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On generalized derivations of lattice implication algebras

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Abstract. In this paper, we introduce the notion of generalized derivation of lattice implication algebra and investigated some related properties. Also, we prove that if D is a generalized derivation associated with a derivation d of L, then $D(x \to y) = x \to D(y)$ for all $x, y \in L$.

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1. Introduction

The concept of lattice implication algebra was proposed by Xu [2], in order to establish an alternative logic knowledge representation. Also, in [3], Xu and Qin discussed the properties lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Xu and Qin [4] introduced the notion of filters in a lattice implication, and investigated their properties. The present author [1, 5] introduced the notion of derivation and f-derivation in lattice implications algebras and obtained some related results. In this paper, we introduce the notion of generalized derivation of lattice implication algebra and investigated some related properties. Also, we prove that if D is a generalized derivation associated with a derivation d of L, then $D(x \to y) = x \to D(y)$ for all $x, y \in L$.

2. Preliminary

A lattice implication algebra is an algebra $(L; \wedge, \vee, \prime, \to, 0, 1)$ of type $(2, 2, 1, 2, 0, 0)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, " $\prime$ " is an order-reversing involution and " $\to$ " is a binary operation, satisfying the following axioms, for all $x, y, z \in L$,

(L1) $x \to (y \to z) = y \to (x \to z)$,
(L2) $x \to x = 1$,
(L3) $x \to y = y' \to x'$,
(L4) $x \to y = y \to x = 1 \Rightarrow x = y$,

(L5) $(x \to y) \to y = (y \to x) \to x$,
(L6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

If L satisfies conditions (L1)−(L5), we say that L is a quasi lattice implication algebra. A lattice implication algebra L is called a lattice H implication algebra, if it satisfies $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in L$ (See [2]).

In the sequel, the binary operation " $\to$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x \leq y$ if and only if $x \to y = 1$ for all $x, y \in L$.

Theorem 2.1. *In a lattice implication algebra L, the following hold:*

(u1) $0 \to x = 1$, $1 \to x = x$ *and* $x \to 1 = 1$,
(u2) $x \to y \leq (y \to z) \to (x \to z)$,
(u3) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,
(u4) $x' = x \to 0$.
(u5) $x \vee y = (x \to y) \to y$,
(u6) $((y \to x) \to y')' = x \wedge y = ((x \to y) \to x')'$,
(u7) $x \leq (x \to y) \to y$.

for all $x, y, z \in L$ (See [2]).

Definition 2.2. In a lattice H implication algebra L, the following hold, for all $x, y, z \in L$,

(u8) $x \to (x \to y) = x \to y$,
(u9) $x \to (y \to z) = (x \to y) \to (x \to z)$ (See [2]).

Definition 2.3. A subset F of a lattice implication algebra L is called a filter of L, if it satisfies:

(F1) $1 \in F$,
(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$, for all $x, y \in L$ (See [4]).

Definition 2.4. Let L_1 and L_2 be lattice implication algebras.

(i) A mapping $f : L_1 \to L_2$ is an implication homomorphism, if $f(x \to y) = f(x) \to f(y)$ for all $x, y \in L_1$.
(ii) A mapping $f : L_1 \to L_2$ is an lattice implication homomorphism, if $f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y), f(x') = f(x)'$ for all $x, y \in L_1$ (See [3])

Definition 2.5. Let L be a lattice implication algebra. A mapping $d : L \to L$ is called a derivation of L, if

$$d(x \to y) = (x \to d(y)) \vee (d(x) \to y)$$

for all $x, y \in L$ (See [1]).

Proposition 2.6. *Let d be a derivation on L. Then the following conditions hold:*

(1) $d(1) = 1$,
(2) $d(x) = d(x) \vee x$ *for every* $x \in L$,
(3) $x \leq d(x)$ *for every* $x \in L$,
(4) $x \vee y \leq d(x) \vee d(y)$ *for every* $x, y \in L$,
(5) $d(x \to y) = x \to d(y)$ *for every* $x, y \in L$ (See [1]).

3. Generalized derivations of lattice implication algebras

In what follows, let L denote a lattice implication algebra unless otherwise specified.

Definition 3.1. Let L be a lattice implication algebra. A map $D : L \to L$ is a generalized derivation of L, if there exists a derivation $d : L \to L$ such that

$$D(x \to y) = (x \to D(y)) \vee (d(x) \to y)$$

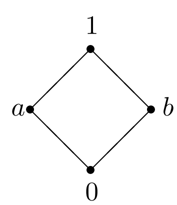
for all $x, y \in L$.

Let L be a lattice implication algebra. If $D = d$, then D is a derivation on L.

Example 3.2. Let $X = \{x, y\}$. Then

$$L = \mathcal{P}(X) = \{\varnothing, \{x\}, \{y\}, X\}.$$

Let $0 = \varnothing$, $a = \{x\}$, $b = \{y\}$, $1 = X$. Then $L = \{0, a, b, 1\}$ is a bounded lattice with above Hasse diagram.



We can make an implication $\to$ on L such as

$$a \to b = \{x\}^C \cup \{y\} = \{y\} \cup \{y\} = \{y\} = b.$$

Hence we have the operation table of the implication :

x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Define a map $d : L \to L$ and $D : L \to L$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 0, 1, b \\ b & \text{if } x = a \end{cases} \qquad D(x) = \begin{cases} b & \text{if } x = 0 \\ 1 & \text{if } x = a, b, 1. \end{cases}$$

Then it is easy to check that d is a derivation on L and D is a generalized derivation associated with d.

Example 3.3. In Example 3.2, define a map $d : L \to L$ and $D : L \to L$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 0, 1, b \\ b & \text{if } x = a \end{cases} \qquad D(x) = \begin{cases} a & \text{if } x = 0 \\ b & \text{if } x = a \\ 1 & \text{if } x = b, 1. \end{cases}$$

Then it is easy to check that d is a derivation on L and D is a generalized derivation associated with d.

Proposition 3.4. *Let d be a derivation on L and let D be a generalized derivation associated with d. Then the following conditions hold:*

(1) $D(1) = 1$,
(2) $D(x) = D(x) \vee x$ *for every* $x \in L$,
(3) $x \leq D(x)$ *for every* $x \in L$,
(4) $x \to y \leq D(x) \to y$ *for every* $x, y \in L$.

Proof. (1) Let D be a generalized derivation associated with d. Then

$$\begin{aligned} D(1) &= D(1 \to 1) = (1 \to D(1) \vee (d(1) \to 1) \\ &= (1 \to D(1)) \vee (1 \to 1) = D(1) \to 1 = 1. \end{aligned}$$

(2) For every $x \in L$, we have

$$\begin{aligned} D(x) &= D(1 \to x) = (1 \to D(x)) \vee (d(1) \to x) \\ &= (1 \to D(x)) \vee (1 \to x) = D(x) \vee x. \end{aligned}$$

(3) For all $x \in L$, by part (2), we obtain

$$\begin{aligned} x \to D(x) &= x \to (D(x) \vee x) = x \to (D(x) \to x \to x) \\ &= (D(x) \to x) \to (x \to x) == (D(x) \to x) \to 1 \\ &= 1. \end{aligned}$$

This implies $D(x) \leq x$ for every $x \in L$.

(4) For every $x, y \in L$, we have $D(x) \leq x$ for every $x \in L$ by part (3). Then we get $x \to y \leq D(x) \to y$ for every $x, y \in L$ by (u3). □

Proposition 3.5. *Let D be a generalized derivation associated with a derivation d of L. Then the following conditions hold:*

(1) $D(x) \to D(y) \leq D(x \to y)$ *for all* $x, y \in L$,
(2) $D(x) \to y \leq x \to D(y)$ *for all* $x, y \in L$,
(3) $x \to y \leq D(x \to y)$ *for all* $x, y \in L$,
(4) $D(D(x) \to x) = 1$ *for all* $x \in L$.

Proof. (1) For all $x, y \in L$, we have $x \to D(y) \leq (x \to D(y)) \vee (d(x) \to y) = D(x \to y)$ from (u7). Now from $x \leq D(x)$, we get $D(x) \to D(y) \leq x \to D(y)$ by using (u3). Hence $D(x) - D(y) \leq D(x \to y)$.

(2) For any $x, y \in L$, from $x \leq D(x)$ and $y \leq D(y)$, we get $D(x) \to y \leq x \to y$ and $x \to y \leq x \to D(y)$ by using (u3). Hence we obtain $D(x) \to y \leq x \to D(y)$ for all $x, y \in L$.

(3) From Definition 3.1 and (u7), for all $x, y \in L$, we have $x \to D(y) \leq (x \to D(y)) \vee (d(x) \to y) = D(x \to y)$ for all $x, y \in L$. Since $y \leq D(y)$, we get $x \to y \leq x \to D(y)$, which implies $x \to y \leq D(x \to y)$.

(4) For every $x, y \in L$, we get

$$D(D(x) \to x) = (D(x) \to D(x)) \vee (d(D(x)) \to d(x)) \tag{3.1}$$
$$= 1 \vee (d(D(x)) \to d(x)) \tag{3.2}$$
$$= 1. \tag{3.3}$$

□

Proposition 3.6. *Let $D_1, D_2, ..., D_n$ are generalized derivations associated with $d_1, d_2, ..., d_n$, respectively on L. Then*

$$x \leq D_n(D_{n-1}(...(D_2(D_1(x)))...))$$

for $n \in \mathbb{N}$.

Proof. For $n = 1$, we have

$$\begin{aligned} D_1(x) &= D_1(1 \to x) = 1 \to D_1(x)) \vee (d_1(1) \to x) \\ &= (1 \to D_1(x)) \vee (1 \to x) = D_1(x) \vee x = (x \to D_1(x)) \to D_1(x), \end{aligned}$$

which implies $x \to D_1(x) = 1$. Then $x \leq D_1(x)$.

Let $n \in \mathbb{N}$ and $x \leq D_n(D_{n-1}(...(D_2(D_1(x)))...))$. For simplicity, let

$$T_n = D_n(D_{n-1}(...(D_2(D_1(x)))...)).$$

Then

$$\begin{aligned} D_{n+1}(T_n) &= D_{n+1}(1 \to T_n) = (1 \to D_{n+1}(T_n)) \vee (d_{n+1}(1) \to T_n) \\ &= (1 \to D_{n+1}(T_n)) \vee (1 \to T_n) \\ &= D_{n+1}(T_n) \vee T_n = (T_n \to D_{n+1}(T_n)) \to D_{n+1}(T_n). \end{aligned}$$

Thus $T_n \to T_{n+1} = 1$, which implies $T_n \leq T_{n+1}$. So by assumption, $x \leq T_n \leq T_{n+1}$. □

Theorem 3.7. *Let d be a derivation on L. If D is a generalized derivation associated with d on L, then $D(x \to y) = x \to D(y)$ for all $x, y \in L$.*

Proof. Suppose that D is a generalized derivation associated with a derivation d on L. Then for any $x, y \in L$, we have $d(x) \to y \leq x \to y$ since $x \leq d(x)$ and $x \to y \leq x \to D(y)$ since $y \leq D(y)$. Thus we have $d(x) \to y \leq x \to D(y)$ and

$$\begin{aligned} D(x \to y) &= (x \to D(y)) \vee (d(x) \to y) \\ &= ((x \to D(y)) \to (d(x) \to y)) \to (d(x) \to y) \\ &= ((d(x) \to y) \to (x \to D(y))) \to (x \to D(y)) \\ &= 1 \to (x - D(y)) = x - D(y) \end{aligned}$$

from (I5) and (u3). This completes the proof. □

Theorem 3.8. *Let d be a derivation on L and let D be a generalized derivation associated with d. If it satisfies $D(x \to y) = D(x) \to y$ for every $x, y \in L$, then $D(x) = x$.*

Proof. Let d be a derivation on L and let D be a generalized derivation associated with d. Suppose it satisfies $D(x \to y) = D(x) \to y$ for all $x, y \in L$. Then we have

$$\begin{aligned} D(x) &= D(1 \to x) = D(1) \to x \\ &= 1 \to x = x. \end{aligned}$$

This completes the proof. □

Theorem 3.9. *Let D be a generalized derivation associated with a derivation d on L. Then we have $D(x \vee y) = D(x) \vee D(y)$ for every $x, y \in L$.*

Proof. For every $x, y \in L$, we obtain

$$\begin{aligned} D(x \vee y) &= D(x'' \vee y'') = D((x' \wedge y') \to 0) \\ &= (x' \wedge y') \to D(0) = (x' \to D(0)) \vee (y' \to D(0)) \\ &= D(x' \to 0) \vee D(y' \to 0) = D(x) \vee D(y). \end{aligned}$$

□

Theorem 3.10. *Let D be a generalized derivation associated with a derivation d on L. Then D is an isotone generalized derivation on L.*

Proof. Let $x_1, x_2 \in L$ be such that $x_1 \leq x_2$. Then by Theorem 3.9, we get $D(x_1 \vee x_2) = D(x_1) \vee D(x_2)$. Since $x_1 \leq x_2$, we have $x_1 \vee x_2 = x_2$. Thus $D(x_1) \vee D(x_2) = D(x_2)$, which implies $D(x_1) \leq D(x_2)$. □

Definition 3.11. Let d be a derivation on L and let D be a generalized derivation associated with d.

- (i) D is called a monomorphic generalized derivation associate with d, if D is one-to- one.
- (ii) D is called an epic generalized generalized derivation associate with d, if D is onto.

Theorem 3.12. *Let D be a generalized derivation associated with a derivation d on L and let D is idempotent, that is, $D^2 = D$. Then the following conditions are equivalent:*

- (1) *$D(x) = x$ for all $x \in L$,*
- (2) *D is a monomorphic generalized derivation associate with a derivation d of L,*
- (3) *D is an epic generalized derivation associate with a derivation d of L.*

Proof. (1) ⇒(2) is clear.

(2) ⇒(1) Let D be a monomorphic generalized derivation associate with d and $x \in L$. Then by the hypothesis, we have $D(D(x)) = D(x)$ for every $x \in L$. Since D is monomorphic, we get $D(x) = x$ for all $x \in L$.

(1) ⇒(3) is trivial.

(3) ⇒(1) Let D be an epic generalized derivation associate with d and $x \in L$. Then there exists $y \in L$ such that $D(y) = x$. Thus we have

$$D(x) = D(D(y)) = D^2(y) = D(y) = x.$$

□

Let d be a derivation of L and let D be a generalized derivation associated with d. Define a set $Fix_D(L)$ by

$$Fix_D(L) := \{x \in L \mid D(x) = x\}$$

for all $x \in L$. Clearly, $1 \in Fix_D(L)$.

Proposition 3.13. *Let d be a derivation on L and let D be a generalized derivation associated with d. Then the following properties hold:*

- (1) *if $x \in L$ and $y \in Fix_D(L)$, then $x \to y \in Fix_D(L)$,*

(2) *if* $x \in L$ *and* $y \in Fix_D(L)$, *then* $x \vee y \in Fix_D(L)$.

Proof. (1) Let $x \in L$ and $y \in Fix_D(L)$. Then we have $D(y) = y$. Thus we get

$$D(x \to y) = x \to D(y) = x \to y$$

from Theorem 3.7. This completes the proof.

(2) Let $x, y \in Fix_D(L)$. Then we get

$$\begin{aligned} D(x \vee y) &= D((x \to y) \to y) = (x \to y) \to D(y) \\ &= (x \to y) \to y) = (x \to y) \to y = x \vee y \end{aligned}$$

from Theorem 3.7. This completes the proof. □

Proposition 3.14. *Let* d *be a derivation of* L *and let* D *be a generalized derivation associated with* d. *If* $x \leq y$ *and* $x \in Fix_D(L)$, *then* $y \in Fix_D(L)$.

Proof. Let $x \leq y$ and $x \in Fix_D(L)$. Then we have $x \to y = 1$, and so $x \to f(y) = f(x \to y) = f(1) = 1$. This means $x \leq f(y)$. Thus by the hypothesis, $d(x) = x$ for every $x \in L$. So we get

$$\begin{aligned} D(y) &= D((1 \to y) = D((x \to y) \to y) \\ &= D((y \to x) \to x) = (y \to x) \to D(x) \\ &= (y \to x) \to x = (x \to y) \to y = 1 \to y = y, \end{aligned}$$

from Theorem 3.7. Hence $y \in Fix_D(L)$. □

Definition 3.15. Let L be a lattice implication algebra. A non-empty set F of L is called a normal filter, if it satisfies the following conditions:

(i) $1 \in F$,
(ii) $x \in L$ and $y \in F$ imply $x \to y \in F$.

Example 3.16. In Example 3.2, let $F = \{1, a\}$. Then F is a normal filter of a lattice implication algebra L.

Proposition 3.17. *Let* d *be a derivation on* L *and let* D *be a generalized derivation associated with* d. *Then* $Fix_D(L)$ *is a normal filter of* L.

Proof. Clearly, $1 \in Fix_D(L)$. By Proposition 3.13 (1), we know tat $x \in L$ and $y \in Fix_D(L)$ imply $x \to y \in Fix_D(L)$. This completes the proof. □

Let d be a derivation on L and let D be a generalized derivation associated with d of L. Define a set $KerD$ by

$$KerD = \{x \in L \mid D(x) = 1\}.$$

Proposition 3.18. *Let* d *be a derivation on* L *and let* D *be a generalized derivation associated with* d. *Then*

(1) *if* $y \in KerD$, *then* $x \vee y \in KerD$ *for all* $x \in L$,
(2) *if* $x \leq y$ *and* $x \in KerD$, *then* $y \in KerD$,
(3) *if* $y \in KerD$, *then* $x \to y \in KerD$ *for all* $x \in L$.

Proof. (1) Let D be a generalized derivation on L and $y \in KerD$. Then we get $D(y) = 1$. Thus

$$D(x \vee y) = D((x \to y) \to y) = (x \to y) \to D(y) = (x \to y) \to 1 = 1$$

from Theorem 3.9. So we have $x \vee y \in KerD$.

(2) Let $x \leq y$ and $x \in KerD$. Then we get $x \to y = 1$ and $D(x) = 1$. Thus

$$\begin{aligned} D(y) &= D(1 \to y) = D((x \to y) \to y) \\ &= D((y \to x) \to x) = (y \to x) \to D(x) \\ &= (y \to x) \to 1 = 1 \end{aligned}$$

from Theorem 3.9. So we have $y \in KerD$.

(3) Let $y \in KerD$. Then $D(y) = 1$. Thus we have

$$D(x \to y) = x \to D(y) = x \to 1 = 1$$

from Theorem 3.9. So we get $x \to y \in KerD$. □

Theorem 3.19. *Let d be a derivation on L and let D be a generalized derivation associated with a derivation d. Then $KerD$ is a normal filter of L.*

Proof. Clearly, $1 \in KerD$. Let $x \in L$ and $y \in KerD$. Then we have $d(y) = 1$. Thus

$$\begin{aligned} D(x \to y) &= x \to D(y) \\ &= x \to 1 = 1, \end{aligned}$$

which implies $x \to y \in KerD$ from Theorem 3.9. So $KerD$ is a normal filter of L. □

Definition 3.20. Let d be a derivation on L and let D be a generalized derivation associated with d. A normal filter F of L is called a D-normal filter, if $D(F) = F$.

Since $D(1) = 1$, it can be easily observed that the zero normal filter $\{1\}$ is a D-normal filter of L. If L is onto, then $D(L) = L$, which implies L is a D-normal filter of L.

Lemma 3.21. *Let d be a derivation on L and let D be a generalized derivation associated with d and let I, J be any two D-normal filters of L. Then we have $I \subseteq J$ implies $D(I) \subseteq D(J)$.*

Proof. Let $I \subseteq J$ and $x \in D(I)$. Then we have $x = D(y)$ for some $y \in I \subseteq J$. Thus we get $x = D(y) \in D(J)$. So $D(I) \subseteq D(J)$. □

Proposition 3.22. *Let d be a derivation on L and let D be a generalized derivation associated with a derivation d of L. Then an intersection of any two D-normal filters is also a D-normal filter of L.*

Proof. Let $x \in D(I \cap J)$. Then $x = D(a)$ for some $a \in I$ and $a \in J$. Thus $x = D(a) \in D(I) = I$ and $x = D(a) \in D(J) = J$, which implies $x \in I \cap J$. Now let $x \in I \cap J$. Then $x \in I = D(I)$ and $x \in J = D(J)$. Thus we have $x \in D(I) \cap D(J)$. So $I \cap J$ is a D-normal filter of L. □

Definition 3.23. Let D be a generalized derivation associated with a derivation d of L. A normal filter F of L is called an injective normal filter with respect to D, if for $x, y \in L$, $D(x) = D(y)$ and $x \in F$ implies $y \in F$.

Evidently, $KerD$ is an injective normal filter of L. Though the zero normal filter $\{1\}$ is a D-normal filter, there is no guarantee that it is injective normal filter.

Theorem 3.24. *Let D be a generalized derivation associated with a derivation d of L. Then the following conditions are equivalent:*

(1) $\{1\}$ *is injective with respect to* D,
(2) $KerD = \{1\}$,
(3) $D(x) = 1$ *implies that* $x = 1$ *for all* $x \in L$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\{1\}$ is injective with respect to D. Let $x \in KerD$. Then $D(x) = D(1)$. Since $\{1\}$ is injective, we can get $x \in \{1\}$. Thus $KerD = \{1\}$.

(2) $\Rightarrow$ (3) The proof is trivial.

(3) $\Rightarrow$ (1) Let $D(x) = D(y)$ and $x \in \{1\}$. Then $D(y) = D(x) = D(1) = 1$, which implies $y = 1 \in \{1\}$. □

Theorem 3.25. *Let D be a generalized derivation associated with a derivation d of L and let D be idempotent. Then a D-normal filter F of L is injective with respect to D if and only if for any $x \in L, D(x) \in F$ implies $x \in F$.*

Proof. Let F be a D-normal filter of L and let F be injective with respect to D. Suppose that $D(x) \in F = D(F)$ and $x \in L$. Then $D(x) = D(a)$ for some $a \in F$. Since F is injective and $a \in F$, we get that $x \in F$.

Conversely, let $x, y \in L, D(x) = D(y)$ and $x \in F$. Since $x \in D(F)$, we get $x = D(a)$ for some $a \in F$. Then $D(y) = D(x) = D(D(a)) = D(a) \in D(F)$. Thus $y \in F$. So F is an injective normal filter of L with respect to D. □

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