Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Fuzzy ideals in demi-pseudocomplemented MS-algebras

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Received 28 January 2019; Revised 18 March 2019; Accepted 11 April 2019

Abstract. In this paper, we study fuzzy congruences, kernel fuzzy ideals and $(\circ,*)$-fuzzy ideals of a demi-pseudocomplemented MS-algebra L. In particular, we study the fuzzy congruence relation generated by fuzzy relation and its properties. Also we prove that $(\circ,*)$-fuzzy ideals form a sublattice of fuzzy ideals of L, and the set of these fuzzy ideals is isomorphic to the closed interval G_F and χ_ι of the fuzzy congruence lattice of L where G_F is Glievenko fuzzy congruence and χ_ι is the universal fuzzy congruence.

2010 AMS Classification: 06D15, 06D30, 06D72

Keywords: Demi-pseudocomplemented MS-algebra, Fuzzy ideals, Fuzzy congruence.

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1. Introduction

Zadeh [16] introduced the concepts of a fuzzy set, the philosophy behind this notion has permeated various disciplines of human knowledge including those of logic and reasoning, which is the foundation stone of all mathematical science. Among various branches of pure and applied mathematics abstract algebra was one of the first few subjects where research was carried out using the notion of fuzzy sets. Rosenfeld [11] extended the notion of group theory to introduce a new discipline of fuzzy groups. A similar treatment [1, 13, 15] introduced fuzzy ideals of distributive lattice. Recently, Alaba and Alemayehu [2] introduced the notion of clouser fuzzy ideals of MS-algebras. Also Alaba, Taye and Alemayehu [3] introduced the concept of δ- fuzzy ideals in MS-algebras.

On the other hand, Blyth and Varlet [6] introduced the notion of MS-algebras as a common abstraction of de Morgan algebras and Stone algebras. Sankappanavar [12] introduced the notion of demi-pseudocomplemented algebrs, and Blyth, Fang and Wang [4] studied on ideals and congruences of distributive demi-pseudocomplemented algebras. More recently, Fang and Tan [7] characterized kernel ideals and $(\circ,*)$-ideals in demi-pseudocomplemented MS-algebras.

These studies motivated us to study fuzzy congruences, kernel fuzzy ideals and $(\circ,*)$-fuzzy ideals in demi-pseudocomplemented MS-algebras L. In particular, we study the fuzzy congruence relation generated by fuzzy relation and its properties. Also we prove that $(\circ,*)$-fuzzy ideals form a sublattice of fuzzy ideals of L, and the set of these fuzzy ideals is isomorphic to the interval $[G_F, \chi_\iota]$ of the fuzzy congruence lattice of L where G_F is Glievenko fuzzy congruence, and χ_ι is the universal fuzzy congruence.

2. Preliminaries

In this section, we recall basic concepts frequently used in this article.

Definition 2.1 ([12])**.** A demi-pseudocomplemented algebra is an algebra $(L,\wedge,\vee,^*,0,1)$ in which $(L,\wedge,\vee,0,1)$ is a bounded lattice and a unary operation $x\longrightarrow x^*$ satisfying the following properties: for $\forall x,y\in L$,

(1) $(x\vee y)^* = x^*\wedge y^*$,
(2) $(x\wedge y)^{**} = x^{**}\wedge y^{**}$,
(3) $0^* = 1$ and $1^* = 0$,
(4) $x^{***} = x^*$,
(5) $x^*\wedge x^{**} = 0$.

As shown by Sankappanavara [12], in a demi-pseudocomplemented algebra, the following property holds:
(6) $x^*\wedge(x^*\wedge y)^* = x^*\wedge y^*$, for $\forall x,y\in L$.

Definition 2.2 ([6])**.** An Ockham algebra is a bounded distributive lattice L together with a dual endomorphism $f: L\longrightarrow L$. An MS-algebra is an Ockham algebra in which dual endomorphism $x\longrightarrow f(x)$ is determined by the inequality $x\leq f^2(x)$. As usual, we shall write x° for $f(x)$.

Definition 2.3 ([8])**.** A demi-pseudocomplemented Ockham algebra is an algebra $(L,\wedge,\vee,f,^*,0,1)$ of type $(2,2,1,1,0,0)$ where $(L,\wedge,\vee,f,0,1)$ is an Ockham algebra, $(L,\wedge,\vee,^*,0,1)$ is a demi-p-lattice and the operations $x\longrightarrow f(x)$ and $x\longrightarrow x^*$ are linked by the identity $f(x^*) = [f(x)]^*$.

Specially, if $(L,^\circ)$ is an MS-algebra, then $(L,\wedge,\vee,^\circ,^*,0,1)$ is called demi-pseudoco -mplemented MS-algebra. We shall denote the class of demi-pseudocomplemented MS-algebra by dpMS-algebra.

Theorem 2.4 ([8])**.** *If $(L,^\circ,^*)$ is a dpMS-algebra then the flowing statements hold:*

(1) $x^* = x^{\circ\circ*}$ *for* $\forall x\in L$,
(2) $x^{\circ\circ} = x^{**} = x$ *for* $\forall x\in L^*$,
(3) $(x\wedge y)^* = x^*\vee y^*$ *for* $\forall x,y\in L$,
(4) $x^*\vee x^{**} = 1$ *for for* $\forall x\in L$,
(5) $x^*\wedge y = 0\Rightarrow x^*\leq y^*$ *for* $\forall x,y\in L$,
(6) $(L^*,^*)$ *is a boolean algebra.*

Definition 2.5 ([9]). Let L be a lattice and let $H \subseteq L \times L$. We denote by $\Theta(H)$ the smallest congruence relation containing H, and call it the congruence relation generated by H. If $H = I \times I$, where I is an ideal, we write $\Theta[I]$,for all $\Theta(H)$.

Definition 2.6 ([7]). An equivalence relation θ is a congruence relation in dpMS-algebra L, if it is a lattice congruence and $(a, b) \in \theta$ implies $(a^\circ, b^\circ) \in \theta$ and $(a^*, b^*) \in \theta$ for all $a, b \in L$. To distinguish lattice congruence of dpMS-algebra L from congruence of dpMS-algebra L, we shall use the subscript $'lat'$ to denote lattice congruence.

As shown [9], if I is an ideal of a distributive lattice L, then

$$(x, y) \in \Theta[I] \Leftrightarrow (\exists i \in I)\; x \vee i = y \vee i. \tag{2.1}$$

Dully, if F is a filter of a distributive lattice L, then

$$(x, y) \in \Theta[F] \Leftrightarrow (\exists j \in F)\; x \wedge j = y \wedge j. \tag{2.2}$$

For an ideal I of dpMS-algebra $(L,^\circ,^*)$, We shall write

$$I_\circ^{\geq} = \{x \in L : (\exists i \in I)\; i^\circ \leq x, \},$$

$$I_{\circ\circ} = \{x \in L : (\exists i \in I)\; x \leq i^{\circ\circ}\}$$

and

$$I^\circ = \{y \in L, x^\circ = y : \exists x \in I\}.$$

Clearly, $I^\circ \subseteq I_\circ^{\geq}$, $I_\circ^{\geq}$ is a filter of L and $I_{\circ\circ}$ is ideal of L. By Definition of $I_\circ^{\geq}$ and I° and by equation (2.2) Fang and Tan [7], characterize as

$$(x, y) \in \Theta_{lat}[I_\circ^{\geq}] \Leftrightarrow (\exists i \in I)\; x \wedge i^\circ = y \wedge i^\circ. \tag{2.3}$$

Theorem 2.7 ([6]). *Let I be an Ideal of the dpMS-algebra L. Then*

$$\Theta[I] = \Theta_{lat}[I_\circ^{\geq}] \vee \Theta_{lat}[I_{\circ\circ}].$$

We recall that for any nonempty set L, the characteristic function of L, denoted by χ_L, is defined as follows: for each $x \in L$,

$$\chi_L(x) = \begin{cases} 1 & if\ x \in L, \\ 0 & if\ \ x \notin L. \end{cases}$$

Definition 2.8 ([1]). Let μ be a fuzzy subset of $(L, \wedge, \vee, 0, 1)$. For any $\alpha \in [0, 1]$, we shall denote the level subset $\mu^{-1}([\alpha, 1])$ by simply μ_α, i.e.

$$\mu_\alpha = \{x \in L : \alpha \leq \mu(x)\}.$$

Theorem 2.9 ([13]). *Let μ be a fuzzy subset of L. Then μ is a fuzzy ideal of L if and only if any one of the following conditions is satisfied:*

(1) $\mu(0) = 1$ *and* $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ *for all* $x, y \in L$,
(2) $\mu(0) = 1$ *and* $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ *and* $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$ *for all* $x, y \in L$.

A fuzzy relation θ on a set X is map $\theta : X \times X \to [0, 1]$. For any $x, y \in X$ and fuzzy relations θ and ϕ on x, $(\theta \cap \phi)(x, y) = min\{\theta(x, y), \phi(x, y)\}$, $(\theta \cup \phi)(x, y) = max\{\theta(x, y), \phi(x, y)\}$, $\theta \subseteq \phi$ means $\theta(x, y) \leq \phi(x, y)$.

Definition 2.10 ([10]). Suppose that θ and ϕ are two fuzzy relations on a set X. Then $(\theta \circ \phi)(x,y) = \sup_{z\in X}((\theta(x,z) \wedge (\phi)(z,y))$.

Definition 2.11 ([10]). A fuzzy relation ϕ on X is said to be a fuzzy equivalence relation on X, if

(1) $\phi(x,x) = 1$ for all $x \in X$ (reflexive),
(2) $\phi(x,y) = \phi(y,x)$ for all $x,y \in L$ (symmetric),
(3) $\phi(x,z) \geq \phi(x,y) \wedge \phi(y,z)$ for all $x,y,z \in L$ (transitive).

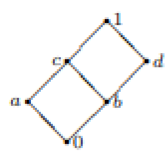
Through out the next sections, L stands for demi-pseudomplemented MS-algebras.

3. Fuzzy congruences in demi-pseudomplemented MS-algebras

Definition 3.1. A fuzzy relation ϕ on a demi-pseudomplemented MS-algebra $(L,^\circ,^*)$ is called fuzzy congruence relation on $(L,^\circ,^*)$, if the following are satisfied:

(1) $\phi(x,x) = 1$ for all $x \in L$,
(2) $\phi(x,y) = \phi(y,x)$ for all $x,y \in L$,
(3) $\phi(x,z) \geq \phi(x,y) \wedge \phi(y,z)$ for all $x,y,z \in L$,
(4) $\phi(x \wedge z, y \wedge w) \wedge \phi(x \vee z, y \vee w) \geq \phi(x,y) \wedge \phi(z,w)$ for all $x,y,z,w \in L$,
(5) $\phi(x^\circ, y^\circ) \wedge \phi(x^*, y^*) \geq \phi(x,y)$ for all $x,y \in L$.

Example 3.2. Consider the dpMS-algebra $(L,^\circ,^*)$ given in Hasse diagram 1 below:



	0	a	b	c	d	1
°	1	d	a	0	a	0
*	1	0	1	0	1	0

Diagram 1

Define a fuzzy relation $\varphi : L \times L \longrightarrow [0,1]$ as $\varphi(0,0) = \varphi(a,a) = \varphi(b,b) = \varphi(c,c) = \varphi(d,d) = \varphi(1,1) = 1$, $\varphi(0,a) = \varphi(a,0) = \varphi(0,b) = \varphi(b,0) = \varphi(c,0) = \varphi(0,c) = \varphi(d,0) = \varphi(0,d) = \varphi(1,0) = \varphi(0,1) = \varphi(b,a) = \varphi(a,b) = \varphi(a,c) = \varphi(c,a) = \varphi(a,d) = \varphi(d,a) = \varphi(a,1) = \varphi(1,a) = \varphi(b,c) = \varphi(c,b) = \varphi(b,d) = \varphi(d,b) = \varphi(b,1) = \varphi(1,b) = \varphi(c,d) = \varphi(d,c) = \varphi(c,1) = \varphi(1,c) = \varphi(1,d) = \varphi(d,1) = 0.8$. Then φ is a fuzzy congruence on L.

Theorem 3.3. *A fuzzy equivalence relation is a fuzzy congruence on $(L,^\circ,^*)$ if and only if $\phi(x,y) \leq \phi(x\wedge z, y\wedge z)\wedge\phi(x\vee z, y\vee z)\wedge\phi(x^\circ,y^\circ)\wedge\phi(x^*,y^*)$, for all $x,y,z \in L$.*

Proof. The forward proof is clear. Conversely, let ϕ be a fuzzy equivalence relation satisfying the following: for all $x,y,z \in L$,

$$\phi(x,y) \leq \phi(x \wedge z, y \wedge z) \wedge \phi(x \vee z, y \vee z) \wedge \phi(x^\circ, y^\circ) \wedge \phi(x^*, y^*).$$

Then for all $x,y,z \in L$,

$$\phi(x,y) \leq \phi(x \wedge z, y \wedge z),\ \ \phi(x,y) \leq \phi(x \vee z, y \vee z),$$
$$\phi(x,y) \leq \phi(x^\circ, y^\circ),\ \ \phi(x,y) \leq \phi(x^*, y^*).$$

Thus or all $x,y,z,w \in L$,

$$\phi(x,y) \wedge \phi(z,w) \leq \phi(x \wedge z, y \wedge z) \wedge \phi(y \wedge z, y \wedge w) \leq \phi(x \wedge z, y \wedge w).$$

Similarly, $\phi(x,y) \wedge \phi(z,w) \leq \phi(x \vee z, y \vee w)$, for all $x,y,z,w \in L$. So ϕ is a fuzzy congruence relation of $(L,^\circ,^*)$. □

Example 3.4. Let $(L,^\circ,^*)$ be an dpMS-algebra. Define the fuzzy relations ϕ and G_F on L by:

$$\phi(x,y) = \begin{cases} 1 & if x^\circ = y^\circ \\ 0 & otherwise \end{cases}$$

and

$$G_F(x,y) = \begin{cases} 1 & if x^* = y^* \\ 0 & otherwise, \end{cases}$$

for any $x,y \in L$. Then

(1) ϕ and G_F are fuzzy congruences of $(L,^\circ,^*)$,
(2) $\phi \subseteq \Theta$.

We call the fuzzy congruence G_F is Glievenko fuzzy congruence.

Theorem 3.5. *A fuzzy relation ϕ on $(L,^\circ,^*)$ is a fuzzy congruence relation if and only if every level subsets ϕ_α of $(L,^\circ,^*)$, $\alpha \in [0,1]$ is a congruence relation on $(L,^\circ,^*)$.*

Corollary 3.6. *An equivalence relation ϕ is a congruence relation on L if and only if its characteristic function χ_ϕ is a fuzzy congruence on L.*

Lemma 3.7. *If $\{\phi_i : i \in \Delta\}$ is a family of fuzzy congruence of $(L,^\circ,^*)$, then $\cap_{i\in\Delta}\phi_i$ is a fuzzy congruence on $(L,^\circ,^*)$.*

The set of all fuzzy congruences of L is denoted by $\mathcal{FC}(L)$ and the set of all congruences of L is denoted by $\mathcal{C}(L)$. $\omega = \{(x,y) \in L \times L : x = y\}$ and $\iota = L \times L$ are the smallest and the largest elements of $\mathcal{C}(L)$ respectively and

$$\chi_\omega(x,y) = \begin{cases} 1 & if (x,y) \in \omega \\ 0 & otherwise, \end{cases}$$

for all $x,y \in L$ and $\chi_\iota(x,y) = 1$ for all $x,y \in L$ are the smallest and the largest elements of $\mathcal{FC}(L)$ respectively.

Theorem 3.8. *$(\mathcal{FC}(L), \subseteq)$ is a complete lattice.*

Proof. Clearly, we note that both fuzzy congruence relations χ_ω and χ_ι are the least and the greatest elements of $\mathcal{FC}(L)$, respectively. Then clearly, $\cap_{i\in\Delta}\phi_i$ is a lower bound of any family $\{\phi_i : i \in \Delta\}$ of fuzzy congruences of L and $(\mathcal{FC}(L), \subseteq)$ is poset. Let Θ be any lower bound of $\{\phi_i : i \in \Delta\}$. Then $\Theta \subseteq \phi_i$, for all $i \in \Delta$. Thus $\Theta \subseteq \cap_{i\in\Delta}\phi_i$. So $\cap_{i\in\Delta}\phi_i$ is a greatest lower bound of $\{\phi_i : i \in \Delta\}$. Hence $(\mathcal{FC}(L), \subseteq)$ is a complete lattice. □

Next we define the fuzzy quotient demi-pseudocomplemented MS-algebra induced by fuzzy congruence relation.

Definition 3.9. Let $(L, \wedge, \vee,^\circ,^*, 0, 1)$ be a demi-pseudocomplemented MS-algebra, $x \in L$ and θ be a fuzzy congruence on L. The fuzzy congruence determined by x and θ, denoted by θ_x, is the fuzzy subset of L defined by $\theta_x(y) = \theta(x,y)$, $\forall\, y \in L$.

Let L/θ denote the set of all fuzzy congruence class, that is $L/\theta = \{\theta_x : x \in L\}$.

Remark 3.10. If θ is a fuzzy congruence of L and $x, y \in L$, then $\theta_x = \theta_y \Leftrightarrow \theta(x, y) = 1$.

Theorem 3.11. *Let θ be a congruence of a demi-Pseudocomplemented MS-algebra $(L, \wedge, \vee, ^\circ, ^*, 0, 1)$. For any $\theta_x,\ \theta_y \in L/\theta$, define*

$$\theta_x \wedge \theta_y = \theta_{x\wedge y},\ \theta_x \vee \theta_y = \theta_{x\vee y},\ (\theta_x)^\circ = \theta_{x^\circ},\ \text{and}\ (\theta_x)^* = \theta_{x^*}.$$

Then $(L/\theta, \wedge, \vee, ^\circ, ^*, \theta_0, \theta_1)$ is a demi-pseudocomplemented MS-algebra, where θ_0, and θ_1 are the smallest and largest elements of L/θ respectively.

Proof. we should first prove that the operation on L/θ is well defined. To do this, for any any $x, y, w\ and\ z \in L$, suppose $\theta_x = \theta_w$ and $\theta_y = \theta_z$. Then

$$\begin{aligned} &\theta(x, w) = 1\ and\ \theta(y, z) = 1 \\ \Longrightarrow\ &\theta(x \wedge y, w \wedge z) \geq \theta(x, w) \wedge \theta(y, z) = 1 \\ \Longrightarrow\ &\theta_{x\wedge y} = \theta_{w\wedge z}. \end{aligned}$$

Similarly, $\theta_{x\vee y} = \theta_{w\vee z}$. Thus the operations of L/θ are well defined.

For any $x \in L$, $\theta_x \wedge \theta_0 = \theta_{x\wedge 0} = \theta_0$. Then $\theta_0 \leq \theta_x$, for all $\theta_x \in L/\theta$. Thus θ_0 is smallest element of L/θ. Similarly, θ_1 is largest element of L/θ.

Now let $\theta_x, \theta_y \in L/\theta$. Then

$$(\theta_x \vee \theta_y)^* = (\theta_{x\vee y})^* = \theta_{(x\vee y)^*} = \theta_{x^*\wedge y^*} = \theta_{x^*} \wedge \theta_{y^*} = (\theta_x)^* \wedge (\theta_x)^*.$$

Thus

$$\begin{aligned} (\theta_x \wedge \theta_y)^{**} &= (\theta_{x\wedge y})^{**} = \theta_{(x\wedge y)^{**}} = \theta_{x^{**}\wedge y^{**}} \\ &= \theta_{x^{**}} \wedge \theta_{y^{**}} = (\theta_x)^{**} \wedge (\theta_x)^{**}. \end{aligned}$$

Similarly, we can easily show that $(\theta_0)^* = \theta_1$, $(\theta_1)^* = \theta_0$, $(\theta_x)^{***} = (\theta_x)^*$ and $(\theta_x)^* \wedge (\theta_y)^{***} = \theta_0$ and $(\theta_x)^* \wedge ((\theta_x)^* \wedge \theta_y)^* = (\theta_x)^* \wedge (\theta_y)^*$. HsO $(L/\theta, \wedge, \vee, ^*, \theta_0, \theta_1)$ is a demi-Pseudocomplemented algebra. Similarly, we can see that $(L/\theta, \wedge, \vee, ^\circ, \theta_0, \theta_1)$ is an MS-algebra. Hence $(L/\theta, \wedge, \vee, ^\circ, ^*, \theta_0, \theta_1)$ is a demi-pseudocomplemented MS-algebra. □

L/θ is called the fuzzy quotient a demi-pseudocomplemented MS-algebra L induced by θ. It is clear that the map $x \longrightarrow L/\theta$ is a homomorphism from L onto L/θ.

Example 3.12. As shown in Example 3.2, we can easily verified that $L/\varphi = \{\varphi_0, \varphi_a, \varphi_b, \varphi_c, \varphi_d, \varphi_1\}$, and $(L/\varphi, \wedge, \vee, ^\circ, ^*, \varphi_0, \varphi_1)$ is a demi-Pseudocomplemented MS-algebra, where φ_0, and φ_1 are the smallest and largest elements of L/φ respectively.

Now we define the join of two fuzzy congruence of L.

Definition 3.13. Let ϕ and φ be any two fuzzy congruence relations of a dpMS-algebra $(L, ^\circ, ^*)$. Then define $\phi \vee \varphi = \cap\{\Theta \in \mathcal{FC}(L) : \phi \subseteq \Theta$ and $\varphi \subseteq \Theta\}$, i.e., $\phi \vee \varphi$ is the smallest fuzzy congruence containing $\phi \cup \varphi$.

Theorem 3.14. *Let ϕ and φ be any fuzzy congruence relations on a dpMS-algebra $(L, ^\circ, ^*)$. Then $\phi \vee \varphi = \cup_{n=1}^{\infty}\Theta_n$, where $\Theta_1 = \phi \circ \varphi \circ \phi$, $\Theta_2 = \phi \circ \varphi \circ \phi \circ \varphi \circ \phi$, $\Theta_3 = \phi \circ \varphi \circ \phi \circ \varphi \circ \phi \circ \varphi \circ \phi$,... .*

Proof. Let $\kappa = \cup_{n=0}^{\infty}\Theta_n$. We prove that κ is the smallest fuzzy congruence relation in a dpMS-algebra $(L, ^\circ, ^*)$ containing ϕ and φ. It can be easily verified that $\phi \subseteq \Theta_1 \subseteq \Theta_2 \subseteq$, ... and $\varphi \subseteq \Theta_1 \subseteq \Theta_2 \subseteq$, ... and so $\Theta_n \subseteq \phi \vee \varphi$.

Now we see that κ is a fuzzy congruence relation in dpMS-algebra $(L,^{\circ},^{*})$.

(1) $1 = \phi(x,x) \leq \Theta_1(x,x) \leq \cup_{n=0}^{\infty}\Theta_n = \kappa(x,x)$. Then $\kappa(x,x) = 1$.

(2) Symmetric is straightforward.

(3) $\kappa(x,y) \wedge \kappa(y,z) = \cup_{n=1}^{\infty}\Theta_n(x,y) \wedge \cup_{n=1}^{\infty}\Theta_n(y,z) = \sup_n \Theta_n(x,y) \wedge \sup_n \Theta_n(y,z) \leq \cup_{n=1}^{\infty}\Theta_n(x,z)$ since $\Theta_n(x,y) \wedge \Theta_m(y,z) \leq \Theta_{n+m}(x,z)$, for any real number n and m.

(4)
$$\begin{aligned}\kappa(x,y) &= \cup_{n=1}^{\infty}\Theta_n(x,y)\\ &= \sup_n(\sup_{z_1,z_2,\ldots,z_{2n}}(\phi(x,z_1) \wedge \varphi(z_1,z_2) \wedge \phi(z_2,z_3) \wedge \ldots \wedge \phi(z_{2n},y)))\\ &\leq \sup_n(\sup_{z_1\wedge c,z_2\wedge c,\ldots,z_{2n}\wedge c}(\phi(x\wedge c,z_1\wedge c) \wedge \varphi(z_1\wedge c,z_2\wedge c)\\ &\qquad \wedge\phi(z_2\wedge c,z_3\wedge c)\ldots \wedge \phi(z_{2n}\wedge c,y\wedge c)))\\ &= \cup_{n=1}^{\infty}\Theta_n(x\wedge c,y\wedge c) = \kappa(x\wedge c,y\wedge c).\end{aligned}$$

Similarly, we can show that $\kappa(x,y) \leq \kappa(x\vee c,y\vee c)$.

$$\begin{aligned}(5)\ \kappa(x,y) &= \cup_{n=1}^{\infty}\Theta_n(x,y)\\ &= \sup_n(\sup_{z_1,z_2,\ldots,z_{2n}}(\phi(x,z_1) \wedge \varphi(z_1,z_2) \wedge \phi(z_2,z_3) \wedge \ldots \wedge \phi(z_{2n},y)))\\ &\leq \sup_n(\sup_{z_1^{\circ},z_2^{\circ},\ldots,z_{2n}^{\circ}}(\phi(x^{\circ},z_1^{\circ}) \wedge \varphi(z_1^{\circ},z_2^{\circ}) \wedge \phi(z_2^{\circ},z_3^{\circ})\ldots \wedge \phi(z_{2n}^{\circ},y^{\circ})))\\ &= \cup_{n=1}^{\infty}\Theta_n(x^{\circ},y^{\circ}) = \kappa(x^{\circ},y^{\circ}).\\ (6)\ \kappa(x,y) &= \cup_{n=1}^{\infty}\Theta_n(x,y)\\ &= \sup_n(\sup_{z_1,z_2,\ldots,z_{2n}}(\phi(x,z_1) \wedge \varphi(z_1,z_2) \wedge \phi(z_2,z_3) \wedge \ldots \wedge \phi(z_{2n},y)))\\ &\leq \sup_n(\sup_{z_1^{*},z_2^{*},\ldots,z_{2n}^{*}}(\phi(x^{*},z_1^{*}) \wedge \varphi(z_1^{*},z_2^{*}) \wedge \phi(z_2^{*},z_3^{*})\ldots \wedge \phi(z_{2n}^{*},y^{*})))\\ &= \cup_{n=1}^{\infty}\Theta_n(x^{*},y^{*}) = \kappa(x^{*},y^{*}).\end{aligned}$$

This implies κ is fuzzy congruence of a dpMS-algebra $(L,^{\circ},^{*})$.

Finally, let τ be any fuzzy congruence relation such that $\phi \subseteq \tau$ and $\varphi \subseteq \tau$. Then we prove that $\kappa \subseteq \tau$.

$$\begin{aligned}\kappa(x,y) &= \cup_{n=1}^{\infty}\Theta_n(x,y)\\ &= \sup_n(\sup_{z_1,z_2,\ldots,z_{2n}}(\phi(x,z_1) \wedge \varphi(z_1,z_2) \wedge \phi(z_2,z_3)\ldots \wedge \phi(z_{2n},y)))\\ &\leq \sup_n(\sup_{z_1,z_2,\ldots,z_{2n}}(\tau(x,z_1) \wedge \tau(z_1,z_2) \wedge \ldots \wedge \tau(z_{2n},y)))\\ &= \sup_n \tau(x,y) = \tau(x,y).\end{aligned}$$

Thus κ is the smallest fuzzy congruence such that $\phi \subseteq \tau$ and $\varphi \subseteq \tau$. So $\phi \vee \varphi = \cup_{n=0}^{\infty}\Theta_n$. □

Definition 3.15. Let L be an dpMS-algebra. The fuzzy congruence generated by the fuzzy relation ϕ of L is defined by $\overline{\Theta}(\phi) = \cap\{\vartheta \in FC(L) : \phi \subseteq \vartheta\}$. If $\phi = \mu \times \mu$ is the product of fuzzy ideal μ by itself, where $(\mu \times \mu)(x,y) = \mu(x) \wedge \mu(y)$ for all $(x,y) \in L \times L$. We write $\overline{\Theta}[\mu]$ instead of $\overline{\Theta}(\phi)$ i.e $\overline{\Theta}[\mu]$ is the smallest fuzzy congruence containing $\mu \times \mu$.

Theorem 3.16. *Let ϕ is a fuzzy relation of dpMS-algebra $(L,^{\circ},^{*})$. Then $\overline{\Theta}(\phi)(x,y) = \sup\{\alpha : (x,y) \in \Theta(\phi_\alpha)\}$, for any $(x,y) \in L \times L$ and $\alpha \in [0,1]$.*

Proof. Let $\varphi(x,y)=\sup\{\alpha:(x,y)\in\Theta(\phi_\alpha)\}$, for any $(x,y)\in L\times L$ and $\alpha\in[0,1]$. Then we prove that $\varphi=\overline{\Theta}(\phi)$. First we see that φ is a fuzzy congruence of $(L,^\circ,^*)$.

(1) $\varphi(x,x)=\sup\{\alpha:(x,x)\in\Theta(\phi_\alpha)\}=1$.

(2) Symmetric is Straightforward.

(3)
$$\begin{aligned}\varphi(x,y)\wedge\varphi(y,z)&=\sup\{\alpha:(x,y)\in\Theta(\phi_\alpha)\}\wedge\sup\{\lambda:(y,z)\in\Theta(\phi_\lambda)\}\\&=\sup\{\alpha\wedge\lambda:(x,y)\in\Theta(\phi_\alpha),\ (y,z)\in\Theta(\phi_\lambda)\}\\&\le\sup\{\alpha\wedge\lambda:(x,y)\in\Theta(\phi_{\alpha\wedge\lambda}),\ (y,z)\in\Theta(\phi_{\alpha\wedge\lambda})\}\\&\le\sup\{\alpha\wedge\lambda:(x,z)\in\Theta(\phi_{\alpha\wedge\lambda})\}\\&=\varphi(x,z).\end{aligned}$$

(4)
$$\begin{aligned}\varphi(x,y)\wedge\varphi(w,z)&=\sup\{\alpha:(x,y)\in\Theta(\phi_\alpha)\}\wedge\sup\{\lambda:(w,z)\in\Theta(\phi_\lambda)\}\\&=\sup\{\alpha\wedge\lambda:(x,y)\in\Theta(\phi_\alpha),\ (w,z)\in\Theta(\phi_\lambda)\}\\&\le\sup\{\alpha\wedge\lambda:(x,y)\in\Theta(\phi_{\alpha\wedge\lambda}),\ (w,z)\in\Theta(\phi_{\alpha\wedge\lambda})\}\\&\le\sup\{\alpha\wedge\lambda:(x\wedge w,y\wedge z)\in\Theta(\phi_{\alpha\wedge\lambda})\}\\&=\varphi(x\wedge w,y\wedge z).\end{aligned}$$

Similarly, we can prove that $\varphi(x,y)\wedge\varphi(y,z)\le\varphi(x\vee w,y\vee z)$.

(5) $\varphi(x,y)=\sup\{\alpha:(x,y)\in\Theta(\phi_\alpha)\}\le\sup\{\alpha:(x^\circ,y^\circ)\in\Theta(\phi_\alpha)\}=\varphi(x^\circ,y^\circ)$. Similarly, $\varphi(x,y)\le\varphi(x^*,y^*)$. Then φ is a fuzzy congruence of a dpMS-algebra $(L,^\circ,^*)$. Now $\phi(x,y)=\{\alpha:(x,y)\in\phi_\alpha\}\le\{\alpha:(x,y)\in\Theta(\phi_\alpha)\}=\varphi(x,y)$, for any $x,y\in L$. Thus $\phi\subseteq\varphi$. Finally, let τ be any fuzzy congruence of a dpMS-algebra $(L,^\circ,^*)$ such that $\phi\subseteq\tau$. We see that $\varphi\subseteq\tau$. If $\phi\subseteq\tau$, then $\phi_\alpha\subseteq\tau_\alpha$ and thus $\Theta(\phi_\alpha)\subseteq\Theta(\tau_\alpha)=\tau_\alpha$. Now,

$$\varphi(x,y)=\{\alpha:(x,y)\in\Theta(\phi_\alpha)\}\le\{\alpha:(x,y)\in\tau_\alpha\}=\tau(x,y).$$

So $\varphi\subseteq\tau$. Hence $\varphi=\overline{\Theta}(\phi)$. □

4. Kernel fuzzy ideals and fuzzy congruences

Definition 4.1. Kernel fuzzy ideal of a dpMS-algebra $(L,^\circ,^*)$ is a fuzzy ideal μ of L for which there exists a fuzzy congruence φ on L such that $\mu=Ker\varphi$ i.e., μ is the kernel fuzzy congruence φ , where $Ker\varphi(x)=\varphi(x,0)$, for all $x\in L$.

Definition 4.2. Cokernel fuzzy filter of a dpMS-algebra $(L,^\circ,^*)$ is a fuzzy filter η of L for which there exists a fuzzy congruence ψ on L such that $\eta=CoKer\psi$ i.e η is the Cokernel fuzzy congruence ψ , where $Coker\psi(x)=\psi(x,1)$, for all $x\in L$.

Example 4.3. In Example 3.2, φ is a fuzzy congruence of a dpMS-algebra $(L,^\circ,^*)$. Define a fuzzy subset μ and η of L as follows:

$$\mu(0)=1,\ \mu(a)=\mu(b)=\mu(c)=\mu(d)=\mu(1)=0.8$$

and

$$\eta(1)=1,\ \eta(a)=\eta(b)=\eta(c)=\eta(d)=\mu(1)=0.8.$$

Then $\mu(x)=\varphi(0,x)$ for all $x\in L$ and $\eta(x)=\varphi(1,x)$, for all $x\in L$. Thus μ is a kernel fuzzy ideal and η is a cokernel fuzzy filter of a dpMS-algebra $(L,^\circ,^*)$.

In the the following Theorem, we characterized a kernel fuzzy ideal of a dpMS-algebra.

Theorem 4.4. *Let $(L,^\circ,^*)$ be a dpMS-algebra. If μ be a kernel fuzzy ideal, then* $\mu(x)\le\mu(x^{\circ\circ})\wedge\mu(x^{\circ*})\wedge\mu(x^{**})$, $\forall\, x\in L$.

Proof. The proof follows from the Definition 4.1. □

For a fuzzy ideal μ of a dpMS-algebra $(L,^\circ,^*)$, define

$$\mu_\circ^{\geq}(x) = \sup\{\mu(i) : i^\circ \leq x,\ i \in L\},\ for\ \forall x \in L,$$
$$\mu^\circ(x) = \sup\{\mu(i) : i^\circ = x,\ i \in L\},\ for\ \forall x \in L,$$
$$\mu_{\circ\circ}(x) = \sup\{\mu(i) : x \leq i^{\circ\circ},\ i \in L\},\ for\ \forall x \in L.$$

Then it can be easily verified that $\mu_\circ^{\geq}$ is a fuzzy filter of L, $\mu_{\circ\circ}$ is a fuzzy ideal of L and the α-level subsets of $\mu_\circ^{\geq}$, μ°, and $\mu_{\circ\circ}$ are

$$(\mu_\circ^{\geq})_\alpha = \{x \in L : (\exists i \in (\mu_\circ^{\geq})_\alpha)\ i^\circ \leq x\},$$

$$\mu_\alpha^\circ = \{y \in L, x^\circ = y : \exists x \in \mu_\alpha\}$$

and

$$(\mu_{\circ\circ})_\alpha = \{x \in L : (\exists i \in \mu_\alpha)\ x \leq i^{\circ\circ}\},\ \text{respectively.}$$

By Definition 3.15, Theorem 3.16 and equations (2.1), (2.2), we can write

$$\begin{aligned}\overline{\Theta}_{lat}[\mu](x,y) &= \sup\{\alpha : (x,y) \in \Theta_{lat}[\mu_\alpha]\},\\ \overline{\Theta}_{lat}[\mu_{\circ\circ}](x,y) &= \sup\{\alpha : (x,y) \in \Theta_{lat}[(\mu_{\circ\circ})_\alpha]\}\\ &= \sup\{\alpha : x \vee i = y \vee i,\ i \in (\mu_{\circ\circ})_\alpha]\}\\ &= \sup\{\alpha : x \vee i = y \vee i,\ i \leq j^{\circ\circ}\ for\ some\ j \in \mu_\alpha\}\end{aligned}$$

and

$$\begin{aligned}\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x,y) &= \sup\{\alpha : (x,y) \in \Theta_{lat}[(\mu_\circ^{\geq})_\alpha]\}\\ &= \sup\{\alpha : x \wedge i = y \wedge i,\ i \in (\mu_\circ^{\geq})_\alpha]\}\\ &= \sup\{\alpha : x \wedge i = y \wedge i,\ j^\circ \leq i\ for\ some\ j \in \mu_\alpha\}.\end{aligned}$$

Also by equation (2.3), we have $\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x,y) = \sup\{\alpha : x \wedge i^\circ = y \wedge i^\circ,\ i \in \Theta[\mu_\alpha]\}$.

Corollary 4.5. *If μ is a kernel fuzzy ideal of a dpMS-algebra $(L,^\circ,^*)$, then $\mu = \mu_{\circ\circ}$ (i.e $\mu(x^{\circ\circ}) \geq \mu(x)$), for $\forall\, x \in L$.*

Proof. $\mu_{\circ\circ}(x) = \sup\{\mu(i) : x \leq i^{\circ\circ}\} \leq \sup\{\mu(i^{\circ\circ}) : x \leq i^{\circ\circ}\} \leq \mu(x)$. It is obvious that $\mu \subseteq \mu_{\circ\circ}$. This implies $\mu = \mu_{\circ\circ}$. □

The description of $\overline{\Theta}_{lat}[\mu_\circ^{\geq}]$ can be characterized as follows.

Lemma 4.6. *Let $(L,^\circ,^*)$ be a dpMS-algebra and μ be a kernel fuzzy ideal of L. Then following conditions hold.*

(1) $\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x,y) \leq \overline{\Theta}_{lat}[\mu](x^\circ,y^\circ] \wedge \overline{\Theta}_{lat}[\mu](x^*,y^*)$,
(2) $\overline{\Theta}_{lat}[\mu](x,y) \leq \overline{\Theta}_{lat}[\mu_\circ^{\geq}](x^\circ,y^\circ) \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](x^*,y^*)$.

Proof. (1)
$$\begin{aligned}\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x,y) &= \sup\{\alpha : (x,y) \in \Theta_{lat}[(\mu_\circ^{\geq})_\alpha]\}\\ &= \sup\{\alpha : x \wedge i = y \wedge i,\ i \in (\mu_\circ^{\geq})_\alpha\}\\ &= \sup\{\alpha : x \wedge i = y \wedge i,\ j^\circ \leq i\ for\ some\ j \in \mu_\alpha\}\\ &\leq \sup\{\alpha : x^\circ \vee i^\circ = y^\circ \vee i^\circ,\ i^\circ \leq j^{\circ\circ}\ for\ some\ j \in \mu_\alpha\}\\ &= \overline{\Theta}_{lat}[\mu_{\circ\circ}](x^\circ,y^\circ) = \overline{\Theta}_{lat}[\mu](x^\circ,y^\circ),\end{aligned}$$

since μ is a kernel fuzzy ideal of L. Also

$$\begin{aligned}\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x,y) &= \sup\{\alpha : (x,y) \in \Theta_{lat}[(\mu_\circ^{\geq})_\alpha]\}\\ &= \sup\{\alpha : x \wedge i = y \wedge i,\ i \in (\mu_\circ^{\geq})_\alpha\}\end{aligned}$$

$$\leq \sup\{\alpha : x^* \vee i^* = y^* \vee i^*,\ i^* \leq j^{\circ *} = j^{\circ * \circ\circ}\ for\ some\ j^{\circ *} \in \mu_\alpha\}$$
$$= \overline{\Theta}_{lat}[\mu_{\circ\circ}](x^*, y^*) = \overline{\Theta}_{lat}[\mu](x^*, y^*).$$

Thus $\overline{\Theta}_{lat}(\mu_\circ^{\geq})(x, y) \leq \overline{\Theta}_{lat}(\mu)(x^\circ, y^\circ) \wedge \overline{\Theta}_{lat}(\mu)(x^*, y^*)$.

(2)
$$\begin{aligned}\overline{\Theta}_{lat}[\mu](x, y) &= \sup\{\alpha : (x, y) \in \Theta_{lat}[\mu_\alpha]\}\\ &= \sup\{\alpha : x \vee i = y \vee i,\ i \in \mu_\alpha\}\\ &= \sup\{\alpha : x \vee i = y \vee i,\ i \in (\mu_{\circ\circ})_\alpha\}\\ &= \sup\{\alpha : x \vee i = y \vee i,\ i \leq j^{\circ\circ},\ j \in \mu_\alpha\}\\ &\leq \sup\{\alpha : x^\circ \wedge i^\circ = y^\circ \wedge i^\circ,\ j^\circ \leq i^\circ,\ j \in \mu_\alpha\}\\ &= \overline{\Theta}_{lat}[\mu_\circ^{\geq}](x^\circ, y^\circ),\end{aligned}$$

$$\begin{aligned}\overline{\Theta}_{lat}[\mu](x, y) &= \sup\{\alpha : (x, y) \in \Theta_{lat}[\mu_\alpha]\}\\ &= \sup\{\alpha : x \vee i = y \vee i,\ i \in \mu_\alpha\}\\ &= \sup\{\alpha : x \vee i = y \vee i,\ i \in (\mu_{\circ\circ})_\alpha\}\\ &= \sup\{\alpha : x \vee i = y \vee i,\ i \leq j^{\circ\circ},\ j \in \mu_\alpha\}\\ &\leq \sup\{\alpha : x^* \wedge i^* = y^* \wedge i^*,\ j^{\circ\circ *} \leq i^* = i^{*\circ\circ},\ j \in \mu_\alpha\}\\ &= \overline{\Theta}_{lat}(\mu_\circ^{\geq})(x^*, y^*).\end{aligned}$$

Thus $\overline{\Theta}_{lat}[\mu](x, y) \leq \overline{\Theta}_{lat}[\mu_\circ^{\geq}](x^\circ, y^\circ) \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](x^*, y^*)$. □

Theorem 4.7. *Let $(L, ^\circ, ^*)$ be a dpMS-algebra, and μ be a kernel fuzzy ideal of L. Then we have $\overline{\Theta}[\mu] = \overline{\Theta}_{lat}[\mu] \vee \overline{\Theta}_{lat}[\mu_\circ^{\geq}]$.*

Proof. Let $\varphi = \overline{\Theta}_{lat}[\mu_\circ^{\geq}] \vee \overline{\Theta}_{lat}[\mu]$. We see that φ is the smallest fuzzy congruence containing $\mu \times \mu$. Clearly, $\mu \times \mu \subseteq \varphi$. Now we prove that φ is a fuzzy congruence of L. Since $\overline{\Theta}_{lat}[\mu_\circ^{\geq}]$ and $\overline{\Theta}_{lat}[\mu]$ are lattice fuzzy congruences, φ is a lattice fuzzy congruence of L.

By Theorem 3.16 and Lemma 4.6 ,

$$\begin{aligned}&\varphi(x, y)\\ &= \sup_n(\sup_{z_1, z_2, \ldots, z_{2n}}(\overline{\Theta}_{lat}[\mu_\circ^{\geq}](x, z_1) \wedge \overline{\Theta}_{lat}[\mu](z_1, z_2) \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](z_2, z_3) \wedge\\ &\qquad \ldots \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](z_{2n}, y))\\ &\leq \sup_n(\sup_{z_1^\circ, z_2^\circ, \ldots, z_{2n}^\circ}(\overline{\Theta}_{lat}[\mu](x^\circ, z_1^\circ) \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](z_1, z_2) \wedge \overline{\Theta}_{lat}[\mu](z_2^\circ, z_3^\circ) \wedge\\ &\qquad \ldots \wedge \overline{\Theta}_{lat}[\mu_\circ^{\geq}](z_{2n}^\circ, y)^\circ))\\ &= \varphi(x^\circ, y^\circ).\end{aligned}$$

Similarly, $\varphi(x, y) \leq \varphi(x^*, y^*)$. Thus φ is a fuzzy congruence of a dpMS-algebra L.

Finally, we see that φ is the smallest fuzzy congruence containing $\mu \times \mu$. Since φ is a fuzzy congruence containing $\mu \times \mu$, $\overline{\Theta}[\mu] \subseteq \varphi$.
Clearly $\overline{\Theta}_{lat}[\mu] \subseteq \overline{\Theta}[\mu]$ and also by Theorem 2.7 and Corollary 4.5, $\overline{\Theta}_{lat}[(\mu_\circ^{\geq})](x, y) = \sup\{\alpha : (x, y) \in \Theta_{lat}[(\mu_\circ^{\geq})_\alpha]\} \leq \sup\{\alpha : (x, y) \in \Theta[\mu_\alpha]\} = \overline{\Theta}[\mu](x, y)$. Thus $\overline{\Theta}_{lat}[\mu_\circ^{\geq}] \vee \overline{\Theta}_{lat}[\mu] \subseteq \overline{\Theta}[\mu]$. sO $\varphi \subseteq \overline{\Theta}[\mu]$. Hence $\varphi = \overline{\Theta}[\mu]$. □

5. $(\circ, *)$-fuzzy ideals

Definition 5.1. A fuzzy ideal μ of a dpMS-algebra $(L, ^\circ, ^*)$ is said to be a $(\circ, *)$-fuzzy ideal if $\mu(x) = \mu(x^{\circ *})$ for all $x \in L$.

Example 5.2. In Example 3.2, μ is kernel fuzzy ideal of φ but not $(\circ, *)$-fuzzy ideal, because $0.8 = \mu(b) \neq \mu(b^{\circ *}) = \mu(a^*) = \mu(0) = 1$.

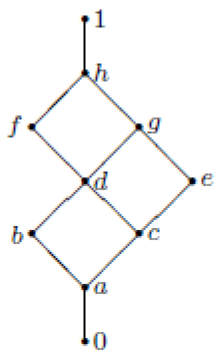
Theorem 5.3. *Let $(L, ^\circ, ^*)$ be a dpMS-algebra. If μ is $(\circ, *)$-fuzzy ideal of L, then $\mu(x^{**}) = \mu(x)\ \forall\ x \in L$.*

Proof. Suppose that μ is a $(\circ,*)$-fuzzy ideal of L. Then $\mu(x)=\mu(x^{\circ *})=\mu(x^{(\circ *)\circ *})=\mu(x^{**})$. $\square$

Theorem 5.4. *μ is a $(\circ,*)$-fuzzy ideal of L if and only if μ_α, $\forall\ \alpha\in[0,1]$ is a $(\circ,*)$-ideal of L.*

Corollary 5.5. *I is a $(\circ,*)$-ideal of L if and only if χ_I is $(\circ,*)$-fuzzy ideal of L.*

Example 5.6. Consider the dpMS-algebra $(L,^\circ,^*)$ given in Hasse diagram 2 below:



	0	a	b	c	d	e	f	g	h	1
$\circ$	1	h	h	h	h	f	g	f	d	0
$*$	1	1	1	1	1	0	1	0	0	0

.

diagram 2

Define a fuzzy subset μ of L as follows:

$$\mu(0)=\mu(a)=\mu(b)=\mu(c)=\mu(d)=\mu(f)=1$$

and

$$\mu(e)=\mu(g)=\mu(h)=\mu(1)=0.7.$$

Then it can be easily verified that μ is a $(\circ,*)$-fuzzy ideal.

As indicated in Example 5.6 , $KerG_F=\mu$ is a $(\circ,*)$-fuzzy ideal. The following Theorem shows that it is true in general case.

Theorem 5.7. *If $(L,^\circ,^*)$ is a dpMS-algebra, then $KerG_F$ is the smallest a $(\circ,*)$-fuzzy ideal of L.*

Proof. We prove that $KerG_F(x^{*\circ})=G_F(x^{*\circ},0)=KerG_F(x)$,for any $x\in L$.

Suppose $G_F(x^{*\circ},0)=1$. Then $x^{*\circ *}=0^*$. This implies $x^{*\circ *\circ}=0^{*\circ}$. Thus we have $x^{**}=x^{*\circ *\circ}=0$. So $x^{***}=x^{*\circ *\circ}=0^*=x^*$. Hence $G_F(x^{**},0)=G_F(x,0)=1$. Therefore $KerG_F(x^{*\circ})=KerG_F(x^{**})=KerG_F(x)=1$.

Suppose $G_F(x^{*\circ},0)=0$. Then $x^{*\circ *}\neq 0^*$. We see that $x^{***}\neq 0^*$ and $x^*\neq 0^*$. Now we prove by contradiction. Assume that $x^*=0^*$. Then $x^{*\circ *}=0^*$. This contradicts the hypothesis. Also assume that $x^{***}=0^*$. Then $x^{*\circ *}=x^{*\circ *\circ *}=0^{*\circ *}=0^*$. It also contradicts the hypothesis. Thus

$$KerG_F(x^{*\circ})=KerG_F(x^{**})=kerG_F(x)=0.$$

So in either cases, for any $x\in L$, $KerG_F(x^{*\circ})=KerG_F(x^{**})=KerG_F(x)$. Hence $KerG_F$ is $(\circ,*)$-fuzzy ideal of L.

Let μ be any $(\circ,*)$-fuzzy ideal. If $KerG_F(x) = G_F(x,0) = 1$, then $x^* = 0^*$. We have $x^{**} = 0^{**} = 0$. This implies

$$KerG_F(x) = KerG_F(x^{**}) = 1 = \mu(0) = \mu(x^{**}) = \mu(x).$$

If $KerG_F(x) = 0$, then $KerG_F(x) = 0 \leq \mu(x)$. Thus for any $x \in L$, $KerG_F(x) \leq \mu(x)$. So $KerG_F \subseteq \mu$. Hence $KerG_F$ is the smallest $(\circ,*)$-fuzzy ideal of L. □

Corollary 5.8. *$(L,^\circ,^*)$ be a dpMS-algebra. The $KerG_F = \chi_{\{0\}}$ if and only if every fuzzy congruence kernel of L is a $(\circ,*)$-fuzzy ideal.*

Proof. (Necessity). Suppose $KerG_F = \chi_{\{0\}}$. Then for every $x \in L$, we have $x \wedge x^* = 0$. Let μ be the kernel of any fuzzy congruence φ of L, i.e., $\mu = Ker\varphi$ and let $x \in L$. Then $\mu(x) = Ker\varphi(x) = \varphi(x,0) \leq \varphi(x^{*\circ},0) = \mu(x^{*\circ})$. Conversely, $\mu(x^{*\circ}) = \varphi(x^{*\circ},0) \leq \varphi(x^*,1) = \varphi(x^*,1) \wedge \varphi(x,x) \leq \varphi(x \wedge x^*, x) = \varphi(0,x) = \mu(x)$. Thus $\mu(x) = \mu(x^{*\circ})$, $\forall\, x \in L$ and so μ is a $(\circ,*)$-fuzzy ideal of L.

(Sufficiency). Suppose every fuzzy congruence kernel of L is a $(\circ,*)$-fuzzy ideal. Then by Theorem 5.7 , $KerG_F(x) = 1 = Ker\chi_\omega(x)$. Thus $KerG_F = \chi_{\{0\}}$. □

Corollary 5.9. *Let $(L,^\circ,^*)$ be a dpMS-algebra. If φ is a fuzzy congruence on L such that $G_F \subseteq \varphi$, then $Ker\varphi$ is a $(\circ,*)$-fuzzy ideal of L.*

We denote the set of all $(\circ,*)$-fuzzy ideals of L by $FI_{\circ*}(L)$ and the set of all fuzzy ideals of L by $FI(L)$.

Theorem 5.10. *If $(L,^\circ,^*)$ a dpMS-algebra, then $FI_{\circ*}(L)$ is a sublattice of the lattice $FI(L)$ of fuzzy ideals of L.*

Proof. It is clear that $KerG_F \in FI_{\circ*}(L)$. Then $FI_{\circ*}(L) \neq \varnothing$. Let $\mu,\nu \in FI_{\circ*}(L)$. Then clearly, $\mu \wedge \nu \in FI_{\circ*}(L)$. We see that $\mu \vee \nu \in FI_{\circ*}(L)$. Thus

$$\begin{aligned}(\mu \vee \nu)(x) &= \sup\{\mu(i) \wedge \nu(j) : x = i \vee j\}\\ &\leq \sup\{\mu(i^{*\circ}) \wedge \nu(j^{*\circ}) : x^{*\circ} = i^{*\circ} \vee j^{*\circ}\}\\ &= (\mu \vee \nu)(x^{*\circ}).\end{aligned}$$

Conversely,

$$\begin{aligned}(\mu \vee \nu)(x^{*\circ}) &= \sup\{\mu(i) \wedge \nu(j) : x^{*\circ} = i \vee j\}\\ &\leq \sup\{\mu(i^{*\circ}) \wedge \nu(j^{*\circ}) : x^{*\circ*\circ} = i^{*\circ} \vee j^{*\circ}\}\\ &= \sup\{\mu(i^{*\circ}) \wedge \nu(j^{*\circ}) : x^{**} = i^{*\circ} \vee j^{*\circ}\}\\ &= (\mu \vee \nu)(x^{**}).\end{aligned}$$

Clearly, $KerG_F(x \wedge x^*) = 1$, $KerG_F \subseteq \mu$ and $KerG_F \subseteq \nu$. So $KerG_F \subseteq \mu \vee \nu$. Since $\mu \vee \nu$ is anti-tone,

$$\begin{aligned}(\mu \vee \nu)(x^{**}) &= (\mu \vee \nu)(x^{**}) \wedge (\mu \vee \nu)(x \wedge x^*)\\ &= (\mu \vee \nu)(x^{**} \vee (x \wedge x^*))\\ &= (\mu \vee \nu)(x^{**} \vee x)\\ &\leq (\mu \vee \nu)(x).\end{aligned}$$

Hence $\mu \vee \nu \in FI_{\circ*}(L)$. Therefore $FI_{\circ*}(L)$ is a sublattice of $FI(L)$. □

Theorem 5.11. *If $(L,^\circ,^*)$ is a dpMS-algebra then for each $\mu \in FI_{\circ*}(L)$, there exists a smallest fuzzy congruence on L with kernel μ given by the followings:*

$$\delta(\mu)(x,y) = \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^\circ = (y \vee i^{\circ\circ}) \wedge i^\circ\}.$$

Proof. Reflexive and symmetric are clear. Next, we prove that $\delta(\mu)$ is transitive. Suppose that $(x\vee i^{\circ\circ})\wedge i^{\circ}=(y\vee i^{\circ\circ})\wedge i^{\circ},\ (y\vee j^{\circ\circ})\wedge j^{\circ}=(z\vee j^{\circ\circ})\wedge j^{\circ}$, for any $x,y,i,j\in L$. Put $r=i\vee j$. Then we have

$$\begin{aligned}(x\vee r^{\circ\circ})\wedge r^{\circ} &= (x\vee(i^{\circ\circ}\vee j^{\circ\circ}))\wedge(i^{\circ}\wedge j^{\circ})\\ &= ((x\vee i^{\circ\circ})\vee j^{\circ\circ}))\wedge(i^{\circ}\wedge j^{\circ})\\ &= ((x\vee i^{\circ\circ})\wedge(i^{\circ}\wedge j^{\circ}))\vee(j^{\circ\circ}\wedge(i^{\circ}\wedge j^{\circ})\\ &= ((y\vee i^{\circ\circ})\wedge(i^{\circ}\wedge j^{\circ}))\vee(j^{\circ\circ}\wedge(i^{\circ}\wedge j^{\circ})\\ &= ((y\vee j^{\circ\circ})\wedge(i^{\circ}\wedge j^{\circ}))\vee(i^{\circ\circ}\wedge(i^{\circ}\wedge j^{\circ})\\ &= ((z\vee j^{\circ\circ})\wedge(i^{\circ}\wedge j^{\circ}))\vee(i^{\circ\circ}\wedge(i^{\circ}\wedge j^{\circ})\\ &= (z\vee(i^{\circ\circ}\vee j^{\circ\circ}))\wedge(i^{\circ}\wedge j^{\circ})\\ &= (z\vee r^{\circ\circ})\wedge r^{\circ}.\end{aligned}$$

Thus $\delta(\mu)(x,y)\wedge\delta(\mu)(y,z)$
$=\sup\{\mu(i):(x\vee i^{\circ\circ})\wedge i^{\circ}=(y\vee i^{\circ\circ})\wedge i^{\circ}\}\wedge\sup\{\mu(j):(y\vee j^{\circ\circ})\wedge j^{\circ}=(z\vee j^{\circ\circ})\wedge j^{\circ}\}$
$=\sup\{\mu(i)\wedge\mu(j):(x\vee i^{\circ\circ})\wedge i^{\circ}=(y\vee i^{\circ\circ})\wedge i^{\circ},\ (y\vee j^{\circ\circ})\wedge j^{\circ}=(z\vee j^{\circ\circ})\wedge j^{\circ}\}$
$\leq\sup\{\mu(r):(x\vee r^{\circ\circ})\wedge r^{\circ}=(z\vee r^{\circ\circ})\wedge r^{\circ}\}$
$=\delta(\mu)(x,z)$.
So $\delta(\mu)(x,y)\wedge\delta(\mu)(y,z)\leq\delta(\mu)(x,z)$. Hence $\delta(\mu)$ is transitive.

For any $z\in L$,
$\delta(\mu)(x,y)$
$=\delta(\mu)(x,y)\wedge\delta(\mu)(z,z)$
$=\sup\{\mu(i):(x\vee i^{\circ\circ})\wedge i^{\circ}=(y\vee i^{\circ\circ})\wedge i^{\circ}\}\wedge\sup\{\mu(j):(z\vee j^{\circ\circ})\wedge j^{\circ}=(z\vee j^{\circ\circ})\wedge j^{\circ}\}$
$=\sup\{\mu(i)\wedge\mu(j):(x\vee i^{\circ\circ})\wedge i^{\circ}=(y\vee i^{\circ\circ})\wedge i^{\circ},\ (z\vee j^{\circ\circ})\wedge j^{\circ}=(z\vee j^{\circ\circ})\wedge j^{\circ}\}$.
Put $r=i\vee j$. Then we have $r^{\circ}=i^{\circ}\wedge j^{\circ},\ r^{\circ\circ}=i^{\circ\circ}\vee j^{\circ\circ}$. Thus

$$\begin{aligned}(x\vee z)\vee r^{\circ\circ})\wedge r^{\circ} &= ((x\vee z)\vee(i^{\circ\circ}\vee j^{\circ\circ}))\wedge(i^{\circ}\wedge j^{\circ})\\ &= ((x\vee i^{\circ\circ})\vee(z\vee j^{\circ\circ}))\wedge(i^{\circ}\wedge j^{\circ})\\ &= ((x\vee i^{\circ\circ})\wedge(i^{\circ}\wedge j^{\circ}))\vee((z\vee j^{\circ\circ})\wedge(i^{\circ}\wedge j^{\circ}))\\ &= ((y\vee i^{\circ\circ})\wedge(i^{\circ}\wedge j^{\circ}))\vee((z\vee j^{\circ\circ})\wedge(j^{\circ}\wedge j^{\circ}))\\ &= ((y\vee z)\vee(i^{\circ\circ}\vee j^{\circ\circ}))\wedge(i^{\circ}\wedge j^{\circ})\\ &= (y\vee z)\vee r^{\circ\circ})\wedge r^{\circ}.\end{aligned}$$

This implies $\delta(\mu)(x,y)\leq\delta(\mu)(x,y)\wedge\delta(\mu)(z,z)\leq\delta(\mu)(x\vee z,y\vee z)$. Similarly, $\delta(\mu)(x,y)\leq\delta(\mu)(x\wedge z,y\wedge z)$. So $\delta(\mu)$ is lattice fuzzy congruence of L.

Next, we see that $\delta(\mu)(x,y)\leq\delta(\mu)(x^{\circ},y^{\circ})\wedge\delta(\mu)(x^{*},y^{*})$.

$$\begin{aligned}&(x\vee i^{\circ\circ})\wedge i^{\circ}=(y\vee i^{\circ\circ})\wedge i^{\circ} \qquad (*)\\ \Rightarrow\ &(x^{\circ}\wedge i^{\circ})\vee i^{\circ\circ}=(y^{\circ}\wedge i^{\circ})\vee i^{\circ\circ}\\ \Rightarrow\ &(x^{\circ}\vee i^{\circ\circ})\wedge(i^{\circ}\vee i^{\circ\circ})=(y^{\circ}\vee i^{\circ\circ})\wedge(i^{\circ}\vee i^{\circ\circ})\\ \Rightarrow\ &(x^{\circ}\vee i^{\circ\circ})\wedge(i^{\circ}\vee i^{\circ\circ})\wedge i^{\circ}=(y^{\circ}\vee i^{\circ\circ})\wedge(i^{\circ}\vee i^{\circ\circ})\wedge i^{\circ}\\ \Rightarrow\ &(x^{\circ}\vee i^{\circ\circ})\wedge i^{\circ}=(y^{\circ}\vee i^{\circ\circ})\wedge i^{\circ}.\end{aligned}$$

Also from (*), we have
$(x^{*}\wedge i^{\circ\circ *})\vee i^{\circ *}=(y^{*}\wedge i^{\circ\circ *})\vee i^{\circ *}$
$\Rightarrow(x^{*}\vee i^{\circ *})\wedge(i^{\circ\circ *}\vee i^{\circ *})=(y^{*}\vee i^{\circ *})\wedge(i^{\circ\circ *}\vee i^{\circ *})$

$\Rightarrow (x^* \vee i^{\circ\circ\circ *}) \wedge (i^{\circ\circ *} \vee i^{\circ\circ\circ *}) = (y^* \vee i^{\circ\circ\circ *}) \wedge (i^{\circ\circ\circ *} \vee i^{\circ *})$.

Put $j = i^{\circ *}$. Then

$(x^* \vee j^{\circ\circ}) \wedge (j^{\circ} \vee j^{\circ\circ} = (y^* \vee j^{\circ\circ}) \wedge (j^{\circ} \vee j^{\circ\circ})$
$\Rightarrow (x^* \vee j^{\circ\circ}) \wedge (j^{\circ} \vee j^{\circ\circ}) \wedge j^{\circ} = (y^* \vee j^{\circ\circ}) \wedge (j^{\circ} \vee j^{\circ\circ}) \wedge j^{\circ}$
$\Rightarrow (x^* \vee j^{\circ\circ}) \wedge j^{\circ} = (y^* \vee j^{\circ\circ}) \wedge j^{\circ}$.

On the other hand,

$$\begin{aligned}\delta(\mu)(x,y) &= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\}\\ &\leq \sup\{\mu(i) : (x^{\circ} \vee i^{\circ\circ}) \wedge i^{\circ} = (y^{\circ} \vee i^{\circ\circ}) \wedge i^{\circ}\}\\ &= \delta(\mu)(x^{\circ}, y^{\circ}).\end{aligned}$$

Also,

$$\begin{aligned}\delta(\mu)(x,y) &= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\}\\ &\leq \sup\{\mu(j) : (x^* \vee j^{\circ\circ}) \wedge j^{\circ} = (y^* \vee j^{\circ\circ}) \wedge j^{\circ}\}\\ &= \delta(\mu)(x^*, y^*).\end{aligned}$$

Thus $\delta(\mu)$ is fuzzy congruence of L.

Next, we show that $Ker\delta(\mu) = \mu$. It is obvious that $\mu \subseteq Ker\delta(\mu)$. Conversely,

$$\begin{aligned}Ker\delta(\mu)(x) &= \delta(\mu)(x, 0)\\ &= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = i^{\circ\circ} \wedge i^{\circ}\}\\ &\leq \sup\{\mu(i) : (x^{**} \vee i^{**}) \wedge i^{\circ **} = i^{**} \wedge i^{\circ **}\}.\end{aligned}$$

Now $(x^{**} \vee i^{**}) \wedge i^{\circ **} = i^{**} \wedge i^{\circ **}$. Then $(x^{**} \wedge i^{\circ **}) \vee (i^{**} \wedge i^{\circ **}) = i^{**} \wedge i^{\circ **}$. We observe that $x^{**} \wedge i^{\circ **} \leq i^{**} \wedge i^{\circ **} \leq i^{**}$. Since $\mu \in FI_{\circ *}(L)$,

$$\mu(i) = \mu(i^{**}) = \mu(i^{\circ *}) \leq \mu(x^{**} \wedge i^{\circ **}).$$

Thus

$$\begin{aligned}Ker\delta(\mu)(x) &\leq \sup\{\mu(i) : (x^{**} \vee i^{**}) \wedge i^{\circ **} = i^{**} \wedge i^{\circ **}\}\\ &= sup\{\mu(i^{\circ *}) \wedge \mu(x^{**} \wedge i^{\circ **}) : (x^{**} \vee i^{**}) \wedge i^{\circ **} = i^{**} \wedge i^{\circ **}\}\\ &= sup\{\mu(i^{\circ *} \vee (x^{**} \wedge i^{\circ **})) : (x^{**} \vee i^{**}) \wedge i^{\circ **} = i^{**} \wedge i^{\circ **}\}\\ &= sup\{\mu(i^{\circ *} \vee x^{**}) : (x^{**} \vee i^{**}) \wedge i^{\circ **} = i^{**} \wedge i^{\circ **}\}\\ &\leq \mu(x^{**}) = \mu(x).\end{aligned}$$

So $\mu = Ker\delta(\mu)$.

Let ϕ be any fuzzy congruence on L with $Ker\phi = \mu$. We prove that $\delta(\mu) \subseteq \phi$. Now

$$\begin{aligned}\delta(\mu)(x,y) &= \sup\{\mu(i) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\}\\ &= \sup\{\phi(i,0) : (x \vee i^{\circ\circ}) \wedge i^{\circ} = (y \vee i^{\circ\circ}) \wedge i^{\circ}\}.\end{aligned}$$

Since ϕ is fuzzy congruence on L, $\phi(i,0) \leq \phi(i^{\circ},1) \leq \phi(i^{\circ\circ},0)$. Then

$$\phi(i,0) \leq \phi(i^{\circ},1) \leq \phi(x, (x \vee i^{\circ\circ}) \wedge i^{\circ})) \wedge \phi((y \vee i^{\circ\circ}) \wedge i^{\circ}, y) \leq \phi(x,y).$$

Thus $\delta(\mu) \subseteq \phi$. So $\delta(\mu)$ is as our required. □

Corollary 5.12. *Let $(L,^{\circ},^{*})$ be a dpMS-algebra. If $\mu \in FI_{\circ *}(L)$, then $\overline{\Theta}[\mu] = \delta(\mu)$.*

Proof. By Theorem 5.11, it is enough to show that $Ker\overline{\Theta}[\mu] = \mu$. It is clear that $\mu \subseteq Ker\overline{\Theta}[\mu]$. Conversely, Theorem 4.7,

$$\begin{aligned}Ker\overline{\Theta}[\mu](x) &= \overline{\Theta}[\mu](0,x)\\ &= \sup_n(\sup_{z_1,z_2,\ldots,z_{2n}}(\overline{\Theta}_{lat}[\mu](0,z_1) \wedge \overline{\Theta}_{lat}[\mu_{\circ}^{\geq}](z_1,z_2)\\ &\quad \wedge\overline{\Theta}_{lat}[\mu](z_2,z_3) \wedge \ldots \wedge \overline{\Theta}_{lat}[\mu](z_{2n},x)))\end{aligned}$$

$$\begin{aligned}
&\leq \overline{\Theta}_{lat}[\mu](0, z_1)\\
&= \sup\{\mu(i) : i = i \vee z_1\}\\
&= \sup\{\mu(i \vee z_1) : i = i \vee z_1\}\\
&\leq \mu(z_1).
\end{aligned}$$

Or

$$\begin{aligned}
Ker\overline{\Theta}[\mu](x) &= \overline{\Theta}[\mu](0, x)\\
&\leq \overline{\Theta}_{lat}[\mu^{\geq}_{\circ}](0, z_1)\\
&= \sup\{\mu(j) : 0 = z_1 \wedge j^{\circ},\ j \in L\}\\
&\leq \sup\{\mu(j) : 0 = z_1^{\circ *} \wedge j^{*},\ j \in L\}\\
&= \sup\{\mu(j^{**}) : 0 = z_1^{\circ *} \wedge j^{*},\ j \in L\}\\
&\leq \mu(z_1^{\circ *})\\
&= \mu(z_1),
\end{aligned}$$

since $z_1^{\circ *} \leq j^{**}$.

Suppose that for $r \leq k,\ \mu(i) \leq \mu(z_r)$, for $z_r \vee i = z_{r+1} \vee i$. Then

$$\begin{aligned}
&\overline{\Theta}[\mu](0, x)\\
&= \sup_{z_1, z_2, \ldots, z_{2n}} (\overline{\Theta}_{lat}[\mu](0, z_1) \wedge \overline{\Theta}_{lat}[\mu^{\geq}_{\circ}](z_1, z_2) \wedge \overline{\Theta}_{lat}[\mu](z_2, z_3) \wedge \ldots \wedge \overline{\Theta}_{lat}[\mu](z_{2n}, x))\\
&\leq \overline{\Theta}_{lat}[\mu](z_k, z_{k+1})\\
&= \sup\{\mu(i) : i \vee z_k = i \vee z_{k+1}\}\\
&= \sup\{\mu(i) \wedge \mu(z_k) : i \vee z_k = i \vee z_{k+1}\}\\
&= \sup\{\mu(i \vee z_k) : i \vee z_k = i \vee z_{k+1}\}\\
&= \sup\{\mu(i \vee z_{k+1}) : i \vee z_k = i \vee z_{k+1}\}\\
&\leq \mu(z_{k+1}).
\end{aligned}$$

On the other hand, since $j^{\circ} \wedge z_k = j^{\circ} \wedge z_{k+1}$ and $j^{**} \vee j^{*} = 1$, we have

$$j^{**} \wedge z_k^{\circ *} = j^{**} \wedge z_{k+1}^{\circ *} \geq z_{k+1}^{\circ *}.$$

Thus

$$\begin{aligned}
\overline{\Theta}[\mu](0, x) &= \overline{\Theta}_{lat}[\mu^{\geq}_{\circ}](z_k, z_{k+1})\\
&= \sup\{\mu(j) : j^{\circ} \wedge z_k = j^{\circ} \wedge z_{k+1},\ j \in L\}\\
&\leq \mu(z_k^{\circ *})\\
&= \mu(z_k)
\end{aligned}$$

So it follows by induction that $Ker\overline{\Theta}[\mu](x) \leq \mu(x)$. This implies $\mu = Ker\overline{\Theta}[\mu]$. Hence the required equality holds. □

Theorem 5.13. *If $(L, ^{\circ}, ^{*})$ be a dpMS-algebra, then for each $\mu \in FI_{\circ *}(L)$, the greatest fuzzy congruence on L with kernel μ is the relation $\sigma(\mu)$ given by:*

$$\sigma(\mu)(x, y) = \mu((x^{**} \wedge y^{*}) \vee (x^{*} \wedge y^{**})).$$

Proof. We prove that $\sigma(\mu)$ is a fuzzy congruence on L. It is obvious that $\sigma(\mu)$ is reflexive and symmetric. We show that $\sigma(\mu)$ is transitive.

$$\begin{aligned}
\mu(x^{**} \wedge z^{*}) &= \mu((x^{**} \wedge z^{*}) \wedge (y^{*} \vee y^{**}))\\
&= \mu((x^{**} \wedge z^{*} \wedge y^{*}) \vee ((x^{**} \wedge z^{*} \wedge y^{**}))\\
&= \mu((x^{**} \wedge z^{*} \wedge y^{*}) \wedge \mu((x^{**} \wedge z^{*} \wedge y^{**}))\\
&\geq \mu(x^{**} \wedge y^{*}) \wedge \mu((z^{*} \wedge y^{**})\\
&\geq \mu(x^{**} \wedge y^{*}) \wedge \mu(x^{*} \wedge y^{**}) \wedge \mu(y^{*} \wedge z^{**}) \wedge \mu((z^{*} \wedge y^{**})\\
&= \mu((x^{**} \wedge y^{*}) \vee (x^{*} \wedge y^{**})) \wedge \mu((y^{*} \wedge z^{**}) \vee (z^{*} \wedge y^{**}))
\end{aligned}$$

$= \sigma(\mu)(x,y) \wedge \sigma(\mu)(y,z)$.

Similarly, $\mu(x^* \wedge z^{**}) \geq \sigma(\mu)(x,y) \wedge \sigma(\mu)(y,z)$. This implies

$$\mu(x^{**} \wedge z^*) \wedge \mu(x^* \wedge z^{**}) \geq \sigma(\mu)(x,y) \wedge \sigma(\mu)(y,z).$$

Then $\sigma(\mu)(x,z) \geq \sigma(\mu)(x,y) \wedge \sigma(\mu)(y,z)$. Also

$$\begin{aligned}\sigma(\mu)(x\wedge a, y\wedge a) &= \mu(((x\wedge a)^{**} \wedge (y\wedge a)^*) \vee ((x\wedge a)^* \wedge (y\wedge a)^{**})))\\ &= \mu((x^{**}\wedge y^* \wedge a^{**}) \vee (x^* \wedge y^{**} \wedge a^{**}))\\ &= \mu(((x^{**}\wedge y^*) \vee (x^*\wedge y^{**})) \wedge a^*)\\ &\geq \mu((x^{**}\wedge y^*) \wedge (x^*\wedge y^{**}))\\ &= \sigma(\mu)(x,y).\end{aligned}$$

Similarly, $\sigma(\mu)(x\vee a, y\vee a) \geq \sigma(\mu)(x,y)$. Thus $\sigma(\mu)$ is a lattice of fuzzy congruence.

On the other hand,

$$\begin{aligned}\sigma(\mu)(x^*,y^*) &= \mu((x^{***}\wedge y^{**}) \vee (x^{**}\wedge y^{***}))\\ &= \mu((x^*\wedge y^{**}) \vee (x^{**}\wedge y^*))\\ &= \sigma(\mu)(x,y)\end{aligned}$$

and

$$\begin{aligned}\sigma(\mu)(x,y) &= \mu((x^{**}\wedge y^*) \vee (x^*\wedge y^{**}))\\ &= \mu(((x^{**}\wedge y^*) \vee (x^*\wedge y^{**}))^{\circ *}))\\ &= \mu((x^{\circ ***}\wedge y^{\circ **}) \vee (x^{\circ **}\wedge y^{\circ **}))\\ &= \mu((x^{\circ *}\wedge y^{\circ **}) \vee (x^{\circ **}\wedge y^{\circ *}))\\ &= \sigma(\mu)(x^\circ, y^\circ).\end{aligned}$$

So $\sigma(\mu)$ is a fuzzy congruence of a dpMS-algebra L.

Also $Ker\sigma(\mu)(x) = \sigma(\mu)(x,0) = \mu((x^{**}\wedge 0^*) \vee (x^*\wedge 0^{**})) = \mu(x^{**}) = \mu(x)$. Then $Ker\sigma(\mu) = \mu$. Thus $\sigma(\mu)$ is a fuzzy congruence of L with kernel μ.

Finally, let φ be a congruence on L with $Ker\varphi = \mu$. Then

$$\varphi(x,y) \leq \varphi(x^*,y^*) \leq \varphi(x^{**},y^{**}).$$

Thus

$$\begin{aligned}\varphi(x,y) &\leq \varphi(x^*,y^*)\\ &= \varphi(x^*,y^*) \wedge \varphi(x^{**},y^{**}) \wedge (x^{**},x^{**}) \wedge \varphi(x^*,x^*)\\ &\leq \varphi(x^*\wedge x^{**}, y^*\wedge x^{**}) \wedge \varphi(x^{**}\wedge x^*, y^{**}\wedge x^*)\\ &\leq \varphi(((x^*\wedge x^{**}) \vee (x^{**}\wedge x^*)), ((y^*\wedge x^{**}) \wedge (y^{**}\wedge x^*)))\\ &= \varphi(0, ((y^*\wedge x^{**}) \wedge (y^{**}\wedge x^*)))\\ &= Ker\varphi((y^*\wedge x^{**}) \wedge (y^{**}\wedge x^*))\\ &= \mu((y^*\wedge x^{**}) \wedge (y^{**}\wedge x^*))\\ &= \sigma(\mu)(x,y).\end{aligned}$$

So $\varphi \subseteq \sigma(\mu)$. Hence $\sigma(\mu)$ is the greatest fuzzy congruence of L with kernel μ. □

Theorem 5.14. *If $(L,^\circ,^*)$ be a dpMS-algebra then for each μ, $\nu \in FI_{\circ *}(L)$, the following statements hold:*

(1) $\mu \subseteq \nu \Leftrightarrow \sigma(\mu) \subseteq \sigma(\nu)$,
(2) $\sigma(KerG_F) = G_F$,
(3) $G_F \subseteq \sigma(\mu)$.

As constituted in Theorems 5.11 and 5.13, for every $\mu \in FI_{\circ *}(L)$, there exists the smallest fuzzy congruence $\delta(\mu)$ and the biggest fuzzy congruence $\sigma(\mu)$ on L with kernel μ. The relationship between these two congruences can be characterized as follows.

Theorem 5.15. *Let $(L,^{\circ},^{*})$ be a dpMS-algebra. If $\mu \in FI_{\circ *}(L)$, then $\sigma(\mu) = G_F \vee \delta(\mu)$.*

Proof. By Theorems 5.10 and 5.12, we have clearly seen that $G_F \vee \delta(\mu) \leq \sigma(\mu)$. To show the reverse inequality, put $t = (x^{**} \wedge y^{*}) \vee (x^{*} \wedge y^{**})$. Then $t^{\circ\circ} = t^{**} = t$ and $x^{**} \vee t = y^{**} \vee t = x^{**} \vee y^{**}$. This implies $(x^{**} \vee t) \wedge t^{\circ} = (y^{**} \vee t) \wedge t^{\circ}$ and also $(x^{**} \vee t^{\circ\circ}) \wedge t^{\circ} = (y^{**} \vee t^{\circ\circ}) \wedge t^{\circ}$ as $t^{\circ\circ} = t$. Now

$$\begin{aligned}\sigma(\mu)(x, y) &= \mu((x^{**} \wedge y^{*}) \vee (x^{*} \wedge y^{**}))\\ &= \mu(t)\\ &\leq \sup\{\mu(t) : (x^{**} \vee t^{\circ\circ}) \wedge t^{\circ} = (y^{**} \vee t^{\circ\circ}) \wedge t^{\circ}\}\\ &= \delta(\mu)(x^{**}, y^{**}).\end{aligned}$$

Clearly $G_F(z, z^{**}) = 1$, for all $z \in L$. On the other hand,

$$\begin{aligned}\delta(\mu)(x^{**}, y^{**}) &= \delta(\mu)(x^{**}, y^{**}) \wedge G_F(y^{**}, y) \wedge G_F(x^{**}, x)\\ &\leq (\delta(\mu) \circ G_F)(x, y) \leq (\delta(\mu) \vee G_F)(x, y).\end{aligned}$$

Thus the result is holds. □

Theorem 5.16. *Let $(L,^{\circ},^{*})$ is a dpMS-algebra. A $(\circ, *)$-fuzzy ideal of L is precisely the kernel of the fuzzy congruence φ on L if and only if $(L/\varphi,^{*})$ is boolean.*

Proof. Suppose μ is a $(\circ, *)$-fuzzy ideal. Then by Theorem 5.14, $G_F \leq \sigma(\mu)$. Since for each $x \in L$, $1 = KerG_F(x \wedge x^{*}) \leq Ker\sigma(\mu)(x \wedge x^{*})$ and $1 = CokerG_F(x \vee x^{*}) \leq Coker\sigma(\mu)(x \vee x^{*})$, we have

$$1 = Ker\varphi(x \wedge x^{*}) = \varphi(x \wedge x^{*}, 0)$$

and

$$1 = Coker\sigma(\mu)(x \vee x^{*}) = \sigma(\mu)(x \vee x^{*}, 1).$$

Thus $\sigma(\mu)_x \wedge (\sigma(\mu)_x)^{*} = \sigma(\mu)_0$ and $\sigma(\mu)_x \vee (\sigma(\mu)_x)^{*} = \sigma(\mu)_1$. So $(L/\sigma(\mu),^{*})$ is boolean.

Conversely, suppose that φ is a congruence on L such that $(L/\varphi, *)$ is boolean. We prove that $Ker\varphi$ is a $(\circ, *)$-fuzzy ideal of L. Let $x \in L$. Then

$$Ker\varphi(x) = \varphi(x, 0) \leq \varphi(x^{\circ *}, 0) = Ker\varphi(x^{\circ *}).$$

Also $Ker\varphi(x^{\circ *}) = \varphi(x^{\circ *}, 0) \leq \varphi(x^{*}, 1)$. Since $(L/\sigma(\mu),^{*})$ is boolean, $\varphi_x \wedge (\varphi_x)^{*} = \varphi_0$. Thus $\varphi(x \wedge x^{*}, 0) = 1$. On the other hand,

$$\begin{aligned}Ker\varphi(x^{\circ *}) &\leq \varphi(x^{*}, 1) \wedge \varphi(x, x) \leq \varphi(x \wedge x^{*}, x)\\ &= \varphi(x \wedge x^{*}, x) \wedge 1\\ &\leq \varphi(x, 0) = Ker\varphi(x).\end{aligned}$$

So $Ker\varphi$ is $(\circ, *)$-fuzzy ideal. □

Example 5.17. As indicated in Example 5.6, $KerG_F = \mu \in FI_{\circ *}(L)$. We can easily verified that $(G_F)_0 = (G_F)_a = (G_F)_b = (G_F)_c = (G_F)_d = (G_F)_f$, $L/G_F = \{(G_F)_0, (G_F)_e, (G_F)_g, (G_F)_h, (G_F)_1\}$ and $(L/G_F,^{*})$ is boolean.

Theorem 5.18. *If $(L,^{\circ},^{*})$ is a dpMS-algebra, then $FI_{\circ *}(L) \simeq [G_F, \chi_\iota]$.*

Proof. Let $\mu, \nu \in FI_{\circ *}(L)$. Then $\sigma(\mu \wedge \nu)(x, y) = (\mu \wedge \nu)((x^{**} \wedge y^{*}) \vee (x^{*} \wedge y^{**})) = \mu((x^{**} \wedge y^{*}) \vee (x^{*} \wedge y^{**})) \wedge \nu((x^{**} \wedge y^{*}) \vee (x^{*} \wedge y^{**})) = \sigma(\mu)(x, y) \wedge \sigma(\nu)(x, y)$. Thus $\mu \longrightarrow \sigma(\mu)$ is $\wedge$ -monomorphism.

By Theorem 5.14, we see that the mapping $\mu \longrightarrow \sigma(\mu)$ is an isotone from $FI_{\circ *}(L) \longrightarrow [G_F, \chi_\iota]$, so that for $\mu, \nu \in FI_{\circ *}(L)$, we have

$$\sigma(\mu) \vee \sigma(\nu) \leq \sigma(\mu \vee \nu). \tag{5.1}$$

The converse inequality of (5.1):

$$\begin{aligned}\sigma(\mu \vee \nu)(x,y) &= (\mu \vee \nu)((x^{**} \wedge y^*) \vee (x^* \wedge y^{**}))\\ &= \sup\{\mu(i) \wedge \nu(j) : i \vee j = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})\}, \text{ i.e.,}\end{aligned}$$

$$\sigma(\mu \vee \nu)(x,y) = \sup\{\mu(i) \wedge \nu(j) : i \vee j = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})\}. \tag{5.2}$$

Put $t = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**}) = i \vee j$. Then $t = t^{**} = i^{**} \vee j^{**}$ and $t \wedge i^* = j^{**} \wedge i^*$ and also observe that

$$\begin{aligned}&[(x \vee i)^* \wedge (y \vee i)^{**}] \vee [(x \vee i)^{**} \wedge (y \vee i)^*]\\ &= [x^* \wedge i^* \wedge (y^{**} \vee i^{**})] \vee [(x^{**} \vee i^{**}) \wedge y^* \wedge i^*]\\ &= (x^* \wedge y^{**} \wedge i^*) \vee (x^{**} \wedge y^* \wedge i^*) = [(x^* \wedge y^{**}) \vee (x^{**} \wedge y^*)] \wedge i^*\\ &= t \wedge i^*.\end{aligned} \tag{5.3}$$

$$\begin{aligned}&[z^* \wedge (z \vee i)^{**}] \vee [z^{**} \wedge (z \vee i)^*]\\ &= [(z^* \wedge z^{**}) \vee (z^* \wedge i^{**})] \vee [z^{**} \wedge z^{**} \wedge z^* \wedge i^*]\\ &= z^* \wedge i^{**}, \forall z \in L(z = x \text{ or } z = y).\end{aligned} \tag{5.4}$$

From (5.2) and (5.3),

$$\begin{aligned}\sigma(\mu \vee \nu)(x,y) &= (\mu \vee \nu)((x^{**} \wedge y^*) \vee (x^* \wedge y^{**}))\\ &= \sup\{\mu(i) \wedge \nu(j) : i \vee j = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})\}\\ &\leq \nu(j^{**} \wedge i^*) = \nu(t \wedge i^*)\\ &= \nu([(x \vee i)^* \wedge (y \vee i)^{**})] \vee [(x \vee i)^{**} \wedge (y \vee i)^*])\\ &= \sigma(\nu)(x \vee i, y \vee i).\end{aligned} \tag{5.5}$$

From (5.2) and (5.4),

$$\begin{aligned}\sigma(\mu \vee \nu)(x,y) &= (\mu \vee \nu)((x^{**} \wedge y^*) \vee (x^* \wedge y^{**}))\\ &= \sup\{\mu(i) \wedge \nu(j) : i \vee j = (x^{**} \wedge y^*) \vee (x^* \wedge y^{**})\}\\ &\leq \mu(x^* \wedge i^{**})\\ &= \mu([x^* \wedge (x \vee i)^{**}] \vee [x^{**} \wedge (x \vee i)^*])\\ &= \sigma(\mu)(x, x \vee i).\end{aligned} \tag{5.6}$$

From (5.5) and (5.6),

$$\sigma(\mu \vee \nu)(x,y) \leq \sigma(\nu)(x \vee i, y \vee i) \wedge \sigma(\mu)(x, x \vee i) \leq (\sigma(\nu) \vee \sigma(\mu))(x, y \vee i).$$

Similarly, $\sigma(\mu \vee \nu)(x,y) \leq (\sigma(\nu) \vee \sigma(\mu))(x \vee i, y)$. Thus we obtain

$$\sigma(\mu \vee \nu)(x,y) \leq (\sigma(\nu) \vee \sigma(\mu))(x,y).$$

So the reverse inequality of (5.2) holds. This implies $\mu \longrightarrow \sigma(\mu)$ is a $\vee$-morphism. In addition to this it is an injective by Theorem 5.14 (1).

Next we see that it is a surjective. Let $\varphi \in [G_F, \chi_\iota]$. Then by Corollary 5.9 , $Ker\varphi$ is a $(\circ, *)$-fuzzy ideal of L. Now $\varphi(x,y) \leq \varphi(x^*, y^*) \leq \varphi(x^* \wedge x^{**}, y^* \wedge x^{**}) = \varphi(0, y^* \wedge x^{**})$ and $\varphi(x,y) \leq \varphi(0, x^* \wedge y^{**})$. Then

$$\begin{aligned}\varphi(x,y) &\leq \varphi(0, ((x^* \wedge y^{**}) \wedge (y^* \wedge x^{**}))\\ &= Ker(\varphi)((x^* \wedge y^{**}) \wedge (y^* \wedge x^{**}))\\ &= \sigma(Ker\varphi)(x,y).\end{aligned}$$

Conversely,

$$\begin{aligned}\sigma(Ker\varphi)(x,y)) &= Ker(\varphi)((x^* \wedge y^{**}) \vee (y^* \wedge x^{**}))\\ &\leq Ker(\varphi)(y^* \wedge x^{**})\end{aligned}$$

$= \varphi(0, y^* \wedge x^{**})$
$\leq \varphi(1, y^{**} \vee x^*)$.

Since $G_F \leq \varphi$, we have $\varphi(x, x^{**}) = 1$. Thus

$$\sigma(Ker\varphi)(x,y)) \leq \varphi(1, y^{**} \wedge x^*) \leq \varphi(x, (y^{**} \wedge x^*) \wedge x^{**}) \leq \varphi(x, x^{**} \wedge y^{**}).$$

Similarly, $\sigma(Ker\varphi)(x,y)) \leq \varphi(y, x^{**} \wedge y^{**})$. It follow that $\sigma(Ker\varphi)(x,y) \leq \varphi(x,y)$. So we have got $\sigma(Ker\varphi) = \varphi$. Hence $\mu \longrightarrow \sigma(\mu)$ is a surjective. Therefore it is a lattice isomorphism. □

6. CONCLUSION AND FUTURE WORK

In this paper, we have studied the fuzzy congruences, kernel fuzzy ideals and $(\circ, *)$-fuzzy ideas of demi-Pseudocomplemented MS-algebras and their Properties. In [14], Filters congruences of Pseudocomplemented MS-algebras have been studied. One of the most promising ideas could be the investigation of fuzzy setting applied to filters congruences of Pseudocomplemented MS-algebras.

Acknowledgements. The authors would like to thank the referees for their valuable comments and constructive suggestions.

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