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Some operations of anti-intuitionistic L-fuzzy soft b-ideals of BG-algebras

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Abstract. In the present paper, the notion of anti-intuitionistic L-fuzzy soft set is applied to the b-ideals of BG-algebras. In fact, the notions of anti-intuitionistic L-fuzzy soft b-ideals of BG-algebras are introduced and their related properties are studied. Furthermore, some operations of anti-intuitionistic L-fuzzy soft b-ideals of BG-algebras are investigated. Finally, the products of anti-intuitionistic L-fuzzy soft b-ideals are discussed.

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Keywords: BG-algebra, Soft set, Intuitionistic L-fuzzy soft set ($ILFSS$), Anti-intuitionistic L-fuzzy soft b-ideal ($AILFSBID$), Product of anti-intuitionistic L-fuzzy soft b-ideal ($AILFSBID$).

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1. Introduction

A fuzzy set is a set with a function to a transitive poset. A fuzzy set is, therefore, a sort of generalized characteristic function. The concepts of fuzzy sets were introduced by Zadeh [32]. After two years later, Goguen [12] introduced the idea of L-fuzzy sets as the generalization of Zadeh's fuzzy sets in 1967. We habitually denote the poset by L and call the fuzzy set an L-fuzzy set or an L-set. Because of the generality of the mathematical definition, some important applications of fuzzy sets do not involve the intuitive concept of fuzziness at all. The notion of intuitionistic fuzzy sets introduced by Atanassov [3, 4] is one among them. Later, the "Intuitionistic L-fuzzy sets" was generalized by Atanassov and Stoeva [5]. In intuitionistic fuzzy set theory, existence and non-existence values are drawn from the unit interval and intersection, a union is modeled by minimum and maximum respectively. We replaced the structure of the totally ordered unit interval by an arbitrary bounded lattice L to allow incomparabilities among elements. The notions of L-fuzzy sets and intuitionistic fuzzy sets were introduced by Hatzimichailidis et al. [13]. They summarize some notions on L-fuzzy sets, where L denotes a complete lattice. They studied a special case of L-fuzzy sets is called as an "Intuitionistic fuzzy sets".

Kim et al. [20] introduced the concept of BG-algebras is a generalization of B-algebras. Ahn et al. [2] presented a fuzzy subalgebra of BG-algebras. Later on, intuitionistic L-fuzzy subalgebras and ideals of BG-algebras are studied in [10, 11]. Iséki et al. [14] introduced the ideal theory of BCK-algebras. The soft set theory is a generalization of a fuzzy set theory that was proposed by Molodtsov [23] in 1999 to deal with uncertainty in a parametric manner. Jun [16] introduced soft BCK/BCI-algebras. He applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras. The concept of soft BCK/BCI-algebras and soft subalgebras are introduced, and some of their elementary properties are proved. Maji et al. [21] introduced the concept of fuzzy soft sets as a generalization of the standard soft sets and presented an application of fuzzy soft sets in a decision-making problem. Maji et al. [22] introduced the notions of intuitionistic fuzzy soft set and application of soft sets in a decision-making problem.

Jun et al. [18] introduced a soft p-ideal of soft BCI-algebras. They developed soft p-ideals and p-idealistic soft BCI-algebras, and then investigate their basic properties. They were giving characterizations of (fuzzy) p-ideals in BCI-algebras. Jun et al. [17] applied a fuzzy soft set for dealing with several kinds of theories in BCK/BCI-algebras. They presented fuzzy soft BCK/BCI-algebras, (closed) fuzzy soft ideals and fuzzy soft p-ideals are introduced, and related properties are investigated. Jun [15] introduced doubt fuzzy BCK/BCI-algebras. Bej et al. [8] introduced the notion of a doubt intuitionistic fuzzy H-ideals in BCK/BCI-algebras, and to study some related properties of it. Balamurugan et al. [6] introduced intuitionistic fuzzy soft ideals in BCK/BCI-algebras. Furthermore, Muhiuddin et al. introduced different related concepts and applied them to BCK/BCI-algebras and residuated lattices (see [24, 25, 31]). Also, Muhiuddin et al. studied the concepts of soft set theory on various aspects (see [1, 19, 26, 27, 28, 29, 30]).

Bej et al. [9] introduced the concepts of doubt intuitionistic fuzzy subalgebras and doubt intuitionistic fuzzy ideals in BCK/BCI-algebras. They show that an intuitionistic fuzzy subset of BCK/BCI-algebras is an intuitionistic fuzzy subalgebra and an intuitionistic fuzzy ideal if and only if the complement of this intuitionistic fuzzy subset is an anti-intuitionistic fuzzy subalgebra, and ideal respectively. Recently, Barbhuiya [7] introduced the concept of (α, β)-doubt fuzzy ideals of BG-algebras and investigated some of their related properties. He defined the cartesian product of (α, β)-doubt fuzzy ideals and studied their properties. Motivated by a lot of work as mentined above, in this paper, we applied the notion of the anti-intuitionistic L-fuzzy soft set to ideals and b-ideals of BG-algebras. Further, some concepts and operations of anti- intuitionistic L-fuzzy soft b-ideals are given and discussed their properties in details. Moreover, the product of anti-intuitionistic L-fuzzy soft b-ideals and related properties are studied.

2. Preliminaries

By a BG-algebra [20], we mean an algebra X with a constant 0 and a binary operation " $*$ " satisfies for every $x, y, z \in X$:

(i) $x * x = 0$,

(ii) $x * 0 = x$,
(iii) $(x * y) * (0 * y) = x$.
For concise, we also call X a BG-algebra.

A non-empty subset I of X is an ideal of X, if it satisfies the following conditions:
(I_1) $0 \in I$,
(I_2) $x * y \in I$ and $y \in I \Rightarrow x \in I$.

An ideal I is called b-ideal of X, if it satisfies the condition (I_1) and the following condition:
(I_3) $(x * z) * y \in I$ and $y \in I \Rightarrow x \in I$.

Throughout this paper $(L, \leq, \vee, \wedge)$ denotes a complete distributive lattice with maximal element 1 and minimal element 0, respectively.

Definition 2.1 ([12]). Let X be a non-empty set. A L-fuzzy set ϕ of X is a function $\phi : X \to L$.

Definition 2.2 ([5]). Let $(L, \leq)$ be the lattice with an involutive order reversing operation $N : L \to L$. Let X be a non-empty set. An intuitionistic L-fuzzy set $(briefly, ILFS)$ A in X is defined as an object of the form

$$A = \{(x, \phi_A(x), \psi_A(x)) \mid x \in X\},$$

where $\phi_A : X \to L$ and $\psi_A(x) : X \to L$ define the degree of membership and the degree of non-membership for every $x \in X$ satisfying $\phi_A(x) \leq N(\psi_A(x))$.
For the sake of simplicity, we shall use the symbol $A = (\phi_A, \psi_A)$ for the $ILFS$ $A = \{(x, \phi_A(x), \psi_A(x)) \mid x \in X\}$.

For every two $ILFS(s)$ A and B in X, we define the following relations and operations:
$\oplus A = \{(x, \phi_A(x), 1 - \phi_A(x)) \mid x \in X\}$,
$\otimes A = \{(x, 1 - \psi_A(x), \psi_A(x)) \mid x \in X\}$,
$\boxplus A = \{(x, \frac{\phi_A(x)}{2}, \frac{\psi_A(x)+1}{2}) \mid x \in X\}$,
$\boxtimes A = \{(x, \frac{\phi_A(x)+1}{2}, \frac{\psi_A(x)}{2}) \mid x \in X\}$,
$!A = \left\{(x, \max\left(\frac{1}{2}, \phi_A(x)\right),\ \min\left(\frac{1}{2}, \psi_A(x)\right)\ \mid x \in X\right\}$,
$?A = \left\{(x, \min\left(\frac{1}{2}, \phi_A(x)\right),\ \max\left(\frac{1}{2}, \psi_A(x)\right)\ \mid x \in X\right\}$,
$A + B = \{(x, \phi_A(x) + \phi_B(x) - \phi_A(x).\phi_B(x), \psi_A(x).\psi_B(x)) \mid x \in X\}$,
$A.B = \{(x, \phi_A(x).\phi_B(x), \psi_A(x) + \psi_B(x) - \psi_A(x).\psi_B(x)) \mid x \in X\}$,
$A@B = \{(x, \frac{\phi_A(x)+\phi_B(x)}{2}, \frac{\psi_A(x)+\psi_B(x)}{2}) \mid x \in X\}$,
$A\$B = \{(x, \sqrt{\phi_A(x).\phi_B(x)}, \sqrt{\psi_A(x).\psi_B(x)}) \mid x \in X\}$.

The six Cartesian products of two $ILFS(s)$ A and B in X are defined as follows:
(i) the Cartesian product " $\times_1$ ",

$$A \times_1 B = \{((x, y), \phi_A(x).\phi_B(x), \psi_A(x).\psi_B(x)) \mid (x, y) \in X \times_1 X\},$$

(ii) the Cartesian product " $\times_2$ ",

$$A \times_2 B = \{((x, y), \phi_A(x) + \phi_B(x) - \phi_A(x).\phi_B(x), \psi_A(x).\psi_B(x)) \mid (x, y) \in X \times_2 X\},$$

(iii) the Cartesian product " $\times_3$ ",

$$A \times_3 B = \{((x, y), \phi_A(x).\phi_B(x), \psi_A(x) + \psi_B(x) - \psi_A(x).\psi_B(x)) \mid (x, y) \in X \times_3 X\},$$

(iv) the Cartesian product " $\times_4$ ",

$$A \times_4 B = \{((x,y), \phi_A(x) \wedge \phi_B(x), \psi_A(x) \vee \psi_B(x)) \mid (x,y) \in X \times_4 X\},$$

(v) the Cartesian product " $\times_5$ ",

$$A \times_5 B = \{((x,y), \phi_A(x) \vee \phi_B(x), \psi_A(x) \wedge \psi_B(x)) \mid (x,y) \in X \times_5 X\},$$

(vi) The Cartesian product " $\times_6$ ",

$$A \times_6 B = \{((x,y), \frac{\phi_A(x) + \phi_B(x)}{2}, \frac{\psi_A(x) + \psi_B(x)}{2}) \mid (x,y) \in X \times_6 X\},$$

where $\phi_A(x) \vee \phi_B(x) = \max(\phi_A(x), \phi_B(x))$ and $\psi_A(x) \wedge \psi_B(x) = \min(\psi_A(x), \psi_B(x))$.

Definition 2.3. An $ILFS$ $A = (\phi_A, \psi_A)$ in X is called an anti-intuitionistic L-fuzzy b-ideal ($briefly, AILFBID$) of X, if it satisfies the following conditions: for any $x, y \in X$,

($AILFBID1$) $\phi_A(0) \leq \phi_A(x)$ and $\psi_A(0) \geq \psi_A(x)$,
($AILFBID2$) $\phi_A(x) \leq \phi_A((x * z) * y) \vee \phi_A(y)$,
($AILFBID3$) $\psi_A(x) \geq \psi_A((x * z) * y) \wedge \psi_A(y)$.

3. Operations of anti-intuitionistic L-fuzzy soft b-ideals of BG-algebras

In this section, anti-intuitionistic L-fuzzy soft b-ideals (briefly, $AILFSBID$) of BG-algebras is defined and some operations of $AILFSBID(s)$ of BG-algebras are presented.

Definition 3.1. A pair (R, A) is called a soft set over X, if R is a mapping of A into the set of all subsets of the set X.

Definition 3.2. Let L be a complete Boolean lattice and $L(X)$ be the family of all L-fuzzy sets in X. A pair $(\tilde{R}, A)$ is called a L-fuzzy soft set in X, if $\tilde{R}$ is a mapping given by $\tilde{R} : A \to L(X)$.

Example 3.3. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A = \{\delta_1, \delta_2\}$. Let $(\tilde{R}, A)$ be a L-fuzzy soft set over X, defined as follows:

$\tilde{R}[\delta_1] = \{(x_1, 0.6), (x_2, 0.4), (x_3, 0), (x_4, 0.1), (x_5, 0.1), (x_6, 0.5)\}$
$\tilde{R}[\delta_2] = \{(x_1, 0), (x_2, 0.8), (x_3, 0.9), (x_4, 0.4), (x_5, 0), (x_6, 0.6)\}$.

Then $(\tilde{R}, A)$ is described as

X	x_1	x_2	x_3	x_4	x_5	x_6
$\tilde{R}[\delta_1]$	0.6	0.4	0	0.1	0.1	0.5
$\tilde{R}[\delta_2]$	0	0.8	0.9	0.4	0	0.6

Definition 3.4. Let L be a complete Boolean lattice and $RL(X)$ be the family of all intuitionistic L-fuzzy sets in X. A pair $(\tilde{R}, A)$ is called an intuitionistic L-fuzzy soft set (briefly, ILFSS) in X, if $\tilde{R}$ is a mapping given by $\tilde{R} : A \to RL(X)$.

Example 3.5. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A = \{\delta_1, \delta_2\}$. Let $(\tilde{R}, A)$ be an intuitionistic L-fuzzy soft set over X, defined as follows:

$\tilde{R}[\delta_1] = \{(x_1, [0.6, 0.3]), (x_2, [0.4, 0.5]), (x_3, [0, 0.9]), (x_4, [0.1, 0.8]), (x_5, [0.1, 0.8]),$

$(x_6, [0.5, 0.4])\}$
$\tilde{R}[\delta_2] = \{(x_1, [0, 0.8]), (x_2, [0.8, 0]), (x_3, [0.9, 0.1]), (x_4, [0.4, 0.6]), (x_5, [0, 1]),$
$(x_6, [0.6, 0.3])\}$.
Then $(\tilde{R}, A)$ is described as

X	x_1	x_2	x_3	x_4	x_5	x_6
$\tilde{R}[\delta_1]$	[0.6, 0.3]	[0.4, 0.5]	[0, 0.9]	[0.1, 0.8]	[0.1, 0.8]	[0.5, 0.4]
$\tilde{R}[\delta_2]$	[0, 0.8]	[0.8, 0]	[0.9, 0.1]	[0.4, 0.6]	[0, 1]	[0.6, 0.3]

For the sake of simplicity, we shall use the symbol $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ for the intuitionistic fuzzy set $ILFS$ $\tilde{R}[\delta] = \{(x, \phi_{\tilde{R}[\delta]}(x), \psi_{\tilde{R}[\delta]}(x)) \mid x \in X \text{ and } \delta \in A\}$.

Definition 3.6. An $ILFSS$ $(\tilde{R}, A)$ is called an anti-intuitionistic L-fuzzy soft b-ideal $(briefly, AILFSBID)$ of X, if $\tilde{R}[\delta] = \{(x, \phi_{\tilde{R}[\delta]}(x), \psi_{\tilde{R}[\delta]}(x)) \mid x \in X \text{ and } \delta \in A\}$ is an anti-intuitionistic L-fuzzy b-ideal $(AILFBID)$ of X satisfying the following conditions: for any $x, y \in X$,
$(AILFSBID1)$ $\phi_{\tilde{R}[\delta]}(0) \leq \phi_{\tilde{R}[\delta]}(x)$ and $\psi_{\tilde{R}[\delta]}(0) \geq \phi_{\tilde{R}[\delta]}(x)$,
$(AILFSBID2)$ $\phi_{\tilde{R}[\delta]}(x) \leq \phi_{\tilde{R}[\delta]}((x * z) * y) \vee \phi_{\tilde{R}[\delta]}(y)$,
$(AILFSBID3)$ $\psi_{\tilde{R}[\delta]}(x) \geq \psi_{\tilde{R}[\delta]}((x * z) * y) \wedge \psi_{\tilde{R}[\delta]}(y)$.

Example 3.7. Consider a BG-algebra $X = \{0, 1, 2, 3\}$ with Cayley table.

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ be an $ILFS$ in X defined as

X	0	1	2	3
$\tilde{R}[\delta]$	[0.1,0.7]	[0.1,0.7]	[0.2,0.3]	[0.2,0.3]

Then $\tilde{R}[\delta]$ is an $AILFBID$ of X. Thus $(\tilde{R}, A)$ is an $AILFSBID$ of X.

Definition 3.8. Let $(\tilde{R}, A)$ be an $ILFSS$ of X, if $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ is an $ILFS$ of X. Then the operators $\oplus\tilde{R}[\delta]$ and $\otimes\tilde{R}[\delta]$ are defined as $\oplus\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \bar{\phi}_{\tilde{R}[\delta]})$ and $\otimes\tilde{R}[\delta] = (\bar{\psi}_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ in X.

Example 3.9. We define $(\tilde{R}, A)$ is an $ILFSS$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ is an $ILFS$ of X as follows:

X	1	2	3
$\tilde{R}[\delta]$	[0.72,0.23]	[0.64,0.30]	[0.57,0.36]

Then $\oplus\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \bar{\phi}_{\tilde{R}[\delta]})$ and $\otimes\tilde{R}[\delta] = (\bar{\psi}_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ are
and

X	1	2	3
$\oplus\tilde{R}[\delta]$	[0.72,0.28]	[0.64,0.36]	[0.57,0.43]

X	1	2	3
$\otimes\tilde{R}[\delta]$	[0.77,0.23]	[0.70,0.30]	[0.64,0.36]

Theorem 3.10. *Let $(\tilde{R}, A)$ be an $ILFSS$ of X. If $(\tilde{R}, A)$ is an $AILFSBID$ of X, then*
(1) $\oplus(\tilde{R}, A)$ *is an $AILFSBID$ of X,*
(2) $\otimes(\tilde{R}, A)$ *is an $AILFSBID$ of X.*

Proof. (1) It is sufficient to show that $\phi_{\tilde{R}[\delta]}$ satisfies the second part of the conditions $(AILFSBID1)$ and $(AILFSBID2)$. Clearly, we have

$$\bar{\phi}_{\tilde{R}[\delta]}(0) = 1 - \phi_{\tilde{R}[\delta]}(0) \geq 1 - \phi_{\tilde{R}[\delta]}(x) \geq \bar{\phi}_{\tilde{R}[\delta]}(x).$$

Let $x, y, z \in X$ and $\delta \in A$. Then

$$\begin{aligned}\bar{\phi}_{\tilde{R}[\delta]}(x) &= 1 - \phi_{\tilde{R}[\delta]}(x) \\ &\geq 1 - (\phi_{\tilde{R}[\delta]}((x * z) * y) \vee \phi_{\tilde{R}[\delta]}(y)) \\ &= \{1 - \phi_{\tilde{R}[\delta]}((x * z) * y)\} \wedge \{1 - \phi_{\tilde{R}[\delta]}(y)\} \\ &= \bar{\phi}_{\tilde{R}[\delta]}((x * z) * y) \wedge \bar{\phi}_{\tilde{R}[\delta]}(y).\end{aligned}$$

Thus $\oplus\tilde{R}[\delta]$ is an $AILFBID$ of X. So $\oplus(\tilde{R}, A)$ is an $AILFSBID$ of X.

(2) It is sufficient to show that $\psi_{\tilde{R}[\delta]}$ satisfies the second part of the conditions $(AILFSBID1)$ and $(AILFSBID3)$. Clearly, we have

$$\bar{\psi}_{\tilde{R}[\delta]}(0) = 1 - \psi_{\tilde{R}[\delta]}(0) \leq 1 - \psi_{\tilde{R}[\delta]}(x) \leq \bar{\psi}_{\tilde{R}[\delta]}(x).$$

Let $x, y, z \in X$ and $\delta \in A$. Then

$$\begin{aligned}\bar{\psi}_{\tilde{R}[\delta]}(x) &= 1 - \psi_{\tilde{R}[\delta]}(x) \\ &\leq 1 - (\psi_{\tilde{R}[\delta]}((x * z) * y) \wedge \psi_{\tilde{R}[\delta]}(y)) \\ &= \{1 - \psi_{\tilde{R}[\delta]}((x * z) * y)\} \vee \{1 - \psi_{\tilde{R}[\delta]}(y)\} \\ &= \bar{\psi}_{\tilde{R}[\delta]}((x * z) * y) \vee \bar{\psi}_{\tilde{R}[\delta]}(y).\end{aligned}$$

Thus $\otimes\tilde{R}[\delta]$ is an $AILFBID$ of X. So $\otimes(\tilde{R}, A)$ is an $AILFSBID$ of X. □

Definition 3.11. Let $(\tilde{R}, A)$ be an $ILFSS$ of X and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ be an $ILFS$ of X. Then the modal type operator

$$\boxplus\tilde{R}[\delta] = \{(x, \phi_{\boxplus\tilde{R}[\delta]}(x), \psi_{\boxplus\tilde{R}[\delta]}(x)) \mid x \in X \text{ and } \delta \in A\}$$

is defined by $\phi_{\boxplus\tilde{R}[\delta]}(x) = \frac{\phi_{\tilde{R}[\delta]}}{2}(x)$ and $\psi_{\boxplus\tilde{R}[\delta]}(x) = \frac{\psi_{\tilde{R}[\delta]}+1}{2}(x)$,
where $\phi_{\boxplus\tilde{R}[\delta]} : X \to [0, 1]$ and $\psi_{\boxplus\tilde{R}[\delta]} : X \to [0, 1]$.

Example 3.12. Let $X = \{a, b, c\}$ be a fixed universe and let $(\tilde{R}, A)$ be an $ILFSS$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ is an $ILFS$ of X as follows:

X	a	b	c
$\tilde{R}[\delta]$	[0.3,0.2]	[0.7,0.1]	[0.5,0.5]

Then $\boxplus\tilde{R}[\delta]$ is an $ILFS$ given by

X	a	b	c
$\boxplus\tilde{R}[\delta]$	[0.15,0.6]	[0.35,0.55]	[0.25,0.75]

Theorem 3.13. *Let $(\tilde{R}, A)$ be an $ILFSS$ of X. If $(\tilde{R}, A)$ is an $AILFSBID$ of X, then the modal type operator $\boxplus(\tilde{R}, A)$ is an $AILFSBID$ of X.*

Proof. For each $x \in X$,

$$\phi_{\boxplus\tilde{R}[\delta]}(0) = \frac{\phi_{\tilde{R}[\delta]}}{2}(0) \leq \frac{\phi_{\tilde{R}[\delta]}}{2}(x) = \phi_{\boxplus\tilde{R}[\delta]}(x)$$

and

$$\psi_{\boxplus\tilde{R}[\delta]}(0) = \frac{\psi_{\tilde{R}[\delta]}+1}{2}(0) \geq \frac{\psi_{\tilde{R}[\delta]}+1}{2}(x) = \psi_{\boxplus\tilde{R}[\delta]}(x).$$

Let $x, y, z \in X$. Then

$$\begin{aligned}\phi_{\boxplus\tilde{R}[\delta]}(x) &= \tfrac{\phi_{\tilde{R}[\delta]}}{2}(x) \leq \tfrac{\phi_{\tilde{R}[\delta]}}{2}((x*z)*y) \vee \tfrac{\phi_{\tilde{R}[\delta]}}{2}(y)\\ &= \phi_{\boxplus\tilde{R}[\delta]}((x*z)*y) \vee \phi_{\boxplus\tilde{R}[\delta]}(y)\end{aligned}$$

and

$$\begin{aligned}\psi_{\boxplus\tilde{R}[\delta]}(x) &= \tfrac{\psi_{\tilde{R}[\delta]}+1}{2}(x) \geq \tfrac{\psi_{\tilde{R}[\delta]}+1}{2}((x*z)*y) \wedge \tfrac{\psi_{\tilde{R}[\delta]}+1}{2}(y)\\ &= \psi_{\boxplus\tilde{R}[\delta]}((x*z)*y) \wedge \psi_{\boxplus\tilde{R}[\delta]}(y).\end{aligned}$$

Thus $\boxplus\tilde{R}[\delta]$ is an $AILFBID$ of X. So $\boxplus(\tilde{R}, A)$ is an $AILFSBID$ of X. □

Definition 3.14. Let $(\tilde{R}, A)$ be an $ILFSS$ of X and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ be an $ILFS$ of X. Then the modal type operator

$$\boxtimes\tilde{R}[\delta] = \{(x, \phi_{\boxtimes\tilde{R}[\delta]}(x), \psi_{\boxtimes\tilde{R}[\delta]}(x)) \mid x \in X \text{ and } \delta \in A\}$$

is defined by $\phi_{\boxtimes\tilde{R}[\delta]}(x) = \frac{\phi_{\tilde{R}[\delta]}+1}{2}(x)$ and $\psi_{\boxtimes\tilde{R}[\delta]}(x) = \frac{\psi_{\tilde{R}[\delta]}}{2}(x)$,
where $\phi_{\boxtimes\tilde{R}[\delta]} : X \to [0, 1]$ and $\psi_{\boxtimes\tilde{R}[\delta]} : X \to [0, 1]$.

Example 3.15. Let $X = \{a, b, c\}$ be a fixed universe and let $(\tilde{R}, A)$ be an $ILFSS$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ is an $ILFS$ of X as follows:

X	a	b	c
$\tilde{R}[\delta]$	[0.3,0.2]	[0.7,0.1]	[0.5,0.5]

Then $\boxtimes\tilde{R}[\delta]$ is an $ILFS$ given by

X	a	b	c
$\boxtimes\tilde{R}[\delta]$	[0.65,0.1]	[0.85,0.05]	[0.75,0.25]

Theorem 3.16. *Let $(\tilde{R}, A)$ be an $ILFSS$ of X and let $(\tilde{R}, A)$ be an $AILFSBID$ of X. Then the modal type operator $\boxtimes(\tilde{R}, A)$ is an $AILFSBID$ of X.*

Proof. For each $x \in X$,

$$\phi_{\boxtimes\tilde{R}[\delta]}(0) = \frac{\phi_{\tilde{R}[\delta]}+1}{2}(0) \leq \frac{\phi_{\tilde{R}[\delta]}+1}{2}(x) = \phi_{\boxtimes\tilde{R}[\delta]}(x)$$

and

$$\psi_{\boxtimes\tilde{R}[\delta]}(0) = \frac{\psi_{\tilde{R}[\delta]}}{2}(0) \geq \frac{\psi_{\tilde{R}[\delta]}}{2}(x) = \psi_{\boxtimes\tilde{R}[\delta]}(x).$$

Let $x, y, z \in X$. Then

$$\begin{aligned}\phi_{\boxtimes\tilde{R}[\delta]}(x) &= \tfrac{\phi_{\tilde{R}[\delta]}+1}{2}(x) \leq \tfrac{\phi_{\tilde{R}[\delta]}+1}{2}((x*z)*y) \vee \tfrac{\phi_{\tilde{R}[\delta]}+1}{2}(y)\\ &= \phi_{\boxtimes\tilde{R}[\delta]}((x*z)*y) \vee \phi_{\boxtimes\tilde{R}[\delta]}(y)\end{aligned}$$

and

$$\begin{aligned}\psi_{\boxtimes\tilde{R}[\delta]}(x) &= \tfrac{\psi_{\tilde{R}[\delta]}}{2}(x) \geq \tfrac{\psi_{\tilde{R}[\delta]}}{2}((x*z)*y) \wedge \tfrac{\psi_{\tilde{R}[\delta]}}{2}(y)\\ &= \psi_{\boxtimes\tilde{R}[\delta]}((x*z)*y) \wedge \psi_{\boxtimes\tilde{R}[\delta]}(y).\end{aligned}$$

Thus $\boxtimes\tilde{R}[\delta]$ is an $AILFBID$ of X. So $\boxtimes(\tilde{R}, A)$ is an $AILFSBID$ of X. □

Definition 3.17. Let $(\tilde{R}, A)$ be an $ILFSS$ of X and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ be an $ILFS$ of X. Then the level operator

$$!\tilde{R}[\delta] = \{(x, \phi_{!\tilde{R}[\delta]}(x), \psi_{!\tilde{R}[\delta]}(x)) \mid x \in X \text{ and } \delta \in A\}$$

is defined by $\phi_{!\tilde{R}[\delta]}(x) = \frac{1}{2} \vee \phi_{\tilde{R}[\delta]}$ and $\psi_{!\tilde{R}[\delta]}(x) = \frac{1}{2} \wedge \psi_{\tilde{R}[\delta]}$,
where $\phi_{!\tilde{R}[\delta]} : X \to [0, 1]$ and $\psi_{!\tilde{R}[\delta]} : X \to [0, 1]$.

Example 3.18. Let $X = \{a, b, c\}$ be a fixed universe and let $(\tilde{R}, A)$ be an $ILFSS$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ is an $ILFS$ of X as follows:

X	a	b	c
$\tilde{R}[\delta]$	[0.3,0.2]	[0.7,0.1]	[0.5,0.5]

Then $!\tilde{R}[\delta]$ is an $ILFS$ given by

X	a	b	c
$!\tilde{R}[\delta]$	[0.5,0.2]	[0.7,0.1]	[0.5,0.5]

Theorem 3.19. *Let $(\tilde{R}, A)$ be an $ILFSS$ of X. If $(\tilde{R}, A)$ is an $AILFSBID$ of X, then the level operator $!\tilde{R}[\delta]$ is an $AILFSBID$ of X.*

Proof. For each $x \in X$,

$$\phi_{!\tilde{R}[\delta]}(0) = \frac{1}{2} \vee \phi_{\tilde{R}[\delta]}(0) \leq \frac{1}{2} \vee \phi_{\tilde{R}[\delta]}(x) = \phi_{!\tilde{R}[\delta]}(x)$$

and

$$\psi_{!\tilde{R}[\delta]}(0) = \frac{1}{2} \wedge \psi_{\tilde{R}[\delta]}(0) \geq \frac{1}{2} \wedge \psi_{\tilde{R}[\delta]}(x) = \phi_{!\tilde{R}[\delta]}(x).$$

Let $x, y, z \in X$. Then

$$\begin{aligned}\phi_{!\tilde{R}[\delta]}(x) &= \tfrac{1}{2} \vee \phi_{\tilde{R}[\delta]}(x) \leq \tfrac{1}{2} \vee [\phi_{\tilde{R}}((x*z)*y) \vee \tfrac{\phi_{\tilde{R}[\delta]}}{2}(y)]\\ &= [\tfrac{1}{2} \vee \phi_{\tilde{R}[\delta]}((x*z)*y)] \vee [\tfrac{1}{2} \vee \tfrac{\phi_{\tilde{R}[\delta]}}{2}(y)]\end{aligned}$$

$= \phi_{!\tilde{R}[\delta]}((x * z) * y) \vee \phi_{!\tilde{R}[\delta]}(y)$

and

$\psi_{!\tilde{R}[\delta]}(x) = \frac{1}{2} \wedge \psi_{\tilde{R}[\delta]}(x) \geq \frac{1}{2} \wedge [\psi_{\tilde{R}}((x * z) * y) \wedge \frac{\psi_{\tilde{R}[\delta]}}{2}(y)]$
$= [\frac{1}{2} \wedge \psi_{\tilde{R}[\delta]}((x * z) * y)] \wedge [\frac{1}{2} \wedge \frac{\psi_{\tilde{R}[\delta]}}{2}(y)]$
$= \psi_{!\tilde{R}[\delta]}((x * z) * y) \wedge \psi_{!\tilde{R}[\delta]}(y).$

Thus $!\tilde{R}[\delta]$ is an $AILFBID$ of X. So $!(\tilde{R}, A)$ is an $AILFSBID$ of X. □

Definition 3.20. Let $(\tilde{R}, A)$ be an $ILFSS$ of X and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ is an $ILFS$ of X. Then the level operator

$$?\tilde{R}[\delta] = \{(x, \phi_{?\tilde{R}[\delta]}(x), \psi_{?\tilde{R}[\delta]}(x)) \mid x \in X \text{ and } \delta \in A\}$$

is defined by $\phi_{?\tilde{R}[\delta]}(x) = \frac{1}{2} \wedge \phi_{\tilde{R}[\delta]}$ and $\psi_{?\tilde{R}[\delta]}(x) = \frac{1}{2} \vee \psi_{\tilde{R}[\delta]}$,
where $\phi_{?\tilde{R}[\delta]} : X \to [0, 1]$ and $\psi_{?\tilde{R}[\delta]} : X \to [0, 1]$.

Example 3.21. Let $X = \{a, b, c\}$ be a fixed universe and let $(\tilde{R}, A)$ be an $ILFSS$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ is an $ILFS$ of X as follows:

X	a	b	c
$\tilde{R}[\delta]$	[0.3,0.2]	[0.7,0.1]	[0.5,0.5]

Then $?\tilde{R}[\delta]$ is an $ILFS$ given by

X	a	b	c
$?\tilde{R}[\delta]$	[0.3,0.5]	[0.5,0.5]	[0.5,0.5]

Theorem 3.22. *Let $(\tilde{R}, A)$ be an $ILFSS$ of X. If $(\tilde{R}, A)$ is an $AILFSBID$ of X, then the level operator $?(\tilde{R}, A)$ is an $AILFSBID$ of X.*

Proof. For each $x \in X$,

$$\phi_{?\tilde{R}[\delta]}(0) = \frac{1}{2} \wedge \phi_{\tilde{R}[\delta]}(0) \leq \frac{1}{2} \wedge \phi_{\tilde{R}[\delta]}(x) = \phi_{?\tilde{R}[\delta]}(x)$$

and

$$\psi_{?\tilde{R}[\delta]}(0) = \frac{1}{2} \vee \psi_{\tilde{R}[\delta]}(0) \geq \frac{1}{2} \vee \psi_{\tilde{R}[\delta]}(x) = \psi_{?\tilde{R}[\delta]}(x).$$

Let $x, y, z \in X$. Then

$\phi_{?\tilde{R}[\delta]}(x) = \frac{1}{2} \wedge \phi_{\tilde{R}[\delta]}(x) \leq \frac{1}{2} \wedge [\phi_{\tilde{R}}((x * z) * y) \vee \frac{\phi_{\tilde{R}[\delta]}}{2}(y)]$
$= [\frac{1}{2} \wedge \phi_{\tilde{R}[\delta]}((x * z) * y)] \vee [\frac{1}{2} \wedge \frac{\phi_{\tilde{R}[\delta]}}{2}(y)]$
$= \phi_{?\tilde{R}[\delta]}((x * z) * y) \vee \phi_{?\tilde{R}[\delta]}(y)$

and

$\psi_{?\tilde{R}[\delta]}(x) = \frac{1}{2} \vee \psi_{\tilde{R}[\delta]}(x) \geq \frac{1}{2} \vee [\psi_{\tilde{R}}((x * z) * y) \wedge \frac{\psi_{\tilde{R}[\delta]}}{2}(y)]$
$= [\frac{1}{2} \vee \psi_{\tilde{R}[\delta]}((x * z) * y)] \wedge [\frac{1}{2} \vee \frac{\psi_{\tilde{R}[\delta]}}{2}(y)]$
$= \psi_{?\tilde{R}[\delta]}((x * z) * y) \wedge \psi_{?\tilde{R}[\delta]}(y).$

Thus $?\tilde{R}[\delta]$ is an $AILFBID$ of X. So $?(\tilde{R}, A)$ is an $AILFSBID$ of X. □

Definition 3.23. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X, and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ be two $ILFS(s)$ of X. Then

$$\tilde{R}[\delta] + \tilde{S}[\eta] = \{(x, \phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x), \psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x)) \mid x \in X, \delta \in A \text{ and } \eta \in B\}$$

is defined by:

$$\phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x) = \phi_{\tilde{R}[\delta]}(x) + \phi_{\tilde{S}[\eta]}(x) - \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(x)$$

and

$$\psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x) = \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(x),$$

where $\phi_{\tilde{R}[\delta]+\tilde{S}[\eta]} : X \to [0, 1]$ and $\psi_{\tilde{R}[\delta]+\tilde{S}[\eta]} : X \to [0, 1]$.

Example 3.24. We define $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X as follows:

X	a	b	c	d	e
$\tilde{R}[\delta]$	[0.5,0.3]	[0.4,0.4]	[0.3,0.5]	[0.2,0.6]	[0.1,0.7]

and

X	a	b	c	d	e
$\tilde{S}[\eta]$	[0.4,0.2]	[0.3,0.3]	[0.2,0.4]	[0.1,0.5]	[0.0,0.6]

Then $\tilde{R}[\delta] + \tilde{S}[\eta]$ is an $ILFS$ given by

X	a	b	c	d	e
$\tilde{R}[\delta] + \tilde{S}[\eta]$	[0.7,0.06]	[0.58,0.12]	[0.44,0.2]	[0.28,0.3]	[0.1,0.42]

Theorem 3.25. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $AILFSBID(s)$ of X, then $(\tilde{R}, A) + (\tilde{S}, B)$ is an $AILFSBID$ of X.*

Proof. For each $x \in X$, $\delta \in A$ and $\eta \in B$,

$$\begin{aligned} \phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(0) &= \phi_{\tilde{R}[\delta]}(0) + \phi_{\tilde{S}[\eta]}(0) - \phi_{\tilde{R}[\delta]}(0).\phi_{\tilde{S}[\eta]}(0) \\ &\leq \phi_{\tilde{R}[\delta]}(x) + \phi_{\tilde{S}[\eta]}(x) - \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(x) \\ &= \phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x) \end{aligned}$$

and

$$\psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(0) = \psi_{\tilde{R}[\delta]}(0).\psi_{\tilde{S}[\eta]}(0) \geq \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(x) = \psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x).$$

Let $x, y, z \in X, \delta \in A$ and $\eta \in B$. Then

$$\begin{aligned} \phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x) &= \phi_{\tilde{R}[\delta]}(x) + \phi_{\tilde{S}[\eta]}(x) - \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(x) \\ &\leq [\phi_{\tilde{R}[\delta]}((x * z) * y) \vee \phi_{\tilde{R}[\delta]}(y)] + [\phi_{\tilde{S}[\eta]}((x * z) * y) \vee \phi_{\tilde{S}[\eta]}(y)] \\ &\quad -[\phi_{\tilde{R}[\delta]}((x * z) * y) \vee \phi_{\tilde{R}[\delta]}(y)][\phi_{\tilde{S}[\eta]}((x * z) * y) \vee \phi_{\tilde{S}[\eta]}(y)] \\ &= [\phi_{\tilde{R}[\delta]}((x * z) * y) + \phi_{\tilde{S}[\eta]}((x * z) * y) \\ &\quad -\phi_{\tilde{R}[\delta]}((x * z) * y).\phi_{\tilde{S}[\eta]}((x * z) * y)] \\ &\quad \vee[\phi_{\tilde{R}[\delta]}(y) + \phi_{\tilde{S}[\eta]}(y) - \phi_{\tilde{R}[\delta]}(y).\phi_{\tilde{S}[\eta]}(y)] \end{aligned}$$

and

$$\begin{aligned}\psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x) &= \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(x)\\ &\geq [\psi_{\tilde{R}[\delta]}((x*z)*y)\wedge\psi_{\tilde{R}[\delta]}(y)].[\psi_{\tilde{S}[\eta]}((x*z)*y)\wedge\psi_{\tilde{S}[\eta]}(y)]\\ &= [\psi_{\tilde{R}[\delta]}((x*z)*y).\psi_{\tilde{S}[\eta]}((x*z)*y)]\wedge[\psi_{\tilde{R}[\delta]}(y).\psi_{\tilde{S}[\eta]}(y)]\end{aligned}$$

Thus $\phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x) \leq \phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}((x*z)*y) \vee \phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(y)$ and

$\psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(x) \geq \psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(y)$.

So $\tilde{R}[\delta]+\tilde{S}[\eta]$ is an $AILFBID$ of X. Hence $(\tilde{R},A)+(\tilde{S},B)$ is an $AILFSBID$ of X. □

Definition 3.26. Let $(\tilde{R},A)$ and $(\tilde{S},B)$ be two $ILFSS(s)$ of X, and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]},\psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]},\psi_{\tilde{S}[\eta]})$ be two $ILFS(s)$ of X. Then

$$\tilde{R}[\delta].\tilde{S}[\eta] = \{(x,\phi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x),\psi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x)) \mid x\in X, \delta\in A \text{ and } \eta\in B\}$$

is defined by:

$$\phi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x) = \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(x)$$

and

$$\psi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x) = \psi_{\tilde{R}[\delta]}(x)+\psi_{\tilde{S}[\eta]}(x)-\psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(x),$$

where $\phi_{\tilde{R}[\delta].\tilde{S}[\eta]} : X\to[0,1]$ and $\psi_{\tilde{R}[\delta].\tilde{S}[\eta]} : X\to[0,1]$.

Example 3.27. We define $(\tilde{R},A)$ and $(\tilde{S},B)$ are two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]},\psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]},\psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X as follows:

X	a	b	c	d	e
$\tilde{R}[\delta]$	[0.4,0.3]	[0.1,0.7]	[1.0,0.0]	[0.0,0.0]	[0.0,1.0]

and

X	a	b	c	d	e
$\tilde{S}[\eta]$	[0.5,0.1]	[0.3,0.2]	[0.5,0.5]	[0.2,0.2]	[1.0,0.0]

Then $\tilde{R}[\delta].\tilde{S}[\eta]$ is an $ILFS$ given by

X	a	b	c	d	e
$\tilde{R}[\delta].\tilde{S}[\eta]$	[0.2,0.37]	[0.03,0.76]	[0.5,0.5]	[0.0,0.2]	[0.0,0.1]

Theorem 3.28. *Let $(\tilde{R},A)$ and $(\tilde{S},B)$ be two $ILFSS(s)$ of X. If $(\tilde{R},A)$ and $(\tilde{S},B)$ are two $AILFSBID(s)$ of X, then $(\tilde{R},A).(\tilde{S},B)$ is an $AILFSBID$ of X.*

Proof. For each $x\in X,\ \delta\in A$ and $\eta\in B$,

$$\phi_{\tilde{R}[\delta].\tilde{S}[\eta]}(0) = \phi_{\tilde{R}[\delta]}(0).\phi_{\tilde{S}[\eta]}(0) \leq \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(x) = \phi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x)$$

and

$$\begin{aligned}\psi_{\tilde{R}[\delta].\tilde{S}[\eta]}(0) &= \psi_{\tilde{R}[\delta]}(0)+\psi_{\tilde{S}[\eta]}(0)-\psi_{\tilde{R}[\delta]}(0).\psi_{\tilde{S}[\eta]}(0)\\ &\geq \psi_{\tilde{R}[\delta]}(x)+\psi_{\tilde{S}[\eta]}(x)-\psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(x)\\ &= \psi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x).\end{aligned}$$

Let $x,y,z\in X,\delta\in A$ and $\eta\in B$. Then

$$\begin{aligned}\phi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x) &= \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(x)\\ &\leq [\phi_{\tilde{R}[\delta]}((x*z)*y) \vee \phi_{\tilde{R}[\delta]}(y)].[\phi_{\tilde{S}[\eta]}((x*z)*y) \vee \phi_{\tilde{S}[\eta]}(y)]\\ &= [\phi_{\tilde{R}[\delta]}((x*z)*y).\phi_{\tilde{S}[\eta]}((x*z)*y)] \vee [\phi_{\tilde{R}[\delta]}(y).\phi_{\tilde{S}[\eta]}(y)]\end{aligned}$$

and

$$\begin{aligned}\psi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x) &= \psi_{\tilde{R}[\delta]}(x) + \psi_{\tilde{S}[\eta]}(x) - \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(x)\\ &\geq [\psi_{\tilde{R}[\delta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta]}(y)] + [\psi_{\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{S}[\eta]}(y)]\\ &\quad -[\psi_{\tilde{R}[\delta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta]}(y)].[\psi_{\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{S}[\eta]}(y)]\\ &= [\psi_{\tilde{R}[\delta]}((x*z)*y) + \psi_{\tilde{S}[\eta]}((x*z)*y)\\ &\quad -\psi_{\tilde{R}[\delta]}((x*z)*y).\psi_{\tilde{S}[\eta]}((x*z)*y)]\\ &\quad \wedge[\psi_{\tilde{R}[\delta]}(y) + \psi_{\tilde{S}[\eta]}(y) - \psi_{\tilde{R}[\delta]}(y).\psi_{\tilde{S}[\eta]}(y)].\end{aligned}$$

Thus $\phi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x) \leq \phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}((x*z)*y) \vee \phi_{\tilde{R}[\delta]+\tilde{S}[\eta]}(y)$ and
$\psi_{\tilde{R}[\delta].\tilde{S}[\eta]}(x) \geq \psi_{\tilde{R}[\delta]+\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta].\tilde{S}[\eta]}(y)$.
So $\tilde{R}[\delta].\tilde{S}[\eta]$ is an $AILFBID$ of X. Hence $(\tilde{R},A).(\tilde{S},B)$ is an $AILFSBID$ of X. □

Definition 3.29. Let $(\tilde{R},A)$ and $(\tilde{S},B)$ be two ILFSS(s) of X, and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ be two ILFSS(s) of X. Then

$$\tilde{R}[\delta]@\tilde{S}[\eta] = \{(x, \phi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x), \psi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x)) \mid x \in X, \delta \in A \text{ and } \eta \in B\}$$

is defined by:

$$\phi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x) = \frac{\phi_{\tilde{R}[\delta]}(x) + \phi_{\tilde{S}[\eta]}(x)}{2}$$

and

$$\psi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x) = \frac{\psi_{\tilde{R}[\delta]}(x) + \psi_{\tilde{S}[\eta]}(x)}{2},$$

where $\phi_{\tilde{R}[\delta]@\tilde{S}[\eta]} : X \to [0,1]$ and $\psi_{\tilde{R}[\delta]@\tilde{S}[\eta]} : X \to [0,1]$.

Example 3.30. We define $(\tilde{R},A)$ and $(\tilde{S},B)$ are two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X as follows:

X	a	b	c	d	e
$\tilde{R}[\delta]$	[0.5,0.3]	[0.1,0.7]	[1.0,0.0]	[0.0,0.0]	[0.0,1.0]

and

X	a	b	c	d	e
$\tilde{S}[\eta]$	[0.7,0.1]	[0.3,0.2]	[0.5,0.5]	[0.2,0.2]	[1.0,0.0]

Then $\tilde{R}[\delta]@\tilde{S}[\eta]$ is an $ILFS$ given by

X	a	b	c	d	e
$\tilde{R}[\delta]@\tilde{S}[\eta]$	[0.6,0.2]	[0.2,0.45]	[0.75,0.25]	[0.1,0.1]	[0.5,0.5]

Theorem 3.31. *Let $(\tilde{R},A)$ and $(\tilde{S},B)$ be two $ILFSS(s)$ of X. If $(\tilde{R},A)$ and $(\tilde{S},B)$ are two $AILFSBID(s)$ of X, then $(\tilde{R},A)@(\tilde{S},B)$ is an $AILFSBID$ of X.*

Proof. For each $x \in X,\ \delta \in A$ and $\eta \in B$, we have

$$\phi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(0) = \frac{\phi_{\tilde{R}[\delta]}(0) + \phi_{\tilde{S}[\eta]}(0)}{2} \leq \frac{\phi_{\tilde{R}[\delta]}(x) + \phi_{\tilde{S}[\eta]}(x)}{2} = \phi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x)$$

and

$$\psi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(0) = \frac{\psi_{\tilde{R}[\delta]}(0) + \psi_{\tilde{S}[\eta]}(0)}{2} \geq \frac{\psi_{\tilde{R}[\delta]}(x) + \psi_{\tilde{S}[\eta]}(x)}{2} = \psi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x).$$

Let $x, y, z \in X, \delta \in A$ and $\eta \in B$. Then

$$\begin{aligned} \phi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x) &= \frac{\phi_{\tilde{R}[\delta]}(x)+\phi_{\tilde{S}[\eta]}(x)}{2} \\ &\leq \frac{[\phi_{\tilde{R}[\delta]}((x*z)*y) \vee \phi_{\tilde{R}[\delta]}(y)]+[\phi_{\tilde{S}[\eta]}((x*z)*y) \vee \phi_{\tilde{S}[\eta]}(y)]}{2} \\ &= \frac{[\phi_{\tilde{R}[\delta]}((x*z)*y)+\phi_{\tilde{S}[\eta]}((x*z)*y)] \vee [\phi_{\tilde{R}[\delta]}(y)+\phi_{\tilde{S}[\eta]}(y)]}{2} \end{aligned}$$

and

$$\begin{aligned} \psi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x) &= \frac{\psi_{\tilde{R}[\delta]}(x)+\psi_{\tilde{S}[\eta]}(x)}{2} \\ &\geq \frac{[\psi_{\tilde{R}[\delta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta]}(y)]+[\psi_{\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{S}[\eta]}(y)]}{2} \\ &= \frac{[\psi_{\tilde{R}[\delta]}((x*z)*y)+\psi_{\tilde{S}[\eta]}((x*z)*y)] \wedge [\psi_{\tilde{R}[\delta]}(y)+\psi_{\tilde{S}[\eta]}(y)]}{2}. \end{aligned}$$

Thus $\phi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x) \leq \phi_{\tilde{R}[\delta]@\tilde{S}[\eta]}((x*z)*y) \vee \phi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(y)$ and
$\psi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(x) \geq \psi_{\tilde{R}[\delta]@\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta]@\tilde{S}[\eta]}(y)$.
So $\tilde{R}[\delta]@\tilde{S}[\eta]$ is an $AILFBID$ of X. Hence $(\tilde{R}, A)@(\tilde{S}, B)$ is an $AILFSBID$ of X. □

Definition 3.32. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X, and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ be two $ILFS(s)$ of X. Then

$$\tilde{R}[\delta]\$\tilde{S}[\eta] = \{(x, \phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x), \psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x)) \mid x \in X,\ \delta \in A \text{ and } \eta \in B\}$$

is defined by:

$$\phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x) = \sqrt{\phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(x)}$$

and

$$\psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x) = \sqrt{\psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(x)},$$

where $\phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]} : X \to [0,1]$ and $\psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]} : X \to [0,1]$.

Example 3.33. We define $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X as follows:

X	a	b	c	d	e
$\tilde{R}[\delta]$	[0.5,0.3]	[0.1,0.7]	[1.0,0.0]	[0.0,0.0]	[0.0,1.0]

and

X	a	b	c	d	e
$\tilde{S}[\eta]$	[0.7,0.1]	[0.3,0.2]	[0.5,0.5]	[0.2,0.2]	[1.0,0.0]

Then $\tilde{R}[\delta]\$\tilde{S}[\eta]$ is an $ILFS$ given by

X	a	b	c	d	e
$\tilde{R}[\delta]\$\tilde{S}[\eta]$	[0.59...,0.17...]	[0.17...,0.37...]	[0.707...,0.0]	[0.0,0.0]	[0.0,0.0]

Theorem 3.34. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $AILFSBID(s)$ of X, then $(\tilde{R}, A)\$(\tilde{S}, B)$ is an $AILFSBID$ of X.*

Proof. For each $x \in X, \delta \in A$ and $\eta \in B$,

$$\phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(0) = \sqrt{\phi_{\tilde{R}[\delta]}(0).\phi_{\tilde{S}[\eta]}(0)} \leq \sqrt{\phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(x)} = \phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x)$$

and

$$\psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(0) = \sqrt{\psi_{\tilde{R}[\delta]}(0).\psi_{\tilde{S}[\eta]}(0)} \geq \sqrt{\psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(x)} = \psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x).$$

Let $x, y, z \in X, \delta \in A$ and $\eta \in B$. Then

$$\begin{aligned}\phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x) &= \sqrt{[\phi_{\tilde{R}[\delta]}((x*z)*y) \vee \phi_{\tilde{R}[\delta]}(y)].[\phi_{\tilde{S}[\eta]}((x*z)*y) \vee \phi_{\tilde{S}[\eta]}(y)]} \\ &= \sqrt{[\phi_{\tilde{R}[\delta]}((x*z)*y).\phi_{\tilde{S}[\eta]}((x*z)*y)] \vee [\phi_{\tilde{R}[\delta]}(y).\phi_{\tilde{S}[\eta]}(y)]}\end{aligned}$$

and

$$\begin{aligned}\psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x) &= \sqrt{[\psi_{\tilde{R}[\delta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta]}(y)].[\psi_{\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{S}[\eta]}(y)]} \\ &= \sqrt{[\psi_{\tilde{R}[\delta]}((x*z)*y).\psi_{\tilde{S}[\eta]}((x*z)*y)] \wedge [\psi_{\tilde{R}[\delta]}(y).\psi_{\tilde{S}[\eta]}(y)]}.\end{aligned}$$

$\psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x) \geq \psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(y)$.

Thus $\phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x) \leq \phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}((x*z)*y) \vee \phi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(y)$ and

$\psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(x) \geq \psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}((x*z)*y) \wedge \psi_{\tilde{R}[\delta]\$\tilde{S}[\eta]}(y)$.

So $\tilde{R}[\delta]\$\tilde{S}[\eta]$ is an $AILFBID$ of X. Hence $(\tilde{R}, A)\$(\tilde{S}, B)$ is an $AILFSBID$ of X. □

4. Cartesian product of anti-intuitionistic L-fuzzy soft BG-algebras

In this section, product of $AILFSBID$ of BG-algebras is defined and some results are studied.

Definition 4.1. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X, and let $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ be two $ILFS(s)$ of X. Then the Cartesian product

$$\tilde{R}[\delta] \times \tilde{S}[\eta] = (\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})$$

is defined by:

$$(\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})(x, y) = \phi_{\tilde{R}[\delta]}(x) \wedge \phi_{\tilde{S}[\eta]}(y)$$

and

$$(\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})(x, y) = \psi_{\tilde{R}[\delta]}(x) \vee \psi_{\tilde{S}[\eta]}(y),$$

where $\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]} : X \times X \to [0, 1]$ and $\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]} : X \times X \to [0, 1]$, for every $(x, y) \in X \times X$ and $(\delta, \eta) \in A \times B$.

Example 4.2. We define $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X as follows:

X	a	b	c
$\tilde{R}[\delta]$	[0.8,0.2]	[0.7,0.3]	[0.6,0.4]

and

X	1	2
$\tilde{S}[\eta]$	[0.6,0.3]	[0.4,0.5]

Then $\tilde{R}[\delta] \times \tilde{S}[\eta] = (\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})$, where $\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]}$ as
$\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}(a,1) = 0.6, \phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}(a,2) = 0.4, \phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}(b,1) = 0.6,$
$\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}(b,2) = 0.4, \phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}(c,1) = 0.6, \phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}(c,2) = 0.4,$
$\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]}(a,1) = 0.3, \psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]}(a,2) = 0.5, \psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]}(b,1) = 0.3,$
$\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]}(b,2) = 0.5, \psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]}(c,1) = 0.4, \psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]}(c,2) = 0.5.$

Theorem 4.3. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $AILFSBID(s)$ of X, then $(\tilde{R}, A) \times (\tilde{S}, B)$ is an $AILFSBID$ of $X \times X$.*

Proof. For each $(x, y) \in X \times X$ and $(\delta, \eta) \in A \times B$,
$(\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})(0,0) = \phi_{\tilde{R}[\delta]}(0) \vee \phi_{\tilde{S}[\eta]}(0) \leq \phi_{\tilde{R}[\delta]}(x) \vee \phi_{\tilde{S}[\eta]}(y) = (\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})(x,y)$
and
$(\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})(0,0) = \psi_{\tilde{R}[\delta]}(0) \wedge \psi_{\tilde{S}[\eta]}(0) \geq \psi_{\tilde{R}[\delta]}(x) \wedge \psi_{\tilde{S}[\eta]}(y) = (\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})(x,y).$
Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times X$ and $(\delta, \eta) \in A \times B$. Then

$$\begin{aligned}
&(\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})(x_1, y_1) \\
&= \phi_{\tilde{R}[\delta]}(x_1) \vee \phi_{\tilde{S}[\eta]}(y_1) \\
&\leq (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{R}[\delta]}(x_2)) \vee (\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \vee \phi_{\tilde{S}[\eta]}(y_2)) \\
&= (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \vee (\phi_{\tilde{R}[\delta]}(x_2) \vee \phi_{\tilde{S}[\eta]}(y_2))
\end{aligned}$$

and

$$\begin{aligned}
&(\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})(x_1, y_1) \\
&= \psi_{\tilde{R}[\delta]}(x_1) \wedge \psi_{\tilde{S}[\eta]}(y_1) \\
&\geq (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{R}[\delta]}(x_2)) \wedge (\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \wedge \psi_{\tilde{S}[\eta]}(y_2)) \\
&= (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \wedge (\psi_{\tilde{R}[\delta]}(x_2) \wedge \psi_{\tilde{S}[\eta]}(y_2))
\end{aligned}$$

Thus

$$\begin{aligned}
&(\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})(x_1, y_1) \\
&\leq (\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})(x_2, y_2)
\end{aligned}$$

and

$$\begin{aligned}
&(\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})(x_1, y_1) \\
&\geq (\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \wedge (\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})(x_2, y_2).
\end{aligned}$$

So $\tilde{R}[\delta] \times \tilde{S}[\eta]$ is an $AILFBID$ of $X \times X$. Hence $(\tilde{R}, A) \times (\tilde{S}, B)$ is an $AILFSBID$ of $X \times X$. □

Definition 4.4. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X. Then the Cartesian product

$$\oplus(\tilde{R}[\delta] \times \tilde{S}[\eta]) = (\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]}, \bar{\phi}_{\tilde{R}[\delta]} \times \bar{\phi}_{\tilde{S}[\eta]})$$

is defined by:

$$(\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})(x,y) = \phi_{\tilde{R}[\delta]}(x) \wedge \phi_{\tilde{S}[\eta]}(y)$$

and

$$(\bar{\phi}_{\tilde{R}[\delta]} \times \bar{\phi}_{\tilde{S}[\eta]})(x, y) = \bar{\phi}_{\tilde{R}[\delta]}(x) \vee \bar{\phi}_{\tilde{S}[\eta]}(y),$$

where $\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]} : X \times X \to [0, 1]$ and $\bar{\phi}_{\tilde{R}[\delta]} \times \bar{\phi}_{\tilde{S}[\eta]} : X \times X \to [0, 1]$, for every $(x, y) \in X \times X$ and $(\delta, \eta) \in A \times B$.

Theorem 4.5. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are $AILFSBID(s)$ of X, then $\oplus((\tilde{R}, A) \times (\tilde{R}, A))$ is also an $AILFSBID$ of $X \times X$.*

Proof. Since $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are $AILFSBID(s)$ of X, $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are $ILFS(s)$ of X. Then $\tilde{R}[\delta] \times \tilde{S}[\eta]$ is $AILFBID$ of X. Thus for each $(x, y) \in X \times X$ and $(\delta, \eta) \in A \times B$,

$$(\phi_{\tilde{R}[\delta]} \times \phi_{\tilde{S}[\eta]})(x, y) = \phi_{\tilde{R}[\delta]}(x) \vee \phi_{\tilde{S}[\eta]}(y)$$
$$\Rightarrow 1 - (\bar{\phi}_{\tilde{R}[\delta]} \times \bar{\phi}_{\tilde{S}[\eta]})(x, y) = \{1 - \bar{\phi}_{\tilde{R}[\delta]}(x)\} \vee \{1 - \bar{\phi}_{\tilde{S}[\eta]}(y)\}$$
$$\Rightarrow 1 - \{1 - \bar{\phi}_{\tilde{R}[\delta]}(x)\} \vee \{1 - \bar{\phi}_{\tilde{S}[\eta]}(y)\} = (\bar{\phi}_{\tilde{R}[\delta]} \times \bar{\phi}_{\tilde{S}[\eta]})(x, y)$$
$$\Rightarrow (\bar{\phi}_{\tilde{R}[\delta]} \times \bar{\phi}_{\tilde{S}[\eta]})(x, y) = \bar{\phi}_{\tilde{R}[\delta]}(x) \wedge \bar{\phi}_{\tilde{S}[\eta]}(y).$$

So $\oplus(\tilde{R}[\delta] \times \tilde{S}[\eta])$ is an $AILFBID$ of $X \times X$. Hence $\oplus((\tilde{R}, A) \times (\tilde{R}, A))$ is an $AILFSBID$ of $X \times X$. □

Definition 4.6. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X. Then the Cartesian product

$$\otimes(\tilde{R}[\delta] \times \tilde{S}[\eta]) = (\bar{\psi}_{\tilde{R}[\delta]} \times \bar{\psi}_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})$$

is defined by:

$$(\bar{\psi}_{\tilde{R}[\delta]} \times \bar{\psi}_{\tilde{S}[\eta]})(x, y) = \bar{\psi}_{\tilde{R}[\delta]}(x) \wedge \bar{\psi}_{\tilde{S}[\eta]}(y)$$

and

$$(\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})(x, y) = \psi_{\tilde{R}[\delta]}(x) \vee \psi_{\tilde{S}[\eta]}(y),$$

where $\bar{\psi}_{\tilde{R}[\delta]} \times \bar{\psi}_{\tilde{S}[\eta]} : X \times X \to [0, 1]$ and $\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]} : X \times X \to [0, 1]$, for every $(x, y) \in X \times X$ and $(\delta, \eta) \in A \times B$.

Theorem 4.7. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are AILFSB-ideal of X, then $\otimes((\tilde{R}, A) \times (\tilde{S}, B))$ is also an AILFSB-ideal of $X \times X$.*

Proof. Since $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are $AILFSBID(s)$ of X, $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are $ILFS(s)$ of X. Then $\tilde{R}[\delta] \times \tilde{S}[\eta]$ is $AILFSBID$ of X. Thus for each $(x, y) \in X \times X$ and $(\delta, \eta) \in A \times B$,

$$(\psi_{\tilde{R}[\delta]} \times \psi_{\tilde{S}[\eta]})(x, y) = \psi_{\tilde{R}[\delta]}(x) \wedge \psi_{\tilde{S}[\eta]}(y)$$
$$\Rightarrow 1 - (\bar{\psi}_{\tilde{R}[\delta]} \times \bar{\psi}_{\tilde{S}[\eta]})(x, y) = \{1 - \bar{\psi}_{\tilde{R}[\delta]}(x)\} \wedge \{1 - \bar{\psi}_{\tilde{S}[\eta]}(y)\}$$
$$\Rightarrow 1 - \{1 - \bar{\psi}_{\tilde{R}[\delta]}(x)\} \wedge \{1 - \bar{\psi}_{\tilde{S}[\eta]}(y)\} = (\bar{\psi}_{\tilde{R}[\delta]} \times \bar{\psi}_{\tilde{S}[\eta]})(x, y)$$
$$\Rightarrow (\bar{\psi}_{\tilde{R}[\delta]} \times \bar{\psi}_{\tilde{S}[\eta]})(x, y) = \bar{\psi}_{\tilde{R}[\delta]}(x) \vee \bar{\psi}_{\tilde{S}[\eta]}(y).$$

So $\otimes(\tilde{R}[\delta] \times \tilde{S}[\eta])$ is an $AILFBID$ of $X \times X$. Hence $\otimes((\tilde{R}, A) \times (\tilde{S}, B))$ is also an $AILFSBID$ of $X \times X$. □

Definition 4.8. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ be two $ILFS(s)$ of X. Then the Cartesian product

$$(\tilde{R}[\delta] \times_1 \tilde{S}[\eta]) = (\phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]})$$

is defined by:

$$(\phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]})(x, y) = \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(y)$$

and

$$(\psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]})(x, y) = \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(y),$$

where $\phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]} : X \times_1 X \to [0, 1]$ and $\psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]} : X \times_1 X \to [0, 1]$, for every $(x, y) \in X \times_1 X$ and $(\delta, \eta) \in A \times_1 B$.

Theorem 4.9. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are $AILFSBID(s)$ of X, then $(\tilde{R}, A) \times_1 (\tilde{S}, B)$ is also an $AILFSBID$ of $X \times_1 X$.*

Proof. For each $(x, y) \in X \times_1 X$ and $(\delta, \eta) \in A \times_1 B$,

$\phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]}(0, 0) = \phi_{\tilde{R}[\delta]}(0).\phi_{\tilde{S}[\eta]}(0) \leq \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(y) = \phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]}(x, y)$

and

$\psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]}(0, 0) = \psi_{\tilde{R}[\delta]}(0).\psi_{\tilde{S}[\eta]}(0) \geq \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(y) = \psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]})(x, y)$.

Let $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3) \in X \times_1 X$ and $(\delta, \eta) \in A \times_1 B$. Then

$$\begin{aligned} &(\phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]})(x_1, y_1) \\ &= \phi_{\tilde{R}[\delta]}(x_1).\phi_{\tilde{S}[\eta]}(y_1) \\ &\leq (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{R}[\delta]}(x_2)).(\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \vee \phi_{\tilde{S}[\eta]}(y_2)) \\ &= (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2).\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \vee (\phi_{\tilde{R}[\delta]}(x_2).\phi_{\tilde{S}[\eta]}(y_2)) \end{aligned}$$

and

$$\begin{aligned} &(\psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]})(x_1, y_1) \\ &= \psi_{\tilde{R}[\delta]}(x_1).\psi_{\tilde{S}[\eta]}(y_1) \\ &\geq (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{R}[\delta]}(x_2)).(\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \wedge \psi_{\tilde{S}[\eta]}(y_2)) \\ &= (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2).\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \wedge (\psi_{\tilde{R}[\delta]}(x_2).\psi_{\tilde{S}[\eta]}(y_2)). \end{aligned}$$

Thus $(\phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]})(x_1, y_1)$

$\leq (\phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\phi_{\tilde{R}[\delta]} \times_1 \phi_{\tilde{S}[\eta]})(x_2, y_2)$

and

$(\psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]})(x_1, y_1)$

$\geq (\psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]})(x_2, y_2)$.

So $\tilde{R}[\delta] \times_1 \tilde{S}[\eta]$ is an $AILFBID$ of $X \times_1 X$. Hence $(\tilde{R}, A) \times_1 (\tilde{S}, B)$ is an $AILFSBID$ of $X \times_1 X$. □

Definition 4.10. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X. Then the Cartesian product

$$\tilde{R}[\delta] \times_2 \tilde{S}[\eta] = (\phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times_2 \psi_{\tilde{S}[\eta]})$$

is defined by:

$$(\phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]})(x, y) = \phi_{\tilde{R}[\delta]}(x) + \phi_{\tilde{S}[\eta]}(y) - \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(y)$$

and

$$(\psi_{\tilde{R}[\delta]} \times_2 \psi_{\tilde{S}[\eta]})(x, y) = \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(y),$$

where $\phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]} : X \times_2 X \to [0,1]$ and $\psi_{\tilde{R}[\delta]} \times_2 \psi_{\tilde{S}[\eta]} : X \times_2 X \to [0,1]$, for every $(x,y) \in X \times_2 X$ and $(\delta,\eta) \in A \times_2 B$.

Theorem 4.11. *Let $(\tilde{R},A)$ and $(\tilde{S},B)$ are two $ILFSS(s)$ of X. If $(\tilde{R},A)$ and $(\tilde{S},B)$ are $AILFSBID(s)$ of X, then $(\tilde{R},A) \times_2 (\tilde{S},B)$ is also an $AILFSBID$ of $X \times_2 X$.*

Proof. For each $(x,y) \in X \times_2 X$ and $(\delta,\eta) \in A \times_2 B$,

$$\begin{aligned}\phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]}(0,0) &= \phi_{\tilde{R}[\delta]}(0) + \phi_{\tilde{S}[\eta]}(0) - \phi_{\tilde{R}[\delta]}(0).\phi_{\tilde{S}[\eta]}(0)\\ &\leq \phi_{\tilde{R}[\delta]}(x) + \phi_{\tilde{S}[\eta]}(y) - \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(y)\\ &= \phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]}(x,y).\end{aligned}$$

and

$$\begin{aligned}\psi_{\tilde{R}[\delta]} \times_2 \psi_{\tilde{S}[\eta]})(0,0) &= \psi_{\tilde{R}[\delta]}(0).\psi_{\tilde{S}[\eta]}(0)\\ &\geq \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(y)\\ &= \psi_{\tilde{R}[\delta]} \times_1 \psi_{\tilde{S}[\eta]})(x,y).\end{aligned}$$

Let $(x_1,y_1),(x_2,y_2)$ and $(x_3,y_3) \in X \times_2 X$ and $(\delta,\eta) \in A \times_2 B$. Then

$$\begin{aligned}&(\phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]})(x_1,y_1)\\ &= \phi_{\tilde{R}[\delta]}(x_1) + \phi_{\tilde{S}[\eta]}(y_1) - \phi_{\tilde{R}[\delta]}(x_1).\phi_{\tilde{S}[\eta]}(y_1)\\ &\leq (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{R}[\delta]}(x_2)) + (\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \vee \phi_{\tilde{S}[\eta]})(y_2)\\ &\quad -(\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{R}[\delta]}(x_2)).(\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \vee \phi_{\tilde{S}[\eta]}(y_2))\\ &= \{\phi_{\tilde{R}[\delta]}((x_1*x_3)*x_2)+\phi_{\tilde{S}[\eta]}((y_1*y_3)*y_2)-\phi_{\tilde{R}[\delta]}((x_1*x_3)*x_2).\phi_{\tilde{S}[\eta]}((y_1*y_3)*y_2)\}\\ &\quad \vee\{\phi_{\tilde{R}[\delta]}(x_2) + \phi_{\tilde{S}[\eta]}(y_2) - \phi_{\tilde{R}[\delta]}(x_2).\phi_{\tilde{S}[\eta]}(y_2)\}\end{aligned}$$

and

$$\begin{aligned}&(\psi_{\tilde{R}[\delta]} \times_2 \psi_{\tilde{S}[\eta]})(x_1,y_1)\\ &= \psi_{\tilde{R}[\delta]}(x_1).\psi_{\tilde{S}[\eta]}(y_1)\\ &\geq (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{R}[\delta]}(x_2)).(\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \wedge \psi_{\tilde{S}[\eta]}(y_2))\\ &= (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2).\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \wedge (\psi_{\tilde{R}[\delta]}(x_2).\psi_{\tilde{S}[\eta]}(y_2)).\end{aligned}$$

Thus

$$\begin{aligned}&(\phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]})(x_1,y_1)\\ &\leq (\phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\phi_{\tilde{R}[\delta]} \times_2 \phi_{\tilde{S}[\eta]})(x_2,y_2)\end{aligned}$$

and

$$\begin{aligned}&(\psi_{\tilde{R}[\delta]} \times_2 \psi_{\tilde{S}[\eta]})(x_1,y_1)\\ &\geq (\psi_{\tilde{R}[\delta]} \times_2 \psi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\psi_{\tilde{R}[\delta]} \times_2 \psi_{\tilde{S}[\eta]})(x_2,y_2).\end{aligned}$$

So $\tilde{R}[\delta] \times_2 \tilde{S}[\eta]$ is an $AILFBID$ of $X \times_2 X$. Hence $(\tilde{R},A) \times_2 (\tilde{S},B)$ is an $AILFSBID$ of $X \times_2 X$. □

Definition 4.12. Let $(\tilde{R},A)$ and $(\tilde{S},B)$ be two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X. Then the Cartesian product

$$\tilde{R}[\delta] \times_3 \tilde{S}[\eta] = (\phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]})$$

is defined by:

$$(\phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]})(x,y) = \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(y)$$

and

$$(\psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]})(x,y) = \psi_{\tilde{R}[\delta]}(x) + \psi_{\tilde{S}[\eta]}(y) - \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(y),$$

where $\phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]} : X \times_3 X \to [0,1]$ and $\psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]} : X \times_3 X \to [0,1]$, for every $(x,y) \in X \times_3 X$ and $(\delta,\eta) \in A \times_3 B$.

Theorem 4.13. *Let $(\tilde{R},A)$ and $(\tilde{S},B)$ are two $ILFSS(s)$ of X. If $(\tilde{R},A)$ and $(\tilde{S},B)$ are $AILFSBID(s)$ of X, then $(\tilde{R},A) \times_3 (\tilde{S},B)$ is also an $AILFSBID$ of $X \times_3 X$.*

Proof. For each $(x,y) \in X \times_3 X$ and $(\delta,\eta) \in A \times_3 B$,

$$\phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]}(0,0) = \phi_{\tilde{R}[\delta]}(0).\phi_{\tilde{S}[\eta]}(0) \leq \phi_{\tilde{R}[\delta]}(x).\phi_{\tilde{S}[\eta]}(y) = \phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]}(x,y)$$

and

$$\psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]}(0,0) = \psi_{\tilde{R}[\delta]}(0) + \psi_{\tilde{S}[\eta]}(0) - \psi_{\tilde{R}[\delta]}(0).\psi_{\tilde{S}[\eta]}(0)$$
$$\geq \psi_{\tilde{R}[\delta]}(x) + \psi_{\tilde{S}[\eta]}(y) - \psi_{\tilde{R}[\delta]}(x).\psi_{\tilde{S}[\eta]}(y) = \psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]})(x,y).$$

Let $(x_1,y_1),(x_2,y_2)$ and $(x_3,y_3) \in X \times_3 X$ and $(\delta,\eta) \in A \times_3 B$. Then

$$\begin{aligned}
&(\phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]})(x_1,y_1)\\
&= \phi_{\tilde{R}[\delta]}(x_1).\phi_{\tilde{S}[\eta]}(y_1)\\
&\leq (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{R}[\delta]}(x_2)).(\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \vee \phi_{\tilde{S}[\eta]}(y_2))\\
&= (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2).\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \vee (\phi_{\tilde{R}[\delta]}(x_2).\phi_{\tilde{S}[\eta]}(y_2))
\end{aligned}$$

and

$$\begin{aligned}
&(\psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]})(x_1,y_1)\\
&= \psi_{\tilde{R}[\delta]}(x_1) + \psi_{\tilde{S}[\eta]}(y_1) - \psi_{\tilde{R}[\delta]}(x_1).\psi_{\tilde{S}[\eta]}(y_1)\\
&\geq (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{R}[\delta]}(x_2)) + (\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \wedge \psi_{\tilde{S}[\eta]})(y_2)\\
&\quad -(\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{R}[\delta]}(x_2)).(\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \wedge \psi_{\tilde{S}[\eta]}(y_2))\\
&= \{\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) + \psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)\\
&\quad -\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2).\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)\}\\
&\quad \wedge\{\psi_{\tilde{R}[\delta]}(x_2) + \psi_{\tilde{S}[\eta]}(y_2) - \psi_{\tilde{R}[\delta]}(x_2).\psi_{\tilde{S}[\eta]}(y_2)\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&(\phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]})(x_1,y_1)\\
&\leq (\phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\phi_{\tilde{R}[\delta]} \times_3 \phi_{\tilde{S}[\eta]})(x_2,y_2)
\end{aligned}$$

and

$$\begin{aligned}
&(\psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]})(x_1,y_1)\\
&\geq (\psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\psi_{\tilde{R}[\delta]} \times_3 \psi_{\tilde{S}[\eta]})(x_2,y_2).
\end{aligned}$$

So $\tilde{R}[\delta] \times_3 \tilde{S}[\eta]$ is an $AILFBID$ of $X \times_3 X$. Hence $(\tilde{R},A) \times_3 (\tilde{S},B)$ is an $AILFSBID$ of $X \times_3 X$. □

Definition 4.14. Let $(\tilde{R},A)$ and $(\tilde{S},B)$ be two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X. Then the Cartesian product

$$\tilde{R}[\delta] \times_4 \tilde{S}[\eta] = (\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]})$$

is defined by:

$$(\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]})(x,y) = \phi_{\tilde{R}[\delta]}(x) \wedge \phi_{\tilde{S}[\eta]}(y)$$

and

$$(\psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]})(x,y) = \psi_{\tilde{R}[\delta]}(x) \vee \psi_{\tilde{S}[\eta]}(y),$$

where $\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]} : X \times_4 X \to [0,1]$ and $\psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]} : X \times_4 X \to [0,1]$, for every $(x,y) \in X \times_4 X$ and $(\delta,\eta) \in A \times_4 B$.

Theorem 4.15. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are $AILFSBID(s)$ of X, then $(\tilde{R}, A) \times_4 (\tilde{S}, B)$ is also an $AILFSBID$ of $X \times_4 X$.*

Proof. For each $(x, y) \in X \times_4 X$ and $(\delta, \eta) \in A \times_4 B$,

$$\begin{aligned}(\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]})(0,0) &= \phi_{\tilde{R}[\delta]}(0) \wedge \phi_{\tilde{S}[\eta]}(0)\\ &\leq \phi_{\tilde{R}[\delta]}(x) \wedge \phi_{\tilde{S}[\eta]}(y)\\ &= (\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]})(x, y)\end{aligned}$$

and

$$\begin{aligned}(\psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]})(0,0) &= \psi_{\tilde{R}[\delta]}(0) \vee \psi_{\tilde{S}[\eta]}(0)\\ &\geq \psi_{\tilde{R}[\delta]}(x) \vee \psi_{\tilde{S}[\eta]}(y)\\ &= (\psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]})(x, y).\end{aligned}$$

Let $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3) \in X \times_4 X$ and $(\delta, \eta) \in A \times_4 B$. Then

$$\begin{aligned}&(\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]})(x_1, y_1)\\ &= \phi_{\tilde{R}[\delta]}(x_1) \wedge \phi_{\tilde{S}[\eta]}(y_1)\\ &\leq (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{R}[\delta]}(x_2)) \wedge (\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \vee \phi_{\tilde{S}[\eta]}(y_2))\\ &= (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \vee (\phi_{\tilde{R}[\delta]}(x_2) \wedge \phi_{\tilde{S}[\eta]}(y_2))\end{aligned}$$

and

$$\begin{aligned}&(\psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]})(x_1, y_1)\\ &= \psi_{\tilde{R}[\delta]}(x_1) \vee \psi_{\tilde{S}[\eta]}(y_1)\\ &\geq (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{R}[\delta]}(x_2)) \vee (\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \wedge \psi_{\tilde{S}[\eta]}(y_2))\\ &= (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \wedge (\psi_{\tilde{R}[\delta]}(x_2) \vee \psi_{\tilde{S}[\eta]}(y_2)).\end{aligned}$$

Thus

$$\begin{aligned}&(\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]})(x_1, y_1)\\ &\leq (\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\phi_{\tilde{R}[\delta]} \times_4 \phi_{\tilde{S}[\eta]})(x_2, y_2)\end{aligned}$$

and

$$\begin{aligned}&(\psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]})(x_1, y_1)\\ &\geq (\psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \wedge (\psi_{\tilde{R}[\delta]} \times_4 \psi_{\tilde{S}[\eta]})(x_2, y_2).\end{aligned}$$

So $\tilde{R}[\delta] \times_4 \tilde{S}[\eta]$ is an $AILFBID$ of $X \times_4 X$. Hence $(\tilde{R}, A) \times_4 (\tilde{S}, B)$ is an $AILFSBID$ of $X \times_4 X$. □

Definition 4.16. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X. Then the Cartesian product

$$\tilde{R}[\delta] \times_5 \tilde{S}[\eta] = (\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]})$$

is defined by:

$$(\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]})(x, y) = \phi_{\tilde{R}[\delta]}(x) \vee \phi_{\tilde{S}[\eta]}(y)$$

and

$$(\psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]})(x, y) = \psi_{\tilde{R}[\delta]}(x) \wedge \psi_{\tilde{S}[\eta]}(y),$$

where $\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]} : X \times_5 X \to [0, 1]$ and $\psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]} : X \times_5 X \to [0, 1]$, for every $(x, y) \in X \times_5 X$ and $(\delta, \eta) \in A \times_5 B$.

Theorem 4.17. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are $AILFSBID(s)$ of X, then $(\tilde{R}, A) \times_5 (\tilde{S}, B)$ is also an $AILFSBID$ of $X \times_5 X$.*

Proof. For each $(x, y) \in X \times_5 X$ and $(\delta, \eta) \in A \times_5 B$,

$$\begin{aligned}(\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]})(0,0) &= \phi_{\tilde{R}[\delta]}(0) \vee \phi_{\tilde{S}[\eta]}(0)\\ &\leq \phi_{\tilde{R}[\delta]}(x) \vee \phi_{\tilde{S}[\eta]}(y)\\ &= (\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]})(x, y)\end{aligned}$$

and

$$\begin{aligned}(\psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]})(0,0) &= \psi_{\tilde{R}[\delta]}(0) \wedge \psi_{\tilde{S}[\eta]}(0)\\ &\geq \psi_{\tilde{R}[\delta]}(x) \wedge \psi_{\tilde{S}[\eta]}(y)\\ &= (\psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]})(x, y).\end{aligned}$$

Let $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3) \in X \times_5 X$ and $(\delta, \eta) \in A \times_5 B$. Then

$$\begin{aligned}&(\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]})(x_1, y_1)\\ &= \phi_{\tilde{R}[\delta]}(x_1) \vee \phi_{\tilde{S}[\eta]}(y_1)\\ &\leq (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{R}[\delta]}(x_2)) \vee (\phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \vee \phi_{\tilde{S}[\eta]}(y_2))\\ &= (\phi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \vee \phi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \vee (\phi_{\tilde{R}[\delta]}(x_2) \vee \phi_{\tilde{S}[\eta]}(y_2))\end{aligned}$$

and

$$\begin{aligned}&(\psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]})(x_1, y_1)\\ &= \psi_{\tilde{R}[\delta]}(x_1) \wedge \psi_{\tilde{S}[\eta]}(y_1)\\ &\geq (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{R}[\delta]}(x_2)) \wedge (\psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2) \wedge \psi_{\tilde{S}[\eta]}(y_2))\\ &= (\psi_{\tilde{R}[\delta]}((x_1 * x_3) * x_2) \wedge \psi_{\tilde{S}[\eta]}((y_1 * y_3) * y_2)) \wedge (\psi_{\tilde{R}[\delta]}(x_2) \wedge \psi_{\tilde{S}[\eta]}(y_2)).\end{aligned}$$

Thus

$$\begin{aligned}&(\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]})(x_1, y_1)\\ &\leq (\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\phi_{\tilde{R}[\delta]} \times_5 \phi_{\tilde{S}[\eta]})(x_2, y_2)\end{aligned}$$

and

$$\begin{aligned}&(\psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]})(x_1, y_1)\\ &\geq (\psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \wedge (\psi_{\tilde{R}[\delta]} \times_5 \psi_{\tilde{S}[\eta]})(x_2, y_2).\end{aligned}$$

So $\tilde{R}[\delta] \times_5 \tilde{S}[\eta]$ is an $AILFBID$ of $X \times_5 X$. Hence $(\tilde{R}, A) \times_5 (\tilde{S}, B)$ is an $AILFSBID$ of $X \times_5 X$. □

Definition 4.18. Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ be two $ILFSS(s)$ of X which $\tilde{R}[\delta] = (\phi_{\tilde{R}[\delta]}, \psi_{\tilde{R}[\delta]})$ and $\tilde{S}[\eta] = (\phi_{\tilde{S}[\eta]}, \psi_{\tilde{S}[\eta]})$ are two $ILFS(s)$ of X. Then the Cartesian product

$$\tilde{R}[\delta] \times_6 \tilde{S}[\eta] = (\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]}, \psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]})$$

is defined by:

$$(\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]})(x, y) = \frac{\phi_{\tilde{R}[\delta]}(x) + \phi_{\tilde{S}[\eta]}(y)}{2}$$

and

$$(\psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]})(x, y) = \frac{\psi_{\tilde{R}[\delta]}(x) + \psi_{\tilde{S}[\eta]}(y)}{2},$$

where $\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]} : X \times_6 X \to [0,1]$ and $\psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]} : X \times_6 X \to [0,1]$, for every $(x, y) \in X \times_6 X$ and $(\delta, \eta) \in A \times_6 B$.

Theorem 4.19. *Let $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are two $ILFSS(s)$ of X. If $(\tilde{R}, A)$ and $(\tilde{S}, B)$ are $AILFSBID(s)$ of X, then $(\tilde{R}, A) \times_6 (\tilde{S}, B)$ is also an $AILFSBID$ of $X \times_6 X$.*

Proof. For each $(x, y) \in X \times_6 X$ and $(\delta, \eta) \in A \times_6 B$,

$$\begin{aligned}(\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]})(0,0) &= \tfrac{\phi_{\tilde{R}[\delta]}(0)+\phi_{\tilde{S}[\eta]}(0)}{2} \\ &\leq \tfrac{\phi_{\tilde{R}[\delta]}(x)+\phi_{\tilde{S}[\eta]}(y)}{2} \\ &= (\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]})(x,y)\end{aligned}$$

and

$$\begin{aligned}(\psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]})(0,0) &= \tfrac{\psi_{\tilde{R}[\delta]}(0)+\psi_{\tilde{S}[\eta]}(0)}{2} \\ &\geq \tfrac{\psi_{\tilde{R}[\delta]}(x)+\psi_{\tilde{S}[\eta]}(y)}{2} \\ &= (\psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]})(x,y).\end{aligned}$$

Let $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3) \in X \times_6 X$ and $(\delta, \eta) \in A \times_6 B$. Then

$$\begin{aligned}&(\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]})(x_1,y_1) \\ &= \tfrac{\phi_{\tilde{R}[\delta]}(x_1)+\phi_{\tilde{S}[\eta]}(y_1)}{2} \\ &\leq \tfrac{[\phi_{\tilde{R}[\delta]}((x_1*x_3)*x_2)\vee\phi_{\tilde{R}[\delta]}(x_2)]+[\phi_{\tilde{S}[\eta]}((y_1*y_3)*y_2)\vee\phi_{\tilde{S}[\eta]}(y_2)]}{2} \\ &\leq \tfrac{[\phi_{\tilde{R}[\delta]}((x_1*x_3)*x_2)+\phi_{\tilde{S}[\eta]}((y_1*y_3)*y_2)]\vee[\phi_{\tilde{R}[\delta]}(x_2)+\phi_{\tilde{S}[\eta]}(y_2)]}{2}\end{aligned}$$

and

$$\begin{aligned}&(\psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]})(x_1,y_1) \\ &= \tfrac{\psi_{\tilde{R}[\delta]}(x_1)+\psi_{\tilde{S}[\eta]}(y_1)}{2} \\ &\geq \tfrac{[\psi_{\tilde{R}[\delta]}((x_1*x_3)*x_2)\wedge\psi_{\tilde{R}[\delta]}(x_2)]+[\psi_{\tilde{S}[\eta]}((y_1*y_3)*y_2)\wedge\psi_{\tilde{S}[\eta]}(y_2)]}{2} \\ &\geq \tfrac{[\psi_{\tilde{R}[\delta]}((x_1*x_3)*x_2)+\psi_{\tilde{S}[\eta]}((y_1*y_3)*y_2)]\wedge[\psi_{\tilde{R}[\delta]}(x_2)+\psi_{\tilde{S}[\eta]}(y_2)]}{2}.\end{aligned}$$

Thus

$$\begin{aligned}&(\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]})(x_1,y_1) \\ &\leq (\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \vee (\phi_{\tilde{R}[\delta]} \times_6 \phi_{\tilde{S}[\eta]})(x_2, y_2)\end{aligned}$$

and

$$\begin{aligned}&(\psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]})(x_1,y_1) \\ &\geq (\psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]})((x_1 * x_3) * x_2, (y_1 * y_3) * y_2) \wedge (\psi_{\tilde{R}[\delta]} \times_6 \psi_{\tilde{S}[\eta]})(x_2, y_2).\end{aligned}$$

So $\tilde{R}[\delta] \times_6 \tilde{S}[\eta]$ is an $AILFBID$ of $X \times_6 X$. Hence $(\tilde{R}, A) \times_6 (\tilde{S}, B)$ is an $AILFSBID$ of $X \times_6 X$. □

Conclusion

The objective of this paper is to introduce the concept of the anti-intuitionistic L-fuzzy soft b-ideals of BG-algebras. The operations of anti-intuitionistic L-fuzzy soft b-ideals are given and their elementary properties in details are discussed. Finally, the product of anti-intuitionistic L-fuzzy soft b-ideals and some related properties are studied.

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