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Fibonacci lacunary statistical convergence in intuitionistic fuzzy n-normed linear spaces

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Abstract. In the current paper, we investigate the concept of Fibonacci lacunary statistical convergence in intuitionistic fuzzy n-normed linear spaces which removes any ambiguity that could arise from the previously defined notion of the same. Some inclusion relations between the sets of Fibonacci statistically convergent and Fibonacci lacunary statistically convergent sequences are examined in an intuitionistic fuzzy n-normed linear space. We also define Fibonacci lacunary statistical Cauchy sequence in an intuitionistic fuzzy n-normed linear space and prove that it is equivalent to Fibonacci lacunary statistically convergent sequence.

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1. Introduction and Background

The concept of statistical convergence for real number sequences was first originated by Fast [13]. Later, it was further investigated from sequence point of view and linked with summability theory by Fridy [16] and Šalát [53]. Theory of lacunary statistical convergence has become an important working area after the study of Fridy and Orhan [17].

The concept of 2-normed spaces was introduced and studied by Gähler [18]. This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. [67] of USA in 1969 entitled 2-Banach spaces. In the same year Gähler [19] published another paper on this theme. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with Gähler and Gupta of 1975 also provide valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by Siddiqi [62].

Fuzziness has revolutionized many areas such as mathematics, science, engineering, medicine. This concept was given by Zadeh [68]. The concept of fuzziness are using by many researchers for Cybernetics, Artificial Intelligence, Expert System and Fuzzy control, Pattern recognition, Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, Weather forecasting.

Recently, the fuzzy logic became an important area of research in several branches of mathematics like metric and topological spaces, theory of functions etc. It attracted many researchers on sequence spaces and summability theory to introduce various types of sequence spaces and examine their different properties.

The notion of a fuzzy norm on a linear space was first originated by Katsaras [35]. Felbin [14] gave an alternative idea of a fuzzy norm whose concerned metric is of Kaleva and Seikkala [25] type.

Intuitionistic fuzzy sets was first examined by Atanassov [2]. The notion of intuitionistic fuzzy metric space has been introduced by Park [51]. Furthermore, the concept of intuitionistic fuzzy normed space is given by Saadati and Park [52]. A lot of improvement has been made in the area of intuitionistic fuzzy normed space after the studies of [7, 33, 34, 39, 45, 47, 48, 54, 55].

Recently, motivated by the theory of 2-norm and n-normed linear space [21, 22, 46, 56, 57, 58, 64] and fuzzy normed linear space [3, 4, 6, 15], the notions of fuzzy n-normed linear space [49] and intuitionistic fuzzy n-normed linear space [50] were examined. Also, we refer [8, 9, 10, 11, 40, 41, 42, 43, 60, 61, 63] for details in the area of intuitionistic fuzzy n-normed linear space.

Fibonacci sequence was initiated in the book Liber Abaci of Fibonacci which was written in 1202. However, the sequence is based on older history. The sequence had been described earlier as Virahanka numbers in Indian mathematics [20]. In Liber Abaci, the sequence starts with 1, nowadays the sequence begins either with $f_0 = 0$ or with $f_1 = 1$.

The numbers in the bottom row are called Fibonacci numbers, and the number sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

is the Fibonacci sequence [44].

The Fibonacci numbers are a sequence of numbers (f_n) for $n = 1, 2, ...$ defined by the linear recurrence equation $f_n = f_{n+1} - f_{n-2}$, $n \geq 2$. From this definition, it means that the first two numbers in Fibonacci sequence are either 1 and 1 (or 0 and 0) depending on the chosen starting point of the sequence and all subsequent numbers are the sum of the previous two.

The Fibonacci sequence was firstly used in the theory of sequence spaces by Kara and Başarır [27]. Afterward, Kara [26] defined the Fibonacci difference matrix $\widehat{F}$ by using the Fibonacci sequence (f_n) for $n \in \{1, 2, 3, ...\}$ and introduced the new sequence spaces related to the matrix domain of $\widehat{F}$.

Following [27] and [26], high quality papers have been produced on the Fibonacci matrix by many mathematicians [1, 5, 12, 23, 24, 28, 29, 30, 31, 32, 36, 65].

Kirişçi and Karaisa [37] defined Fibonacci type statistical convergence and investigated some fundamental properties. Afterward, Kirişçi [38] examined Fibonacci statistical convergence on intuitionistic fuzzy normed spaces.

In this paper, we have given a new definition of the notion of convergence of a sequence in an intuitionistic fuzzy n-normed linear spaces which is different from the one defined in [63] and other related works.This definition removes any ambiguity that could arise from previous definitions. We have developed all our results based on this new definition.

In view of the recent studies of convergence in intuitionistic fuzzy n-normed, it looks like very natural to extend the interesting concept of Fibonacci lacunary statistical convergence in intuitionistic fuzzy n-normed further by using Fibonacci sequence which we mainly do here.

The aim of the present paper is to introduce and investigate the Fibonacci lacunary statistical convergence and Fibonacci lacunary statistical Cauchy sequence on intuitionistic fuzzy n-normed linear spaces and obtain some important results on them.

We recall some useful definitions and results.

Definition 1.1 ([59])**.** A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm, if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a*1=a$, for all $a\in[0,1]$,
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d\in[0,1]$.

Definition 1.2 ([59])**.** A binary operation $\diamond:[0,1]\times[0,1]\to[0,1]$ is a continuous t-co-norm, if $\diamond$ satisfies the following conditions:

(i) $\diamond$ is commutative and associative,
(ii) $\diamond$ is continuous,
(iii) $a\diamond 0=a$ for all $a\in[0,1]$,
(iv) $a\diamond b\le c\diamond d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d\in[0,1]$.

Definition 1.3 ([66])**.** An intuitionistic fuzzy n-normed linear space or in short i-f-n-NLS is an object of the form

$$\mathcal{A}=\{(\mathcal{X},\phi(\mathbf{x},t),\omega(\mathbf{x},t)):\mathbf{x}=(x_1,x_2,...,x_n)\in\mathcal{X}^n\},$$

where $\mathcal{X}$ is a linear space over a field $\mathbb{F}$ and ϕ,ω are fuzzy sets on $\mathcal{X}^n\times(0,\infty)$, ϕ denotes the degree of membership and ω denotes the degree of non-membership of $(x_1,x_2,...,x_n,t)\in\mathcal{X}^n\times(0,\infty)$ satisfying the following conditions:

(i) $\phi(\mathbf{x},t)+\omega(\mathbf{x},t)\le 1$,
(ii) $\phi(\mathbf{x},t)>0$,
(iii) $\phi(x_1,x_2,...,x_n,t)=1$ if and only if $x_1,x_2,...,x_n$ are linearly dependent,
(iv) $\phi(x_1,x_2,...,x_n,t)$ is invariant under any permutation of $x_1,x_2,...,x_n$,
(v) $\phi(x_1,x_2,...,cx_n,t)=\phi\left(x_1,x_2,...,x_n,\frac{t}{|c|}\right)$ if $c\neq 0$, $c\in\mathbb{F}$,
(vi) $\phi(x_1,x_2,...,x_n,s)*\phi(x_1,x_2,...,x_n',t)\le\phi(x_1,x_2,...,x_n+x_n',s+t)$,
(vii) $\phi(\mathbf{x},\circ):(0,\infty)\to[0,1]$ is continuous in t,
(viii) $\omega(\mathbf{x},t)>0$,

(ix) $\omega\left(x_1, x_2, ..., x_n, t\right) = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(x) $\omega\left(x_1, x_2, ..., x_n, t\right)$ is invariant under any permutation of $x_1, x_2, ..., x_n$,
(xi) $\omega\left(x_1, x_2, ..., cx_n, t\right) = \omega\left(x_1, x_2, ..., x_n, \frac{t}{|c|}\right)$ if $c \neq 0$, $c \in \mathbb{F}$,
(xii) $\omega\left(x_1, x_2, ..., x_n, s\right) * \omega\left(x_1, x_2, ..., x'_n, t\right) \geq \omega\left(x_1, x_2, ..., x_n + x'_n, s + t\right)$,
(xiii) $\omega\left(\mathbf{x}, \circ\right) : (0, \infty) \to [0, 1]$ is continuous in t.

Corollary 1.4. *For convenience we denote the intuitionistic fuzzy n-normed linear space by* $\mathcal{A} = (\mathcal{X}, \phi, \omega, *, \diamond)$.

Definition 1.5 ([50])**.** A sequence $\{x_{n_k}\}$ in an i-f-n-NLS $\mathcal{A}$ is said to convergence to $\xi \in \mathcal{X}$ with respect to the intuitionistic fuzzy n-norm (ϕ, ω), if for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer n_0 such that $\phi\left(x_1, x_2, ..., x_{n-1}, x_{n_k} - \xi, t\right) > 1-\varepsilon$ and $\omega\left(x_1, x_2, ..., x_{n-1}, x_{n_k} - \xi, t\right) < \varepsilon$ for all $k \geq n_0$.

The element ξ is called the limit of the sequence $\{x_{n_k}\}$ with respect to the intuitionistic fuzzy n-norm (ϕ, ω) and is denoted as $(\phi, \omega) - \lim x_{n_k} = \xi$.

Definition 1.6 ([50])**.** A sequence $\{x_{n_k}\}$ in an i-f-n-NLS $\mathcal{A}$ is said to be Cauchy with respect to the intuitionistic fuzzy n-norm (ϕ, ω), if for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer m_0 such that $\phi\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - x_{n_q}, t\right) > 1 - \varepsilon$ and $\omega\left(x_1, x_2, ..., x_{n-1}, x_{n_p} - x_{n_q}, t\right) < \varepsilon$ for all $p, q \geq m_0$.

Definition 1.7 ([17])**.** A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 1.8 ([17])**.** For a lacunary sequence $\theta = \{k_r\}$ the sequence (x_k) is said to be lacunary statistically convergent to ξ, provided that for each $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \{k \in I_r : |x_k - \xi| \geq \varepsilon\} \right| = 0.$$

In this case, we write $S_\theta - \lim x_k = \xi$.

Definition 1.9 ([38])**.** Take an IFNS $(\mathcal{X}, \phi, \omega, *, \diamond)$. A sequence (x_k) is said to be Fibonacci statistical convergence with respect to IFN (ϕ, ω), if there is a number $\xi \in \mathcal{X}$ such that for every $\varepsilon > 0$ and $t > 0$, the set

$$K_\varepsilon(\widehat{F}) := \left\{k \leq n : \phi\left(\widehat{F}x_k - \xi, t\right) \leq 1 - \varepsilon \text{ or } \omega\left(\widehat{F}x_k - \xi, t\right) \geq \varepsilon\right\}$$

has natural density zero, i.e., $d(K_\varepsilon(\widehat{F})) = 0$. That is,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : \phi\left(\widehat{F}x_k - \xi, t\right) \leq 1 - \varepsilon \text{ or } \omega\left(\widehat{F}x_k - \xi, t\right) \geq \varepsilon\right\}\right| = 0.$$

In this case, we write $d(\widehat{F})_{IFN} - \lim x_k = \xi$ or $x_k \to \xi(S(\widehat{F})_{IFN})$.

2. Main results

Definition 2.1. Let $\mathcal{A}$ be an i-f-n-NLS. We define an open ball $B(x,r,t)(\widehat{F})$ with center x on the n^{th} coordinate of $\mathcal{X}^n$ and radius $0<r<1$, as

$$B(x,r,t)(\widehat{F}) = \left\{ \begin{array}{c} y\in\mathcal{X} : \phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x-y),t\right) > 1-r \text{ and} \\ \omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x-y),t\right) < r \end{array} \right\}$$

for $t>0$.

Definition 2.2. Let $\mathcal{A}$ be an i-f-n-NLS. A sequence $\{x_{n_k}\}$ of elements in $\mathcal{X}$ is said to be Fibonacci statistically convergent to $\xi\in\mathcal{X}$ with regards to the i-f-n-norm (ϕ,ω), if for every $\varepsilon>0$ and $t>0$, there exists $p\in\mathbb{N}$ such that

$$\lim_{p\to\infty}\frac{1}{p}\left|\left\{k\le p : \phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi,t\right)\le 1-\varepsilon \text{ and } \omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi,t\right)\ge\varepsilon\right\}\right| = 0.$$

The element ξ is called the statistical limit of the sequence $\{x_{n_k}\}$ with regards to the intuitionistic fuzzy n-norm (ϕ,ω) and is denoted as $\widehat{F}St_{(\phi,\omega)}-\lim x_{n_k}=\xi$ or $x_{n_k}\to\xi\left(\widehat{F}St_{(\phi,\omega)}\right)$.

Definition 2.3. Let $\mathcal{A}$ be an i-f-n-NLS and θ be a lacunary sequence. A sequence $\{x_{n_k}\}$ of elements in $\mathcal{X}$ is said to be Fibonacci lacunary statistically convergent to $\xi\in\mathcal{X}$ with regards to the i-f-n-norm (ϕ,ω), if for every $\varepsilon>0$ and $t>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r : \phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi,t\right)\le 1-\varepsilon \text{ and } \omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi,t\right)\ge\varepsilon\right\}\right| = 0.$$

The element ξ is called the Fibonacci lacunary statistical limit of the sequence $\{x_{n_k}\}$ with regards to the intuitionistic fuzzy n-norm (ϕ,ω) and is denoted as $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi$ or $x_{n_k}\to\xi\left(\widehat{F}S^{\theta}_{(\phi,\omega)}\right)$.

We denote by $\widehat{F}S^{\theta}_{(\phi,\omega)}(\mathcal{X})$, the set of all Fibonacci lacunary statistically convergent sequences in i-f-n-NLS $\mathcal{A}$.

Next we show that for any fixed θ, $\widehat{F}S^{\theta}_{(\phi,\omega)}$-limit is unique provided it exists.

Theorem 2.4. *Let $\mathcal{A}$ be an i-f-n-NLS and θ be a lacunary sequence. If $\{x_{n_k}\}$ is a sequence in $\mathcal{X}$ such that $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi$ exists, then it is unique.*

Proof. Assume that there exist two distinct elements $\xi_1,\xi_2\in X$ such that $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi_1$ and $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi_2$. Let $\varepsilon>0$ be arbitrary. Choose $s>0$ such that

$$(1-s)*(1-s)>1-\varepsilon \text{ and } s\diamondsuit s<\varepsilon. \tag{2.1}$$

For any $t>0$, we take

$$K=\left\{k\in I_r : \phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi_1,t\right)>1-s \text{ and } \omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi_1,t\right)<s\right\}$$

and

$$L=\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_2,t\right)>1-s\text{ and }\\ \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_2,t\right)<s\Big\}.$$

We shall first show that for $\xi_1\neq\xi_2$ and $t>0$, $K\cap L=\varnothing$. For, if $p\in K\cap L$, then

$$\begin{aligned}&\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\xi_1-\xi_2,t\right)\\ &\geq\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi_1,\tfrac{t}{2}\right)*\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi_2,\tfrac{t}{2}\right)\\ &>(1-s)*(1-s)>1-\varepsilon,\end{aligned}$$

by (2.1). Since $\varepsilon>0$ is arbitrary, we have $\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\xi_1-\xi_2,t\right)=1$ for every $t>0$. Similarly, $\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\xi_1-\xi_2,t\right)=0$ for every $t>0$. This implies that $\xi_1-\xi_2=0$, a contradiction to $\xi_1\neq\xi_2$. Thus $K\cap L=\varnothing$. So $K\subset L^c$. Hence, we have

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_1,t\right)>1-s\text{ and }\\ &\qquad\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_1,t\right)<s\Big\}\Big|\\ &\leq\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_2,t\right)\leq 1-s\text{ and }\\ &\qquad\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_2,t\right)\geq s\Big\}\Big|.\end{aligned}$$

Since $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi_2$, it follows that

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_1,t\right)>1-s\text{ and }\\ \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_1,t\right)<s\Big\}\Big|\leq 0.$$

Since this can not be negative, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_1,t\right)>1-s\text{ and }\\ \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi_1,t\right)<s\Big\}\Big|=0.$$

This contradicts the fact that $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi_1$. Therefore $\xi_1=\xi_2$. □

Theorem 2.5. *$\widehat{F}S^{\theta}_{(\phi,\omega)}$ is a linear space.*

Proof. Let $\{x_{n_k}\}$ be a sequence $\mathcal{X}$.

(i) If $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi$ and $\alpha\neq 0\in\mathbb{R}$, then we need to prove that

$$\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim\alpha x_{n_k}=\alpha\xi.$$

Let $\xi>0$ and $t>0$. If we take

$$K=\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)>1-\varepsilon\text{ and }\\ \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)<\varepsilon\Big\}$$

and

$$L=\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(\alpha x_{n_k}\right)-\alpha\xi,t\right)>1-\varepsilon \text{ and } \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(\alpha x_{n_k}\right)-\alpha\xi,t\right)<\varepsilon\Big\}$$

Let $p\in K$. Then, we have

$$\begin{aligned}
&\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(\alpha x_p\right)-\alpha\xi,t\right)\\
&=\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,\tfrac{t}{|a|}\right)\\
&\geq\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,t\right)*\phi\left(0,\tfrac{t}{|a|}-t\right)\\
&\geq\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,t\right)*1\\
&\geq\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,t\right)>1-\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(\alpha x_p\right)-\alpha\xi,t\right)\\
&=\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,\tfrac{t}{|a|}\right)\\
&\leq\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,t\right)\diamond\omega\left(0,\tfrac{t}{|a|}-t\right)\\
&\leq\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,t\right)\diamond 0\\
&\leq\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,t\right)<\varepsilon.
\end{aligned}$$

Thus $p\in L$. So we have $K\subset L$. Hence $L^c\subset K^c$. It follows that

$$\begin{aligned}
&\lim_{r\to\infty}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(\alpha x_{n_k}\right)-\alpha\xi,t\right)\leq 1-\varepsilon \text{ and } \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(\alpha x_{n_k}\right)-\alpha\xi,t\right)\geq\varepsilon\Big\}\Big|\\
&\leq\lim_{r\to\infty}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\leq 1-\varepsilon \text{ and } \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\geq\varepsilon\Big\}\Big|.
\end{aligned}$$

Since $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi$, it follows that $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim \alpha x_{n_k}=\alpha\xi$.

(ii) Let $\{x_{n_k}\}$ and $\{y_{n_k}\}$ be two sequences in $\mathcal{X}$. If $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi$ and $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim y_{n_k}=\eta$, then we have to prove that $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim\left(x_{n_k}+y_{n_k}\right)=\xi+\eta$. Let $\varepsilon>0$ be given. Choose $s>0$ as in (2.1). For $t>0$, we define the following sets:

$$A=\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}+y_{n_k}\right)-\left(\xi+\eta\right),t\right)>1-\varepsilon \text{ and } \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}+y_{n_k}\right)-\left(\xi+\eta\right),t\right)<\varepsilon\Big\},$$

$$B=\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)>1-s \text{ and } \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)<s\Big\},$$

$$C=\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(y_{n_k}\right)-\eta,t\right)>1-s \text{ and } \omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(y_{n_k}\right)-\eta,t\right)<s\Big\}.$$

Let $p\in B\cap C$. Then by (2.1), we have

$$\begin{aligned}
&\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p+y_p\right)-(\xi+\eta),t\right)\\
&=\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi+\widehat{F}\left(y_p\right)-\eta,\tfrac{t}{2}+\tfrac{t}{2}\right)\\
&\geq\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,\tfrac{t}{2}\right)*\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(y_p\right)-\eta,\tfrac{t}{2}\right)\\
&>(1-s)*(1-s)>1-\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p+y_p\right)-(\xi+\eta),t\right)\\
&\leq\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_p\right)-\xi,\tfrac{t}{2}\right)\diamond\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(y_p\right)-\eta,\tfrac{t}{2}\right)\\
&<s\diamond s<\varepsilon.
\end{aligned}$$

Thus $p\in A$. So $(B\cap C)\subset A$. Hence we have $A^c\subset(B^c\cup C^c)$. It follows that

$$\begin{aligned}
&\lim_{r\to\infty}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}+y_{n_k}\right)-(\xi+\eta),t\right)\leq 1-\varepsilon\text{ and}\\
&\qquad\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}+y_{n_k}\right)-(\xi+\eta),t\right)\geq\varepsilon\Big\}\Big|\\
&\leq\lim_{r\to\infty}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\leq 1-s\text{ and}\\
&\qquad\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\geq s\Big\}\Big|\\
&+\lim_{r\to\infty}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(y_{n_k}\right)-\eta,t\right)\leq 1-s\text{ and}\\
&\qquad\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(y_{n_k}\right)-\eta,t\right)\geq s\Big\}\Big|.
\end{aligned}$$

Since $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi$ and $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim y_{n_k}=\eta$, we have

$$\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim\left(x_{n_k}+y_{n_k}\right)=\xi+\eta.$$

This completes the proof. □

Theorem 2.6. *Let $\mathcal{A}$ be an i-f-n-NLS. For any lacunary sequence θ, $\widehat{F}St_{(\phi,\omega)}(\mathcal{X})\subset\widehat{F}S^{\theta}_{(\phi,\omega)}(\mathcal{X})$ iff $\liminf_r q_r>1$.*

Proof. Sufficient part: Assume that $\liminf_r q_r>1$. Then there exists a $\delta>0$ such that $q_r\geq 1+\delta$ for sufficiently large r which implies that $\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta}$. If $\{x_{n_k}\}$ is Fibonacci statistically convergent to ξ with regards to i-f-n-norm (ϕ,ω), then for each $\varepsilon>0$, $t>0$ and sufficiently large r, we have

$$\begin{aligned}
&\tfrac{\delta}{1+\delta}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\leq 1-\varepsilon\text{ and}\\
&\qquad\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\geq\varepsilon\Big\}\Big|\\
&\leq\tfrac{1}{k_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\leq 1-\varepsilon\text{ and}\\
&\qquad\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\geq\varepsilon\Big\}\Big|\\
&+\tfrac{1}{k_r}\Big|\Big\{k\leq k_r:\phi\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\leq 1-\varepsilon\text{ and}\\
&\qquad\omega\left(\widehat{F}\left(x_1,x_2,...,x_{n-1}\right),\widehat{F}\left(x_{n_k}\right)-\xi,t\right)\geq\varepsilon\Big\}\Big|.
\end{aligned}$$

Thus $x_{n_k}\to\xi\left(\widehat{F}S^{\theta}_{(\phi,\omega)}\right)$. So $\widehat{F}St_{(\phi,\omega)}(\mathcal{X})\subset\widehat{F}S^{\theta}_{(\phi,\omega)}(\mathcal{X})$.

Necessary part: Suppose that $\liminf_r q_r=1$. Then we can select a subsequence $\left\{k_{r(j)}\right\}$ of the lacunary sequence θ such that $\frac{k_{r(j)}}{k_{r(j)-1}}<1+\frac{1}{j}$ and $\frac{k_{r(j)-1}}{k_{r(j-1)}}>j$, where

$r(j) \geq r(j-1)+2$. Let $\xi \neq 0 \in \mathcal{X}$. We define a sequence $\{x_{n_k}\}$ as follows:

$$x_{n_k} = \begin{cases} \xi, & \text{if } k \in I_{r(j)} \text{ for some } j = 1,2,... \\ 0, & \text{otherwise.} \end{cases}$$

We shall show that $\{x_{n_k}\}$ is Fibonacci statistically convergent to ξ with regards to the i-f-n-norm (ϕ, ω). Let $\varepsilon > 0$, $t > 0$. Choose $\varepsilon_1 \in (0,1)$ such that $B(0, \varepsilon_1, t)(\widehat{F}) \subset B(0, \varepsilon, t)(\widehat{F})$ and $\xi \notin B(0, \varepsilon, t)(\widehat{F})$. Also for each p we can find a positive number j_p such that $k_{r(j_p)} < p \leq k_{r(j_p)+1}$. Then, we have

$$\begin{aligned}
\lim_{p\to\infty} \tfrac{1}{p} \Big| \Big\{ k \leq p : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1-\varepsilon \text{ and } \\
\omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon \Big\} \Big| \\
\leq \tfrac{1}{k_{r(j_p)}} \Big| \Big\{ k \leq p : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1-\varepsilon_1 \text{ and } \\
\omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon_1 \Big\} \Big| \\
\leq \tfrac{1}{k_{r(j_p)}} \Big[\Big| \Big\{ k \leq k_{r(j_p)} : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1-\varepsilon_1 \text{ and } \\
\omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon_1 \Big\} \Big| \\
+ \Big| \Big\{ k_{r(j_p)} < k \leq p : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1-\varepsilon_1 \text{ and } \\
\omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon_1 \Big\} \Big| \Big] \\
\leq \tfrac{1}{k_{r(j_p)}} \Big| \Big\{ k \leq k_{r(j_p)} : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1-\varepsilon_1 \text{ and } \\
\omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon_1 \Big\} \Big| \\
+ \tfrac{1}{k_{r(j_p)}} \left(k_{r(j_p)+1} - k_{r(j_p)}\right) \\
< \tfrac{1}{j_p} + \tfrac{1}{j_p+1} + 1 - 1 = \tfrac{1}{j_p} + \tfrac{1}{j_p+1}
\end{aligned}$$

for each p. It follows that $\widehat{F}St_{(\phi,\omega)} - \lim x_{n_k} \to \xi$. Next we shall show that $\{x_{n_k}\}$ is not Fibonacci lacunary statistically convergent with regards to the i-f-n-norm (ϕ, ω). Since $\xi \neq 0$ we choose $\varepsilon > 0$ such that $\xi \notin B(0, \varepsilon, t)(\widehat{F})$ for $t > 0$. Thus

$$\begin{aligned}
\lim_{j\to\infty} \tfrac{1}{h_{r(j)}} \Big| \Big\{ k_{r(j)-1} < k \leq k_{r(j)} : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - 0, t\right) \leq 1-\varepsilon \text{ and } \\
\omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - 0, t\right) \geq \varepsilon \Big\} \Big| \\
= \lim_{j\to\infty} \tfrac{1}{h_{r(j)}} \left(k_{r(j)} - k_{r(j-1)}\right) = \lim_{j\to\infty} \tfrac{1}{h_{r(j)}} \left(h_{r(j)}\right) = 1
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\substack{r\to\infty \\ r\neq r(j), j=1,2,..}} \tfrac{1}{h_r} \Big| \Big\{ k_{r-1} < k \leq k_r : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1-\varepsilon \text{ and } \\
\omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon \Big\} \Big| = 1 \neq 0.
\end{aligned}$$

So neither ξ nor 0 can be Fibonacci lacunary statistical limit of the sequence $\{x_{n_k}\}$ with regards to the i-f-n-norm (ϕ, ω). No other point of $\mathcal{X}$ can be Fibonacci lacunary statistical limit of the sequence as well. Hence $\{x_{n_k}\} \notin \widehat{F}S^{\theta}_{(\phi,\omega)}(\mathcal{X})$ completing the proof. □

The following example establishes that Fibonacci lacunary statistical convergence need not imply Fibonacci statistical convergence.

Example 2.7. Let $(\mathcal{X}, \|\centerdot, ..., \centerdot\|)$ be a n-normed linear space, where $\mathcal{X} = \mathbb{R}$. Define $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$,

$$\phi\left(\widehat{F}\left(x_1, x_2, ..., x_{n_k}\right), t\right) = \frac{t}{t + \left\|\widehat{F}\left(x_1, x_2, ..., x_{n_k}\right)\right\|}$$

and

$$\omega\left(\widehat{F}\left(x_1, x_2, ..., x_{n_k}\right), t\right) = \frac{\left\|\widehat{F}\left(x_1, x_2, ..., x_{n_k}\right)\right\|}{t + \left\|\widehat{F}\left(x_1, x_2, ..., x_{n_k}\right)\right\|}.$$

Then $\mathcal{A} = (\mathcal{X}, \phi, \omega, *, \diamond)$ is an i-f-n-NLS. We define a sequence $\{x_{n_k}\}$ by

$$x_{n_k} = \begin{cases} nk, & \text{for } k_r - \left(\sqrt{h_r}\right) + 1 \leq k \leq k_r,\ r \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

For $\varepsilon > 0$ and $t > 0$,

$$\begin{aligned} K_r(\varepsilon, t) = \Big\{k \in \mathbb{N} : \phi\left(\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right), t\right) &\leq 1 - \varepsilon \text{ and} \\ \omega\left(\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right), t\right) &\geq \varepsilon\Big\}. \end{aligned}$$

Thus

$$\begin{aligned} K_r(\varepsilon, t) = \ & \Big\{k \in \mathbb{N} : \tfrac{t}{t + \left\|\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right)\right\|} \leq 1 - \varepsilon \text{ and} \\ & \qquad \tfrac{\left\|\widehat{F}\left(x_1, x_2, ..., x_{n_k}\right)\right\|}{t + \left\|\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right)\right\|} \geq \varepsilon\Big\} \\ = \ & \Big\{k \in \mathbb{N} : \left\|\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right)\right\| \geq \tfrac{\varepsilon t}{1-\varepsilon} > 0\Big\} \\ = \ & \{k \in \mathbb{N} : x_{n_k} = nk\} \\ = \ & \Big\{k \in \mathbb{N} : k_r - \left(\sqrt{h_r}\right) + 1 \leq k \leq k_r,\ r \in \mathbb{N}\Big\}. \end{aligned}$$

So we get

$$\begin{aligned} & \tfrac{1}{h_r}\left|K_r(\varepsilon, t)\right| = \tfrac{1}{h_r}\left|\Big\{k \in \mathbb{N} : k_r - \left(\sqrt{h_r}\right) + 1 \leq k \leq k_r,\ r \in \mathbb{N}\Big\}\right| \leq \tfrac{\sqrt{h_r}}{h_r} \\ & \Rightarrow \lim_{r\to\infty} \tfrac{1}{h_r}\left|K_r(\varepsilon, t)\right| = 0 \\ & \Rightarrow x_{n_k} \to 0\left(\widehat{F}S^{\theta}_{(\phi,\omega)}\right). \end{aligned}$$

On the other hand, $x_{n_k} \nrightarrow 0\left(\widehat{F}St_{(\phi,\omega)}\right)$, since

$$\begin{aligned} \phi\left(\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right), t\right) &= \tfrac{t}{t + \left\|\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right)\right\|} \\ &= \begin{cases} \frac{t}{t+nk}, & \text{for } k_r - \left(\sqrt{h_r}\right) + 1 \leq k \leq k_r,\ r \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \\ &\leq 1 \end{aligned}$$

and

$$\begin{aligned} \omega\left(\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right), t\right) &= \tfrac{\left\|\widehat{F}\left(x_1, x_2, ..., x_{n_k}\right)\right\|}{t + \left\|\widehat{F}\left(x_1, x_2, ..., x_{n-1}, x_{n_k}\right)\right\|} \\ &= \begin{cases} \frac{nk}{t+nk}, & \text{for } k_r - \left(\sqrt{h_r}\right) + 1 \leq k \leq k_r,\ r \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \\ &\geq 0. \end{aligned}$$

Hence, we have $x_{n_k} \nrightarrow 0 \left(\widehat{F}St_{(\phi,\omega)}\right)$.

Theorem 2.8. *Let $\mathcal{A}$ be an i-f-n-NLS. For any lacunary sequence θ, $\widehat{F}S^{\theta}_{(\phi,\omega)}(\mathcal{X}) \subset \widehat{F}St_{(\phi,\omega)}(\mathcal{X})$ iff $\limsup_r q_r < \infty$.*

Proof. Sufficient part: If $\limsup_r q_r < \infty$, then there is a $H > 0$ such that $q_r < H$ for all r. Assume that $x_{n_k} \to \xi \left(\widehat{F}S^{\theta}_{(\phi,\omega)}\right)$, and let

$$K_r = \left|\left\{k \in I_r : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1 - \varepsilon \text{ and } \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon\right\}\right|.$$

By definition of a Fibonacci lacunary statistical convergent sequence, there is a positive number r_0 such that

$$\frac{K_r}{h_r} < \varepsilon \text{ for all } r > r_0. \tag{2.2}$$

Now let $P = \max\{K_r : 1 \leq r \leq r_0\}$ and p be any integer satisfying $k_{r-1} < p \leq k_r$. Then we have

$$\begin{aligned}
&\frac{1}{p}\left|\left\{k \leq p : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1 - \varepsilon \text{ and } \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1 - \varepsilon \text{ and } \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{k_{r-1}}\{K_1 + K_2 + ... + K_{r_0} + K_{r_0+1} + ... + K_r\} \\
&\leq \frac{P}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\frac{K_{r_0+1}}{h_{r_0+1}} + ... + h_r\frac{K_r}{h_r}\right\} \\
&\leq \frac{r_0 P}{k_{r-1}} + \frac{1}{k_{r-1}}\left(\sup_{r>r_0}\frac{K_r}{h_r}\right)\{h_{r_0+1} + ... + h_r\} \\
&\leq \frac{r_0 P}{k_{r-1}} + \varepsilon\frac{k_r - k_{r_0}}{k_{r-1}} \quad \text{(by 2,2)} \\
&\leq \frac{r_0 P}{k_{r-1}} + \varepsilon q_r \leq \frac{r_0 P}{k_{r-1}} + \varepsilon H.
\end{aligned}$$

Thus $\{x_{n_k}\}$ is Fibonacci statistically convergent. So $\widehat{F}S^{\theta}_{(\phi,\omega)}(\mathcal{X}) \subset \widehat{F}St_{(\phi,\omega)}(\mathcal{X})$.

Necessary part: Suppose that $\limsup_r q_r = \infty$. Let $\xi \neq 0 \in \mathcal{X}$. Choose a subsequence $\left\{k_{r(j)}\right\}$ of the lacunary sequence $\theta = \{k_r\}$ such that $q_{r(j)} > j$, $k_{r(j)} > j + 3$. Define a sequence $\{x_{n_k}\}$ as follows:

$$x_{n_k} = \begin{cases} \xi, & \text{if } k_{r(j)-1} < k \leq 2k_{r(j)-1} \text{ for some } j = 1, 2, ... \\ 0, & \text{otherwise.} \end{cases}$$

Since $\xi \neq 0$, we can select $\varepsilon > 0$ such that $\xi \notin B(0, \varepsilon, t)$ for $t > 0$. Now for $j > 1$,

$$\begin{aligned}\frac{1}{h_{r(j)}} \Big| \Big\{ k \le k_{r(j)} : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - 0, t\right) \le 1 - \varepsilon \text{ and} \\ \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - 0, t\right) \ge \varepsilon \Big\} \Big| \\ < \frac{1}{h_{r(j)}}\left(k_{r(j)-1}\right) < \tfrac{1}{\left(k_{r(j)} - k_{r(j)-1}\right)}\left(k_{r(j)-1}\right) < \tfrac{1}{j-1}.\end{aligned}$$

Then we have $\{x_{n_k}\} \in \widehat{F}S^{\theta}_{(\phi,\omega)}(\mathcal{X})$. But $\{x_{n_k}\} \notin \widehat{F}St_{(\phi,\omega)}(\mathcal{X})$. For

$$\begin{aligned}\frac{1}{2k_{r(j)-1}} \Big| \Big\{ k \le 2k_{r(j)-1} : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - 0, t\right) \le 1 - \varepsilon \text{ and} \\ \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - 0, t\right) \ge \varepsilon \Big\} \Big| \\ < \frac{1}{2k_{r(j)-1}}\left(k_{r(1)-1} + k_{r(2)-1} + ... + k_{r(j)-1}\right) > \frac{1}{2}.\end{aligned}$$

This shows that $\{x_{n_k}\}$ cannot be Fibonacci statistically convergent with regards to the intuitionistic fuzzy n-norm (ϕ, ω). □

Theorem 2.6 and Theorem 2.8 immediately give the following Corollary.

Corollary 2.9. *Let $\mathcal{A}$ be an i-f-n-NLS. For any lacunary sequence θ, $\widehat{F}S^{\theta}_{(\phi,\omega)}(\mathcal{X}) = \widehat{F}St_{(\phi,\omega)}(\mathcal{X})$ iff $1 < \liminf_r q_r \le \limsup_r q_r < \infty$.*

Finally, we define Fibonacci lacunary statistical Cauchy sequence in an intuitionistic fuzzy n-normed linear space and prove that it is equivalent to Fibonacci lacunary statistically convergent sequence in an intuitionistic fuzzy n-normed linear space.

Definition 2.10. Let $\mathcal{A}$ be an i-f-n-NLS and θ be a lacunary sequence. A sequence $\{x_{n_k}\}$ in $\mathcal{X}$ is said to be Fibonacci lacunary-θ-statistically Cauchy provided there is subsequence $\left\{x_{n_{k'(r)}}\right\}$ of the sequence $\{x_{n_k}\}$ such that $k'(r) \in I_r$ for each r, $(\phi, \omega) - \lim_{r\to\infty} x_{n_{k'(r)}} = \xi$ and for each $\varepsilon > 0$ and $t > 0$,

$$\begin{aligned}\lim_{r\to\infty} \tfrac{1}{h_r} \Big| \Big\{ k \in I_r : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \widehat{F}\left(x_{n_{k'(r)}}\right), t\right) \le 1 - \varepsilon \text{ and} \\ \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \widehat{F}\left(x_{n_{k'(r)}}\right), t\right) \ge \varepsilon \Big\} \Big| = 0.\end{aligned}$$

Theorem 2.11. *Let $\mathcal{A}$ be an i-f-n-NLS and θ be a lacunary sequence. A sequence $\{x_{n_k}\}$ in $\mathcal{X}$ is Fibonacci lacunary statistically convergent iff it is Fibonacci lacunary-θ-statistically Cauchy.*

Proof. We first suppose that $\widehat{F}S^{\theta}_{(\phi,\omega)} - \lim x_{n_k} = \xi$. For $t > 0$ and $j \in \mathbb{N}$, let

$$\begin{aligned}K(j,t) = \Big\{ k \in \mathbb{N} : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) > 1 - \tfrac{1}{j} \text{ and} \\ \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) < \tfrac{1}{j} \Big\}.\end{aligned}$$

Then we have the following:

(i) $K(j+1, t) \subset K(j, t)$ and

(ii) $\dfrac{|K(j,t) \cap I_r|}{h_r} \to 1$ as $r \to \infty$.

Since this implies that we can select a positive integer $m(1)$ such that for $r \ge m(1)$, we have $\dfrac{|K(1,t)\cap I_r|}{h_r}>0$, i.e., $K(1,t)\cap I_r\neq\varnothing$. Next we can select $m(2)\geq m(1)$ so that $r\geq m(2)$ implies $K(2,t)\cap I_r\neq\varnothing$. Then for each r satisfying $m(1)\leq r\leq m(2)$, choose $k'(r)\in I_r$ such that $k'(r)\in I_r\cap K(1,t)$, i.e.,

$$\phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}\left(x_{n_{k'(r)}}\right)-\xi,t\right)>0$$

and

$$\omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}\left(x_{n_{k'(r)}}\right)-\xi,t\right)<1.$$

In general, we can choose $m(p+1)>m(p)$ such that

$$r>m(p+1)\ \text{ implies } I_r\cap K(p+1,t)\neq\varnothing.$$

Thus for all r satisfying $m(p)\leq r\leq m(p+1)$, choose $k'(r)\in I_r\cap K(p,t)$, i.e.,

$$\begin{array}{c}\phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}\left(x_{n_{k'(r)}}\right)-\xi,t\right)>1-\frac{1}{p}\text{ and}\\ \omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}\left(x_{n_{k'(r)}}\right)-\xi,t\right)<\frac{1}{p}.\end{array}\tag{2.3}$$

So $k'(r)\in I_r$ for each r together with (2.3) implies that $(\phi,\omega)-\lim_{r\to\infty}x_{n_{k'(r)}}=\xi$. For $\varepsilon>0$, choose $s>0$ such that $(1-s)*(1-s)>1-\varepsilon$ and $s\Diamond s<\varepsilon$. For $t>0$, if we take

$$\begin{aligned}A=\Big\{k\in I_r:\ &\phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\widehat{F}\left(x_{n_{k'(r)}}\right),t\right)>1-\varepsilon\text{ and}\\ &\omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\widehat{F}\left(x_{n_{k'(r)}}\right),t\right)<\varepsilon\Big\},\end{aligned}$$

$$\begin{aligned}B=\Big\{k\in I_r:\ &\phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi,t\right)>1-s\text{ and}\\ &\omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi,t\right)<s\Big\},\end{aligned}$$

$$\begin{aligned}C=\Big\{k\in I_r:\ &\phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}\left(x_{n_{k'(r)}}\right)-\xi,t\right)>1-s\text{ and}\\ &\omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}\left(x_{n_{k'(r)}}\right)-\xi,t\right)<s,\Big\},\end{aligned}$$

then we find that $(B\cap C)\subset A$ and therefore $A^c\subset(B^c\cup C^c)$. Thus we get

$$\begin{aligned}&\lim_{r\to\infty}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\widehat{F}\left(x_{n_{k'(r)}}\right),t\right)\leq1-\varepsilon\text{ and}\\ &\qquad\omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\widehat{F}\left(x_{n_{k'(r)}}\right),t\right)\geq\varepsilon\Big\}\Big|\\ &\leq\lim_{r\to\infty}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi,t\right)\leq1-s\text{ and}\\ &\qquad\omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}(x_{n_k})-\xi,t\right)\geq s\Big\}\Big|\\ &+\lim_{r\to\infty}\tfrac{1}{h_r}\Big|\Big\{k\in I_r:\phi\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}\left(x_{n_{k'(r)}}\right)-\xi,t\right)\leq1-s\text{ and}\\ &\qquad\omega\left(\widehat{F}(x_1,x_2,...,x_{n-1}),\widehat{F}\left(x_{n_{k'(r)}}\right)-\xi,t\right)\geq s\Big\}\Big|.\end{aligned}$$

Since $\widehat{F}S^{\theta}_{(\phi,\omega)}-\lim x_{n_k}=\xi$ and $(\phi,\omega)-\lim_{r\to\infty}x_{n_{k'(r)}}=\xi$, it follows that $\{x_{n_k}\}$ is $\widehat{F}S^{\theta}_{(\phi,\omega)}$-Cauchy.

Conversely, assume that $\{x_{n_k}\}$ is a Fibonacci lacunary-θ-statistically Cauchy with regards to the i-f-n-norm (ϕ,ω). By definition, there is a subsequence $\left\{x_{n_{k'(r)}}\right\}$ of the sequence $\{x_{n_k}\}$ such that $k'(r) \in I_r$ for each r, $(\phi,\omega) - \lim_{r\to\infty} x_{n_{k'(r)}} = \xi$ and for each $\varepsilon > 0$ and $t > 0$,

(2.4)

$$\lim_{r\to\infty} \frac{1}{h_r} \Big| \Big\{ k \in I_r : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \widehat{F}\left(x_{n_{k'(r)}}\right), t\right) \leq 1-\varepsilon \text{ and } \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \widehat{F}\left(x_{n_{k'(r)}}\right), t\right) \geq \varepsilon \Big\} \Big| = 0.$$

As before, we have the following inequality:

$$\begin{aligned} &\lim_{r\to\infty} \frac{1}{h_r} \Big| \Big\{ k \in I_r : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \leq 1-\varepsilon \text{ and } \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \xi, t\right) \geq \varepsilon \Big\} \Big| \\ &\leq \lim_{r\to\infty} \frac{1}{h_r} \Big| \Big\{ k \in I_r : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \widehat{F}\left(x_{n_{k'(r)}}\right), t\right) \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}(x_{n_k}) - \widehat{F}\left(x_{n_{k'(r)}}\right), t\right) \geq s \Big\} \Big| \\ &+ \lim_{r\to\infty} \frac{1}{h_r} \Big| \Big\{ k \in I_r : \phi\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}\left(x_{n_{k'(r)}}\right) - \xi, t\right) \leq 1-s \text{ and } \omega\left(\widehat{F}(x_1, x_2, ..., x_{n-1}), \widehat{F}\left(x_{n_{k'(r)}}\right) - \xi, t\right) \geq s \Big\} \Big|. \end{aligned}$$

Since $(\phi,\omega) - \lim_{r\to\infty} x_{n_{k'(r)}} = \xi$, it follows from (2.4) that $\widehat{F}S^{\theta}_{(\phi,\omega)} - \lim x_{n_k} = \xi$. $\square$

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