_2(b) = [0.1, 0.7),\ h_2(c) = [0, 0.8).$$

Let $\lambda = \{0, 0.4, 0.5\} \in P[0,1]$ and consider $a_\lambda \in H_P(X)$. Then clearly, we can easily check that $a_\lambda \in h_1 \widetilde{\cup} h_2$ but $a_\lambda \notin h_1$ and $a_\lambda \notin h_2$.

Definition 3.6. Let $(X, \cdot)$ be a groupoid and let $h_1,\ h_2 \in HS(X)$. Then the hesitant fuzzy product of h_1 and h_2, denoted by $h_1 \circ h_2$, is a hesitant fuzzy set in X defined by: for each $x \in X$,

$$(h_1 \circ h_2)(x) = \begin{cases} \widetilde{\bigcup}_{yz=x} [h_1(y) \cap h_2(z)] & \text{if } yz = x \\ \phi & \text{otherwise.} \end{cases}$$

Proposition 3.7. *Let $h_1,\ h_2 \in HS(X)$ and let $x_\lambda,\ y_\mu \in H_P(X)$. Then*
(1) $x_\lambda \circ y_\mu = (xy)_{\lambda \cap \mu}$,
(2) $h_1 \circ h_2 = \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2} x_\lambda \circ y_\mu$.

Proof. (1) Let $z \in X$ and suppose $z = x'y'$. Then
$(x_\lambda \circ y_\mu)(z) = \widetilde{\bigcup}_{x'y'=z} [x_\lambda(x') \cap y_\mu(y')] = \lambda \cap \mu$

$$(x_\lambda \circ y_\mu)(z) = \begin{cases} \widetilde{\bigcup}_{x'y'=z} [x_\lambda(x') \cap y_\mu(y')] & \text{if } x'y' = z \\ \phi & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lambda \cap \mu & \text{if } xy = z \\ \phi & \text{otherwise} \end{cases}$$
$$= (xy)_{\lambda \cap \mu}.$$

(2) Let $h = \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2} x_\lambda \circ y_\mu$. For each $z \in X$, we assume that there are $u,\ v \in X$ such that $uv = z$, $x_\lambda \neq \phi$ and $y_\mu \neq \phi$, without loss of generality. Then

$$\begin{aligned}(h_1 \circ h_2)(z) &= \widetilde{\bigcup}_{uv=z}[h_1(u) \cap h_2(v)] \\ &\supset \widetilde{\bigcup}_{uv=z}(\widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2}[x_\lambda(u) \cap y_\mu(v)] \\ &= \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2}[x_\lambda \circ y_\mu] \\ &= h(z)\end{aligned}$$

and

$$\begin{aligned}h(z) &= \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2}(\widetilde{\bigcup}_{uv=z}[x_\lambda(u) \cap y_\mu(v)]) \\ &= \widetilde{\bigcup}_{uv=z}(\widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2}[x_\lambda(u) \cap y_\mu(v)]) \\ &\supset \widetilde{\bigcup}_{uv=z}[u_{h_1(u)}(u) \cap v_{h_2(v)}(v)] \\ &= \widetilde{\bigcup}_{uv=z}[h_1(u) \cap h_2(v)] \\ &= (h_1 \circ h_2)(z).\end{aligned}$$

Thus $h_1 \circ h_2 = h$. So $h_1 \circ h_2 = \widetilde{\bigcup}_{x_\lambda \in h_1,\ y_\mu \in h_2} x_\lambda \circ y_\mu$. □

The following is the immediate result of Definition 3.6.

Proposition 3.8. *Let $(X, \cdot)$ be a groupoid and let $\circ$ be the hesitant fuzzy product.*
(1) *If "$\cdot$" is associative [resp. commutative], then so is "$\circ$" in $HS(X)$.*
(2) *If "$\cdot$" has an identity $e \in X$, then $e_{[0,1]}$ is an identity of "$\circ$" in $HS(X)$, i.e.,*

$$h \circ e_{[0,1]} = h = e_{[0,1]} \circ h, \ \textit{for each } h \in HS(X).$$

Definition 3.9. Let $(G, \cdot)$ be a groupoid and let $\phi \neq h \in HS(X)$. Then h is called a hesitant fuzzy subgroupoid (in short, HGP) in G, if $h \circ h \subset h$.

We will denote the set of all HGPs in G as $HGP(G)$.

Example 3.10. Let $G = \{a, b, c, d\}$ be the groupoid in which $\cdot$ is given by:

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	c	a	b
c	c	d	c	a
d	d	b	d	a

Table 3.1

Consider two hesitant fuzzy sets h_1 and h_2 in defined by:

$$h_1(a) = \{0.1, 0.3, 0.7\},\ h_1(b) = \{0.3\},\ h_1(c) = h_1(d) = \{0.3, 0.7\}.$$

and

$$h_2(a) = \{0.1, 0.3\},\ h_2(b) = (0.6, 1],\ h_2(c) = \{0.3, 0.8\},\ h_2(d) = [0.1, 0.4).$$

Then $h_1 \circ h_1$ is a hesitant fuzzy set in G given by:

$(h_1 \circ h_1)(a) = \{0.1, 0.3\},\ (h_1 \circ h_1)(b) = \{0.3\},$

$(h_1 \circ h_1)(c) = (h_1 \circ h_1)(d) = \{0.3, 0.7\}$.
Thus $h_1 \circ h_1 \subset h_1$. So h_1 is a hesitant fuzzy subgroupoid in G.

On the other hand, $(h_2 \circ h_2)(a) = [0.1, 0.4) \cup \{0.8\} \not\subset \{0.1, 0.3\} = h_2(a)$. Then $h_2 \circ h_2 \not\subset h_2$. Thus h_2 is not a hesitant fuzzy subgroupoid in G.

Theorem 3.11. *Let $(G, \cdot)$ be a groupoid and let $\phi \neq h \in HS(X)$. Then the followings are equivalent:*
(1) *$h \in HGP(G)$,*
(2) *for any $x_\lambda,\ y_\mu \in h$, $x_\lambda \circ y_\mu \in h$, i.e., $(h, \circ)$ is a groupoid,*
(3) *for any $x,\ y \in G$, $h(xy) \supset h(x) \cap h(y)$.*

Proof. (1)⇔(2): From Definitions 3.6 and 3.9, the proof is clear.

(1)⇒(3): Suppose $h \in HGP(G)$ and let $x,\ y \in G$. Then
$h(xy) \supset (h \circ h)(xy)$ [By Definition 3.9]
$= \widetilde{\bigcup}_{xy=uv}[h(u) \cap (v)]$ [By Definition 3.6]
$\supset h(x) \cap h(y)$.
Thus $h(xy) \supset h(x) \cap h(y)$.

(3)⇒(1): The proof is obvious. □

From Proposition 3.11, we can define a HGP in a groupoid G as follows.

Definition 3.12. Let G be a groupoid and let $h \in HS(G)$. Then h is called a hesitant fuzzy subgroupoid (in short, HGP) in G, if

$$h(xy) \supset h(x) \cap h(y), \text{ for any } x,\ y \in G.$$

It is obvious that $h^{\mathbf{0}},\ h^{\mathbf{1}} \in HGP(G)$.

Example 3.13. Let $G = \{a, b, c, d\}$ the groupoid in which $\cdot$ is given by:

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	d	c	c
d	d	b	d	d

Table 3.2

Let h be the hesitant fuzzy set in G defined by:

$$h(a) = [0, 0.6],\ h(b) = (0.1, 0.7],\ h(c) = (0, 0.8),\ h(d) = [0.1, 0.9].$$

Then we can easily check that h is a hesitant fuzzy subgroupoid in G.

Proposition 3.14. *If $(h_j)_{j \in J} \subset HGP(G)$, then $\widetilde{\bigcap}_{j \in J} h_j \in HGP(G)$.*

Proof. Let $h = \widetilde{\bigcap}_{j \in J} h_j$ and let $x,\ y \in G$. Then
$h(xy) = \bigcap_{j \in J} h_j(xy)$
$\supset \bigcap_{j \in J}[h_j(x) \cap h_j(y)]$ [Since $h_j \in HGP(G)$]
$= (\bigcap_{j \in J} h_j(x)) \cap (\bigcap_{j \in J} h_j(y))$
$= [\widetilde{\bigcap}_{j \in J} h_j](x) \cap [\widetilde{\bigcap}_{j \in J} h_j](y)$
$= h(x) \cap h(y)$.
Thus $h(xy) \supset h(x) \cap h(y)$. So $\widetilde{\bigcap}_{j \in J} h_j \in HGP(G)$. □

Proposition 3.15. *Let $f : G \to G^{'}$ be a groupoid homomorphism, let $h_G \in HS(G)$ and $h_{G'} \in HS(G^{'})$.*

(1) *$f(x_\lambda \circ y_\mu) = f(x)_\lambda \circ f(y)_\mu$, for any $x_\lambda,\ y_\mu \in H_P(G)$.*

(2) *If f is surjective and $h \in HGP(G)$, then $f(h) \in HGP(G^{'})$.*

(3) *If $h \in HGP(G^{'})$, then $f^{-1}(h) \in HGP(G)$.*

Proof. (1) Let $x_\lambda,\ y_\mu \in H_P(G)$ and let $z \in G^{'}$. Then

$$\begin{aligned} f(x_\lambda \circ y_\mu)(z) &= f((xy)_{\lambda\cap\mu})(z) \text{ [By Proposition 3.7 (1)]} \\ &= \begin{cases} \widetilde{\bigcup}_{z' \in f^{-1}(z)} (xy)_{\lambda\cap\mu}(z^{'}) & \text{if } f^{-1}(z) \neq \phi \\ \phi & \text{otherwise} \end{cases} \\ &= \begin{cases} \lambda \cap \mu & \text{if } z = f(xy) \\ \phi & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} [f(x)_\lambda \circ f(y)_\mu](z) &= \begin{cases} \bigcup_{z=uv}[f(x)_\lambda(u) \cap f(y)_\mu(v) & \text{for } (u,v) \in G^{'} \times G^{'} \text{with } z = uv \\ \phi & \text{otherwise} \end{cases} \\ &= \begin{cases} \lambda \cap \mu & \text{if } z = f(x)f(y) \\ \phi & \text{otherwise.} \end{cases} \end{aligned}$$

Thus $f(x_\lambda \circ y_\mu) = f(x)_\lambda \circ f(y)_\mu$.

(2) Assume that $f(h) \notin HGP(G^{'})$. Then there are $y,\ y^{'} \in G^{'}$ such that

$$f(h)(yy^{'}) \subsetneqq f(h)(y) \cap f(h)(y^{'}).$$

Thus $\bigcup_{f(z)=yy'} h(z) \subsetneqq (\bigcup_{f(x)=y} h(x)) \cap (\bigcup_{f(x')=y'} h(x^{'}))$. Since f is surjective, there are $x,\ x^{'} \in G$ with $f(x) = y$ and $f(x^{'}) = y^{'}$ such that

$$\bigcup_{f(z)=yy'} h(z) \subsetneqq h(x) \cap h(x^{'}).$$

So $h(xx^{'}) \subset \bigcup_{f(z)=yy'} h(z) \subsetneqq h(x) \cap h(x^{'})$. This is a controdiction from the fact that $h \in HGP(G)$. Hence $f(h) \in HGP(G^{'})$.

(3) The proof is easy. □

Definition 3.16. $h \in HS(X)$ is said to have the sup-property, if for each subset T of X, there is $t_0 \in T$ such that $h(t_0) = \bigcup_{t \in T} h(t)$.

Proposition 3.17. *Let $f : G \to G^{'}$ be a groupoid homomorphism and let $h \in HS(G)$ has the sup-property. If $h \in HGP(G)$, then $f(h) \in HGP(G^{'})$.*

Proof. Let $y,\ y^{'} \in G^{'}$. Then we can consider four cases:

(i) $f^{-1}(y) \neq \phi$, $f^{-1}(y^{'}) \neq \phi$,

(ii) $f^{-1}(y) \neq \phi$, $f^{-1}(y^{'}) = \phi$,

(iii) $f^{-1}(y) = \phi$, $f^{-1}(y^{'}) \neq \phi$,

(iv) $f^{-1}(y) = \phi$, $f^{-1}(y^{'}) = \phi$.

We will prove only the case (i) and omit the remainders. Since h has the sup-property, there are $x_0 \in f^{-1}(y)$ and $x_0^{'} \in f^{-1}(y^{'})$ such that

$$h(x_0) = \bigcup_{t \in f^{-1}(y)} h(t) \text{ and } h(x_0^{'}) = \bigcup_{t' \in f^{-1}(y')} h(t^{'}).$$

Then

$$\begin{aligned} f(h)(yy^{'}) &= \bigcup_{z\in f^{-1}} h(z) \\ &\supset h(x_0x_0^{'}) \text{ [Since } f(x_0x_0^{'}) = f(x_0)f(x_0^{'}) = yy^{'}] \\ &\supset h(x_0)\cap h(x_0^{'}) \text{ [Since } h \in HGP(G)] \\ &= (\bigcup_{t\in f^{-1}(y)} h(t)) \cap (\bigcup_{t^{'}\in f^{-1}(y^{'})} h(t^{'})) \\ &= f(h)(y) \cap f(h)(y^{'}). \end{aligned}$$

Thus $f(h) \in HGP(G^{'})$. □

Definition 3.18. Let $f : X \to Y$ be a mapping and let $h \in HS(X)$. Then h is said to be hesitant fuzzy invariant (in short, HF-invariant), if $f(x) = f(y)$ implies $h(x) = h(y)$.

It is clear that if h is HF-invariant, then $f^{-1} \circ f(h) = h$.

The following is the immediate result of Definition 3.18.

Proposition 3.19. *Let $f : X \to Y$ be a mapping and let*

$$\mathcal{H} = \{h \in HS(X) : h \textit{ is } HF-\textit{invariant and has the sup property}\}.$$

Then there is a one-to-one correspondence between $\mathcal{H}$ and $HS(Imf)$, where Imf denotes the image of f.

The following is the immediate result of Propositions 3.17 and 3.19.

Corollary 3.20. *Let $f : G \to G^{'}$ be a groupoid homomorphism and let*

$$\mathcal{H} = \{h \in HGP(G) : h \textit{ is } HF-\textit{invariant and has the sup property}\}.$$

Then Then there is a one-to-one correspondence between $\mathcal{H}$ and $HGP(Imf)$.

4. Hesitant fuzzy sugroups

Definition 4.1. Let G be a group and let $h \in HS(G)$. Then h is called a hesitant fuzzy subgroup (in short, HFG) in G, if it satisfies the following conditions: for any $x,\ y \in G$,

(i) $h(xy) \supset h(x) \cap h(y)$,

(ii) $h(x^{-1}) \supset h(x)$.

We will denote the set of all HFGs in G as $HFG(G)$.

Example 4.2. Consider the additive group $(\mathbb{Z}, +)$. We define $h : \mathbb{Z} \to P[0,1]$ as follows: for each $n \in \mathbb{Z}$,

$$h(n) = [\frac{1}{2}, \frac{4}{5}], \text{ if } n \text{ is odd and } h(n) = [\frac{1}{3}, \frac{2}{3}], \text{ if } n \text{ is even, and } h(0) = [0,1].$$

Then we can easily see that h is a HFG in $(\mathbb{Z}, +)$.

The following is the immediate result of Proposition 3.13 and Definition 4.1.

Proposition 4.3. *Let G be a group and let $(h_j)_{j\in J} \subset HFG(G)$. Then $\widetilde{\bigcap}_{j\in J} h_j \in HFG(G)$.*

Proposition 4.4. *Let G be a group and let $h \in HFG(G)$. Then*
(1) $h(x^{-1}) = h(x)$*, for each* $x \in G$*,*
(2) $h(e) \supset h(x)$*, for each* $x \in G$*, where* e *is the identity of* G*.*

Proof. (1) Let $x \in G$. Then by Definition 4.1, $h(x) = h(x^{-1})^{-1}) \supset h(x^{-1}) \supset h(x)$. Thus $h(x^{-1}) = h(x)$.
(2) Let $x \in G$. Then by Definition 4.1 and (1),

$$h(e) = h(xx^{-1}) \supset h(x) \cap h(x^{-1}) = h(x) \cap h(x) = h(x).$$

Thus $h(e) \supset h(x)$. □

Theorem 4.5. *Let G be a group and let $h \in HS(G)$. Then $h \in HFG(G)$ if and only if $h(xy^{-1}) \supset h(x) \cap h(y)$, for any $x,\ y \in G$.*

Proof. Suppose $h \in HFG(G)$ and let $x,\ y \in G$. Then by Definition 4.1 and Proposition 4.4 (1) $h(xy^{-1}) \supset h(x) \cap h(y^{-1}) = h(x) \cap h(y)$.

Conversely, suppose the necessary condition holds and let $x \in G$. Then $h(e) = h(xx^{-1}) \supset h(x) \cap h(x) = h(x)$. Thus $h(x^{-1}) = h(ex^{-1}) \supset h(e) \cap h(x) = h(x)$. So h satisfies the condition (ii) of Definition 4.1.

Now let $x,\ y \in G$. Then $h(xy) = h(x(y^{-1})^{-1}) \supset h(x) \cap h(y^{-1}) \supset h(x) \cap h(y)$. Thus $h(xy) \supset h(x) \cap h(y)$. So h satisfies the condition (i) of Definition 4.1. Hence $h \in HFG(G)$. □

Proposition 4.6. *Let G be a group and let $h \in HS(G)$. If $h \in HFG(G)$, then $h \circ h = h$.*

Proof. Suppose $h \in HFG(G)$. Then clearly, $h \in HGP(G)$. Thus by Definition 3.9, $h \circ h \subset h$. Let $x \in G$ such that $x = yz$, where $y,\ z \in G$. Then

$(h \circ h)(x) = \widetilde{\bigcup}_{x=yz}[h(y) \cap h(z)]$ [By Definition 3.6]
$\supset h(x) \cap h(e)$ [Since G is group, $x = xe = ex$.]
$= h(x)$. [By Proposition 4.4 (2)]

Thus $h \circ h \supset h$. So $h \circ h = h$. □

Theorem 4.7. *Let G be a group and let $h_1,\ h_2 \in HFG(G)$. Then $h_1 \circ h_2 \in HFG(G)$ if and only if $h_1 \circ h_2 = h_2 \circ h_1$.*

Proof. Suppose $h_1 \circ h_2 \in HFG(G)$ and let $x \in G$ and let $x \in G$. Then

$(h_1 \circ h_2)(x) = \widetilde{\bigcup}_{x=yz}[h_1(y) \cap h_2(z)]$ [By Definition 3.6]
$= \widetilde{\bigcup}_{x=yz}[h_1(xz^{-1}) \cap h_2(y^{-1}x)]$
[Since G is a group, $y = xz^{-1}$ and $z = y^{-1}x$]
$= \widetilde{\bigcup}_{x=yz}[h_1(zx^{-1}) \cap h_2(x^{-1}y]$ [By Proposition 4.4 (1)]
$= \widetilde{\bigcup}_{x^{-1}=z^{-1}y^{-1}}[h_1(y^{-1}) \cap h_2(z^{-1})]$
$= \widetilde{\bigcup}_{x^{-1}=z^{-1}y^{-1}}[h_2(z^{-1}) \cap h_1(y^{-1})]$
$= (h_2 \circ h_1)(x^{-1})$ [By Definition 3.6]
$= (h_2 \circ h_1)(x)$. [By the hypothesis]

Thus $h_1 \circ h_2 = h_2 \circ h_1$.

Conversely, suppose $h_1 \circ h_2 = h_2 \circ h_1$. Then

$(h_1 \circ h_2) \circ (h_1 \circ h_2) = h_1 \circ (h_2 \circ h_1) \circ h_2$ [By Proposition 3.8 (1)]

$= h_1 \circ (h_1 \circ h_2) \circ h_2$ [By the hypothesis]
$= (h_1 \circ h_1) \circ (h_2 \circ h_2)$
$\subset h_1 \circ h_2$. [Since $h_1,\ h_1 \in HFG(G)$]

Thus by Definition 3.9, $h_1 \circ h_2 \in HGP(G)$. So $h_1 \circ h_2$ satisfies the condition (i) of Definition 4.1.

Let $x \in G$. Then

$(h_1 \circ h_2)(x^{-1}) = (h_2 \circ h_1)(x^{-1})$ [the hypothesis]
$= \widetilde{\bigcup}_{x^{-1}=yz}[h_2(y) \cap h_1(z)]$ [By Definition 3.7]
$= \widetilde{\bigcup}_{x^{-1}=yz}[h_2(x^{-1}z^{-1}) \cap h_1(y^{-1}x^{-1})]$
[Since $y = x^{-1}z^{-1}$ and $z = y^{-1}x^{-1}$]
$= \widetilde{\bigcup}_{x^{-1}=yz}[h_2(zx) \cap h_1(xy)]$ [By Proposition 4.4 (1)]
$= \widetilde{\bigcup}_{x^{-1}=yz}[h_2(y^{-1}) \cap h_1(z^{-1})]$
[Since $zx = y^{-1}$ and $xy = z^{-1}$]
$= \widetilde{\bigcup}_{x=z^{-1}y^{-1}}[h_2(y^{-1}) \cap h_1(z^{-1})]$
$= (h_1 \circ h_2)(x)$.

Thus $h_1 \circ h_2$ satisfies the condition (ii) of Definition 4.1. So $h_1 \circ h_2 \in HFG(G)$. □

Proposition 4.8. *Let G be a group and let $h \in HFG(G)$. Then $G_h = \{x \in G : h(x) = h(e)\}$ is a subgroup.*

Proof. Let $x,\ y \in G_h$. Then

$h(xy^{-1}) \supset h(x) \cap h(y)$ [By Theorem 4.5]
$= h(e) \cap h(e)$ [Since $x,\ y \in G_h$]
$= h(e)$.

Thus $h(xy^{-1}) \supset h(e)$. From Proposition 4.4 (2), it is obvious that $h(e) \supset h(xy^{-1})$. So $h(xy^{-1}) = h(e)$. Hence $xy^{-1} \in G_h$. Therefore G_h is a subgroup of G. □

Proposition 4.9. *Let G be a group and let $h \in HFG(G)$. If $h(xy^{-1}) = h(e)$, for any $x,\ y \in G$, then $h(x) = h(y)$.*

Proof. Let $x,\ y \in G$. Then

$h(x) = h((xy^{-1})y)$
$\supset h(xy^{-1}) \cap h(y)$ [Since $h \in HFG(G)$]
$= h(e) \cap h(y)$ [By the hypothesis]
$= h(y)$ [Proposition 4.4 (2)]

and

$h(y) = h((yx^{-1})x)$
$\supset h(yx^{-1}) \cap h(x)$ [Since $h \in HFG(G)$]
$= h((yx^{-1})^{-1}) \cap h(x)$ [By Proposition 4.4 (1)]
$= h(xy^{-1}) \cap h(x)$
$= h(e) \cap h(x)$ [By the hypothesis]
$= h(x)$. [Proposition 4.4 (2)]

Thus $h(x) = h(y)$. □

Proposition 4.10. *Let G be a group and let $h \in HFG(G)$. If G_h is a normal subgroup of G, then h is constant on each coset of G_h.*

Proof. Let $a \in G$ and let $x \in aG_h$. Then there is $y \in G$ such that $x = ay$. Since G_h is normal, $xa^{-1} \in G_h$. Thus by the definition of G_h, $h(xa^{-1}) = h(e)$. So by Proposition 4.9, $h(x) = h(a)$. Hence h is constant on aG_h, for each $a \in G$.

Similarly, we can easily see that h is constant on $G_h a$, for each $a \in G$. This completes the proof. □

Let H be a subgroup of a group G. Then the number of right [resp. left] coset of H in G is called the index of H in G and denoted by $[G : H]$. If G is a finite group, then there is only a finite number of distinct right [resp. left] cosets of H and thus $[G : H]$ is finite. However, if G is an infinite group, then $[G : H]$ may be either finite or infinite.

Proposition 4.11. *Let G be a group, let $h \in HFG(G)$ and let G_h be normal. If $[G_h : G]$ is finite, then h has the sup property.*

Proof. Let $T \subset G$. Since $[G_h : G]$ is finite, let $[G_h : G] = n$, say

$$\mathcal{A} = \{a_1G_h, a_2G_h, \cdots, a_nG_h\},$$

where $a_i \in G$ $(i = 1,\ 2,\ \cdots,\ n)$ and $a_iG_h \cap a_jG_h = \phi$, for any $i \neq j$.
Let $t \in T$. Since $T \subset G = \bigcup \mathcal{A} = \bigcup_{i=1}^{n} a_iG_h$, there is $i \in \{1, 2, \cdots, n\}$ such that $t \in a_iG_h$. Since G_h be normal, by Proposition 4.10, $h(t) = h(a_i)$ on a_iG_h, say $h(t) = \lambda_i \in P[0, 1]$. Thus there is $t_0 \in T$ such that $h(t_0) = \bigcup_{i=1}^{n} \lambda_i = \bigcup_{t\in T} h(t)$. So h has the sup property. □

Proposition 4.12. *A group G cannot be the union of two proper HFGs.*

Proof. Assume that h_1 and h_2 are two proper HFGs of G such that

$$h_1 \widetilde{\cup} h_2 = h^{\mathbf{1}},\ h_1 \neq h^{\mathbf{1}} \text{ and } h_2 \neq h^{\mathbf{1}}.$$

Let $x \in G$. Then $(h_1 \widetilde{\cup} h_2)(x) = h_1(x) \cup h_2(x) = [0, 1]$. Thus $h_1(x) = [0, 1]$ or $h_2(x) = [0, 1]$. This is a contradiction. This completes the proof. □

Proposition 4.13. *Let G be a finite group and let $h \in HGP(G)$. Then $h \in HFG(G)$.*

Proof. Let $x \in G$. Since G is finite, x has the finite order, say n. Then $x^n = e$, where e is the identity of G. Thus $x^{-1} = x^{n-1}$. Since $h \in HGP(G)$,

$$h(x^{-1}) = h(x^{n-1}) = h(x^{n-2}x) \supset h(x).$$

So $h \in HFG(G)$. □

Theorem 4.14. *Let G be a group, let $h \in HFG(G)$ and let $x \in G$. Then $h(xy) = h(y)$, for each $y \in G$ if and only if $h(x) = h(e)$.*

Proof. Suppose $h(xy) = h(y)$, for each $y \in G$. Then clearly, $h(x) = h(e)$.

Conversely, suppose $h(x) = h(e)$ and let $y \in G$. Then by Proposition 4.4 (2), $h(x) \supset h(y)$. Thus $h(xy) \supset h(x) \cap h(y) = h(y)$. On the other hand,

$h(y) = h(x^{-1}xy) \supset h(x) \cap h(xy)$ [By Proposition 4.4 (1)]
$= h(xy)$. [By the hypothesis and Proposition 4.4 (2)

So $h(xy) = h(y)$, for each $y \in G$. □

Proposition 4.15. *Let $f : G \to G^{'}$ be a group homomorphism, let $h_G \in HFG(G)$ and let $h_{G'} \in HFG(G^{'})$.*
(1) *If h_G has the sup-property, then $f(h_G) \in HFG(G^{'})$.*
(2) *$f^{-1}(h_{G'}) \in HFG(G)$.*

Proof. (1) From Proposition 3.17, it is clear that $h_G \in HGP(G)$. Then it is enough to show that $f(h_G)(y^{-1}) \supset f(h_G)(y)$, for each $y \in f(G)$. Let $y \in f(G)$. Then clearly, $\phi \neq f^{-1}(y) \subset G$. Since h_G has the sup-property, there is $x_0 \in^{-1} (y)$ such that

$$h_G(x_0) = \bigcup_{t \in f^{-1}(y)} h_G(t).$$

Thus $f(h_G)(y^{-1}) = \bigcup_{t \in f^{-1}(y^{-1})} h_G(t) \supset h_G(x_0^{-1}) \supset h_G(x) = f(h_G)(y)$. So $f(h_G) \in HFG(G^{'})$.

(2) From Proposition 3.15, it is clear that $f^{-1}(h_{G'}) \in HGP(G)$. Then it is enough to show that $f^{-1}(h_{G'})(x^{-1}) \supset f^{-1}(h_{G'})(x)$, for each $x \in G$. Let $x \in G$. Then

$$\begin{aligned} f^{-1}(h_{G'})(x^{-1}) &= h_{G'}(f(x^{-1})) \\ &= h_{G'}(f(x)^{-1}) \\ &=\supset h_{G'}(f(x)) \\ &= f^{-1}(h_{G'})(x). \end{aligned}$$

Thus $f^{-1}(h_{G'}) \in HFG(G)$. □

Theorem 4.16. *Let G_p be a cyclic group of prime order, say $G_p = \{0, 1, 2, \cdots, p-1\}$. Then $h \in HFG(G_p)$ if and only if $h(x) = h(1) \subset h(0)$, for each $0 \neq x \in G_p$.*

Proof. Suppose $h \in HFG(G_p)$ and let $0 \neq x \in G_p$. Since x is the sum of $1'$s and 1 is the sum of x's, $h(x) \supset h(1) \supset h(x)$. Thus $h(x) = h(1)$. Since 1 is the identity of G_p, $h(0) \supset h(x)$. So $h(x) = h(1) \subset h(0)$, for each $0 \neq x \in G_p$.

Conversely, suppose $h(x) = h(1) \subset h(0)$, for each $0 \neq x \in G_p$ and let $x,\ y \in G_p$. Then we have four cases: $x \neq 0$, $y \neq 0$ and $x = y$ or $x \neq 0$, $y = 0$ or $x = 0$, $y \neq 0$ or $x \neq 0$, $y \neq 0$ and $x \neq y$.

Case (i): Suppose $x \neq 0$, $y \neq 0$. Then by the hypothesis,

$$h(x) = h(y) = h(1) \subset h(0).$$

Thus $h(x - y) = h(0) \supset h(x) \cap h(y)$.

Case (ii): Suppose $x \neq 0$, $y = 0$. Then clearly, $x - y \neq 0$. Thus by the hypothesis, $h(x - y) = h(x) = h(1) \subset h(0) = h(y)$. So $h(x - y) \supset h(x) \cap h(y)$.

Case (iii): The proof is similar to Case (ii).

Case (iv): Suppose $x \neq 0$, $y \neq 0$ and $x \neq y$. Then clearly, $x - y \neq 0$. Thus by the hypothesis, $h(x - y) = h(x) = h(y) = h(1) \subset h(0)$. So $h(x - y) \supset h(x) \cap h(y)$.

In all cases, $h(x - y) \supset h(x) \cap h(y)$. Hence by Theorem 4.5, $h \in HFG(G_p)$. □

Definition 4.17. Let G be a groupoid and let $h \in HS(G)$. Then h is called a:
(i) hesitant fuzzy left ideal (in short, HFLI) of G, if $h(xy) \supset h(y)$, for any $x,\ y \in G$,
(ii) hesitant fuzzy right ideal (in short, HFRI) of G, if $h(xy) \supset h(x)$, for any $x,\ y \in G$,
(iii) hesitant fuzzy ideal (in short, HFI) of G, if it is both a HFLI and a HFRI.

We will denote the set of all HFLIs [resp. HFRIs and HFIs] of a G as $HFLI(G)$ [resp. $HFRI(G)$ and $HFI(G)$].

It is obvious that $h \in HFI(G)$ if and only if $h(xy) \supset h(x) \cup h(y)$, for any $x,\ y \in G$. Furthermore, a HFI [resp. HFLI and HFRI] is a HGP of G.

It is clear that for each $h \in HGP(G)$, $H(x^n) \supset h(x)$ for each $x \in G$ and if h is a constant hesitant fuzzy set in G, then h is a HFI of G.

Example 4.18. Let $G = \{a, b, c, d\}$ the groupoid in which $\cdot$ is given by:

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	b	d	c
c	c	a	c	b
d	c	b	d	d

Table 4.1

Let h_1 and h_2 be the hesitant fuzzy sets in G defined by, respectively:

$$h_1(a) = h_1(b) = [0, 0.6],\ h_1(c) = h_1(d) = [0, 0.7),$$

$$h_2(a) = [0.1, 0.7],\ h_2(b) = h_2(c) = h_2(d) = [0, 0.7],$$

Then we can easily check that h_1 is a HFLI and h_1 is a HFRI of G.

Proposition 4.19. *The HFLIs [resp. HFRIs and HFIs] of a group G are just constant mappings.*

Proof. Suppose $h : G \to P[0, 1]$ be a constant mapping and let $x,\ y \in G$. Then clearly, $h(xy) = h(x) = h(y)$. Thus $h \in HFLI(G)$ [resp. $HFRI(G)$ and $HFI(G)$].

Suppose $h \in HFLI(G)$. Then $h(xy) \supset h(y)$, for any $x,\ y \in G$. In particular, $h(x) = h(xe) \supset h(x)$, for each $x \in G$. On the other hand, $h(e) = h(x^{-1}x) \supset h(x)$, for each $x \in G$. Thus $h(x) = h(e)$, for each $x \in G$. So h is a constant mapping. This completes the proof. □

Definition 4.20. Let G be a group and let $h \in HFG(G)$. Then h is called a hesitant fuzzy normal subgroup (in short, HFNG) of G, if $h(xy) = h(yx)$, for any $x,\ y \in G$.

We will denote the set of all HFNGs of G as $HFNG(G)$. It is obvious that if G is abelian, then $h \in HFNG(G)$, for each $h \in HFG(G)$.

Example 4.21. Consider the general linear group of degree n, $GL(n, \mathbb{R})$. Then clearly, $GL(n, \mathbb{R})$ is not abelian. Let I_n be the unit matrix in $GL(n, \mathbb{R})$. We define the mapping $h : GL(n, \mathbb{R}) \to P[0, 1]$ as follows: for each $I_n \neq M \in GL(n, \mathbb{R})$,

$h(M) = [\frac{1}{5}, \frac{2}{3}]$, if M is not a triangular matrix,

$h(M) = [\frac{1}{3}, \frac{1}{2}]$, if M is a triangular matrix

and

$h(I_n) = [0, 1]$.

Then we can easily see that $h \in HFNG(GL(n, \mathbb{R}))$.

Proposition 4.22. *Let G be a group, let $h_1 \in HS(G)$ and let $h_2 \in HFNG(G)$. Then $h_1 \circ h_2 = h_2 \circ h_1$.*

Proof. Let $x \in G$. Then

$$\begin{aligned}(h_1 \circ h_2)(x) &= \widetilde{\bigcup}_{x=yz}[h_1(y) \cap h_2(z)]\\ &= \widetilde{\bigcup}_{x=yz}[h_1(y) \cap h_2(y^{-1}x)]\\ &= \widetilde{\bigcup}_{x=(xy^{-1})y}[h_1(y) \cap h_2(xy^{-1})] \text{ [Since } h_2 \in HFNG(G)]\\ &= \widetilde{\bigcup}_{x=(xy^{-1})y}[h_2(xy^{-1}) \cap h_1(y)]\\ &= (h_2 \circ h_1)(x).\end{aligned}$$

Thus $h_1 \circ h_2 = h_2 \circ h_1$. □

Proposition 4.23. *Let G be a group, let $h_1 \in HFNG(G)$ and let $h_2 \in HFG(G)$. Then $h_2 \circ h_1 \in HFG(G)$.*

Proof. From Definitions 3.6 and 3.9, we can easily see that $h_2 \circ h_1 \in HGP(G)$. Then it is sufficient to show that $(h_2 \circ h_1)(x^{-1}) \supset (h_2 \circ h_1)(x)$, for each $x \in G$. Let $x \in G$. Then

$$\begin{aligned}(h_2 \circ h_1)(x^{-1}) &= \widetilde{\bigcup}_{x^{-1}=yz}[h_2(y) \cap h_1(z)]\\ &= \widetilde{\bigcup}_{x=z^{-1}y^{-1}}[h_2((y^{-1})^{-1}) \cap h_1((z^{-1})^{-1})]\\ &\supset \widetilde{\bigcup}_{x=z^{-1}y^{-1}}[h_2(y^{-1})) \cap h_1(z^{-1})]\\ &= (h_1 \circ h_2)(x)\\ &= (h_2 \circ h_1)(x). \text{ [By Proposition 4.21]}\end{aligned}$$

Thus $h_2 \circ h_1 \in HFG(G)$. □

Proposition 4.24. *Let G be a group and let $h_1,\ h_2 \in HFNG(G)$. Then $h_1 \circ h_2 \in HFNG(G)$.*

Proof. From Propositions 4.7, 4.22 and 4.23, it is clear that $h_1 \circ h_2 \in HFG(G)$. Let $a,\ b \in G$. Then there are $x,\ y \in G$ such that $ab = xy$. Since $b = a^{-1}xy$, $ba = (a^{-1}xa)(a^{-1}ya)$. Thus

$$\begin{aligned}(h_1 \circ h_2)(ab) &= \widetilde{\bigcup}_{ab=xy}[h_1(x) \cap h_2(y)]\\ &= \widetilde{\bigcup}_{ab=xy}[h_1(a^{-1}xa) \cap h_2(a^{-1}ya)] \text{ [Since } h_1,\ h_2 \in HFNG(G)]\\ &= \widetilde{\bigcup}_{ba=(a^{-1}xa)(a^{-1}ya)}[h_1(a^{-1}xa) \cap h_2(a^{-1}ya)]\\ &= (h_1 \circ h_2)(ba).\end{aligned}$$

So $h_1 \circ h_2 \in HFNG(G)$. □

Proposition 4.25. *Let G be a group and let $h \in HFNG(G)$. Then G_h is a normal subgroup of G.*

Proof. From Proposition 4.8, it is obvious that G_h is a subgroup of G and $G_h \neq \phi$. Let $x \in G_h$ and let $y \in G$. Then

$$\begin{aligned}h(yxy^{-1}) &= h((yx)y^{-1})\\ &= h(y^{-1}(yx)) \text{ [Since } h \in HFNG(G)]\\ &= h((y^{-1}y)x)\\ &= h(x)\\ &= h(e). \text{ [Since } x \in G_h]\end{aligned}$$

Thus $yxy^{-1} \in G_h$. So G_h is a normal subgroup of G. □

The converse of Proposition 4.25 need not to be true as shown in the following example.

Example 4.26. Let $G = \{e, a, b, c\}$ be the group in which $\cdot$ is given by:

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	b	c
b	b	a	e	c
c	c	a	b	e

Table 4.2

Let h be the hesitant fuzzy set in X defined by:

$$h(e) = h(a) = [0, 1],\ h(b) = (0, 1],\ h(c) = [0, 1).$$

Then we can easily check that h is a hesitant fuzzy subgroup of G. Moreover, $G_h = \{e, a\}$ is a normal subgroup of G. But $h(ab) = (0, 1] \neq [0, 1] = h(ba)$. Thus h is not a hesitant fuzzy normal subgroup of G.

Definition 4.27. Let G be a group and let $h \in HFNG(G)$. Then the quotient group G/G_h is called the hesitant fuzzy quotient subgroup (in short, HFQG) of G with respect to h.

Proposition 4.28. *Let G be a group, let $h_1 \in HFNG(G)$, let $h_2 \in HS(G)$ and let $\pi : G \to G/G_h$ be the natural mapping. Then $\pi^{-1}(\pi(h_2)) = G_{h_1} \circ h_2$.*

Proof. Let $x \in G$. Then

$$\begin{aligned}[\pi^{-1}(\pi(h_2))](x) &= \pi(h_2)(\pi(x))\\ &= \widetilde{\bigcup}_{\pi(x)=\pi(y)} h_2(y)\\ &= \widetilde{\bigcup}_{xy^{-1}\in G_{h_1}} h_2(y)\end{aligned}$$

and

$$\begin{aligned}(G_h \circ h_2)(x) &= \widetilde{\bigcup}_{x=zy}[G_{h_1}(z) \cap h_2(y)]\\ &= \widetilde{\bigcup}_{z=xy^{-1}\in G_{h_1}}[h_1(e) \cap h_2(y)] \text{ [By the definition of } G_{h_1}]\\ &= \widetilde{\bigcup}_{z=xy^{-1}\in G_{h_1}} h_2(y). \text{ [By proposition 4.4 (2)]}\end{aligned}$$

Thus $\pi^{-1}(\pi(h_2)) = G_{h_1} \circ h_2$. □

5. Hesitant fuzzy subrings

Definition 5.1. Let $(R, +, \cdot)$ be a ring and let $\phi \neq h \in HS(R)$. Then h is called a hesitant fuzzy subring (in short, HFR), if it satisfies the following conditions:
(i) $h \in HFG(R, +)$,
(ii) $h \in HGP(R, \cdot)$.

We will denote the set of all HFRs of R as $HFR(R)$.

Example 5.2. Consider the ring $(\mathbb{Z}_2, +, \cdot)$, where $\mathbb{Z}_2 = \{0, 1\}$. We define the mapping $h : \mathbb{Z}_2 \to P[0.1]$ as follows: $h(0) = [0.2, 0.7]$ and $h(1) = [0.4, 0.7)$. Then we can easily see that $h \in HFR(\mathbb{Z}_2)$.

The following is the immediate result of Definition 3.12 and Theorem 4.5.

Theorem 5.3. *Let R) be a ring and let $\phi \neq h \in HS(R)$. Then $h \in HFR(R)$ if and only if for any $x,\ y \in R$,*
(1) $h(x-y) \supset h(x) \cap h(y)$,
(2) $h(xy) \supset h(x) \cap h(y)$.

Definition 5.4. Let R) be a ring and let $\phi \neq h \in HFR(R)$. Then h is called a:
(i) hesitant fuzzy left ideal (in short, HFLI) of R, if $h(xy) \supset h(y)$, for any $x,\ y \in R$,
(ii) hesitant fuzzy right ideal (in short, HFRI) of R, if $h(xy) \supset h(x)$, for any $x,\ y \in R$,
(iii) hesitant fuzzy ideal (in short, HFI), if it is both a HFLI and a HFRI of R.

We will denote the set of all HFLIs [resp. HFRIs and HFIs] of R as $HFLI(R)$ [resp. $HFRI(R)$ and $HFI(R)$].

Example 5.5. Consider the ring $(\mathbb{Z}_4, +, \cdot)$, where $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. We define the mapping $h : \mathbb{Z}_4 \to P[0.1]$ as follows:

$$h(0) = [0.2, 0.8),\ h(1) = (0.3, 0.7) = h(3) \text{ and } h(2) = [0.4, 0.5].$$

Then we can easily see that $h \in HFI(\mathbb{Z}_4)$.

The following is the immediate result of Theorem 5.3 and Definition 5.4.

Theorem 5.6. *Let R) be a ring and let $\phi \neq h \in HS(R)$. Then $h \in HFI(R)$ [resp. $HFLI(R)$ and $HFRI(R)$] if and only if for any $x,\ y \in R$,*
(1) $h(x-y) \supset h(x) \cap h(y)$,
(2) $h(xy) \supset h(x) \cup h(y)$ *[resp.* $h(xy) \supset h(y)$ *and* $h(xy) \supset h(x)$*].*

Theorem 5.7. *Let R) be a skew field (also a division ring) and let $\phi \neq h \in HS(R)$. Let 0 BE the identity of R for "+" and let e be the identity of R for "$\cdot$". Then the followings are equivalent:*
(1) $h \in HFI(R)$ *[resp.* $HFLI(R)$ *and* $HFRI(R)$*],*
(2) *for each* $0 \neq x \in R$, $h(x) = h(e) \subset h(0)$.

Proof. (1)⇒(2): Suppose $h \in HFLI(R)$ and let $0 \neq x \in R$. Then

$$h(x) = h(xe) \supset h(e) \text{ and } h(e) = h(x^{-1}x) \supset h(x).$$

Thus $h(x) = h(e)$. On the other hand, $h(0) = h(e-e) \supset h(e) \cap h(e) = h(e)$. So $h(x) = h(e) \subset h(0)$.

(2)⇒(1): Suppose $h(x) = h(e) \subset h(0)$, for each $0 \neq x \in R$ and let $x,\ y \in R$. Then we have four cases: $x \neq 0,\ y \neq 0$ and $x \neq y$ or $x \neq 0,\ y \neq 0$ and $x = y$ or $x \neq 0,\ y = 0$ or $x = 0,\ y \neq 0$.

Case (i): Suppose $x \neq 0,\ y \neq 0$ and $x \neq y$. Then clearly, $x - y \neq 0$ and $xy \neq 0$. Thus by the hypothesis,

$$h(x-y) = h(e) \supset h(x) \cap h(y) \text{ and } h(xy) = h(e) \supset h(x) \cup h(y).$$

Case (ii): Suppose $x \neq 0,\ y \neq 0$ and $x = y$. Then clearly, $x - y = 0$ and $xy \neq 0$. Thus by the hypothesis,

$$h(x-y) = h(0) \supset h(x) \cap h(y) \text{ and } h(xy) = h(e) \supset h(x) \cup h(y).$$

Case (iii): Suppose $x \neq 0$, $y = 0$. Then clearly, $x - y \neq 0$ and $xy = 0$. Thus by the hypothesis,

$$h(x-y) = h(x) = h(e) \supset h(x) \cap h(y) \text{ and } h(xy) = h(0) \supset h(x) \cup h(y).$$

Case (iv): Suppose $x = 0$, $y \neq 0$. Then the proof is similar to case (iii).
So in all cases, $h \in HFI(R)$. This completes the proof. □

Remark 5.8. Proposition 5.7 shows that a HFLI [resp. HFRI] is a HFI in a skew field.

The following gives a characteristic of a (usual) field by a HFI.

Proposition 5.9. *Let R be a commutative ring with a unity e. Suppose for each $h \in HFI(R)$ and each $0 \neq x \in R$, $h(x) = h(e) \subset h(0)$. Then R is a field.*

Proof. Let A be a ideal of R such that $A \neq R$. Then we can consider A as $A = \chi_A$, where $\chi_A : R \to \{0, 1\} \subset [0, 1]$ is the characteristic function of A. Thus $A \in HS(R)$. Moreover, we can easily see that $A \in HFI(R)$ such that $A \neq h^{\mathbf{1}}$. So there is $y \in R$ such that $y \notin A$ and thus $\chi_A(y) = \{0\}$. By the hypothesis, $\chi_A(x) = \chi_A(e) \subset \chi_A(0)$, each $0 \neq x \in R$. Hence $\chi_A(0) = \{1\}$, i.e., $A = \{0\}$. Therefore R is a field. □

6. Conclusions

We introduced the concepts of a hesitant fuzzy subgroupoid, a hesitant fuzzy subgroup, a hesitant fuzzy normal subgroup, a hesitant fuzzy subring and a hesitant fuzzy ideal and obtained some of its properties, respectively. In particular, we gave a characteristic of a (usual) field by a hesitant fuzzy ideal (See Proposition 5.9). In the future, we will apply the concept of hesitant fuzzy set to BCK/BCI-algebras, d-algebras, B-algebras, etc.

References

[1] Halimah Alshehri and Noura Alshehri, Hesitant anti-fuzzy soft set in BCK-algebras, Mathematical Problems in Engineering 2017, Article ID 3634258, 13 pages. Carlin.
343–351.

[2] D. Divakaran and Sunil Jacob John Hesitant fuzzy rough sets through hesitant fuzzy relations, Ann. Fuzzy Math. Inform. 8 (1) (2014) 33–46.

[3] D. Divakaran and Sunil Jacob John Homomorphism of hesitant fuzzy subgroups, International Journal of Scientific and Engineering Research 5 (9) (2014) 9–14.

[4] D. Divakaran and Sunil Jacob John Hesitant fuzzy subgroups, Journal of New Theory 11 (2016) 54–68.

[5] D. Divakaran and Sunil Jacob John Dual hesitant fuzzy subrings and ideals, Ann. Fuzzy Math. Inform. 13 (3) (2017) 437–448.

[6] D. Divakara and M. Mashinchi Hesitant L-fuzzy relations, 2018 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS) 102–104.

[7] Y. B. Jun, K. J. Lee and S. Z. Song Hesitant fuzzy bi-ideals in semigroups, Commun. Korean Math. Soc. 30 (3) (2015) 143–154.

[8] J. H. Kim, P. K. Lim, J. G. Lee, K. Hur, The category of hesitant fuzzy sets, To be submitted in AFMI.

[9] J. H. Kim, P. K. Lim, J. G. Lee, K. Hur, Hesitant fuzzy relations, To be submitted in AFMI.

[10] Zheng Pei and Liangzhong Yi A note on operations of hesitant fuzzy set, International Journal of Computational Intelligence Systems 8 (2) (2015) 226–239.

[11] A. Solariaju and S. Mahalakshmi Hesitant intuitionistic fuzzy soft groups, International Journal of Pure and Applied Mathematics 118 (10 (2018) 223–232.

[12] G. S. Thakur, Rekha Thakur and Ravi Singh New Hesitant fuzzy operatos, Fuzzy Inf. Eng. 6 (2014) 379–392.

[13] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 (2010) 529–539.

[14] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, in Proc. IEEE 18th Int. Fuzzy Syst. (2009) 1378–1382.

[15] M. Xia and Z. Xu Hesitant fuzzy information aggregation in decision making, Internat. J. Approx. Reason. 52 (3) (2011) 395–407.

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