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β-fuzzy set and some relations on it

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Abstract. In this paper, we introduced a new extension of the concept of fuzzy set that will be given the name"β-fuzzy set". β-fuzzy t-norm, β-fuzzy t-conorm and β-fuzzy relations are defined. Some characteristics of β-fuzzy relations like symmetry, transitive and reflexive are intensively studied. Finally, the results on these characteristics are deduced.

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1. Introduction

Zadeh [25], in 1965, introduced the notion of fuzzy set as an extension of a crisp set. In a crisp set, the characteristic function takes value from {0,1}, but every element in fuzzy set has grade of membership lies on the interval [0,1]. Fuzziness appears in many areas of daily life, for example in engineering, medicine, meteorology, manufacturing, control systems, modeling, signal processing and expert systems, for more details see [4, 5, 6, 8, 7, 9, 14, 15, 16].

Goguen [10], in 1967, generalized the work of Zadeh where initiated L-fuzzy set to develop a new point of view toward optimization problems. L might be a semigroup, a posset, a lattice, a Boolean σ-ring, or subintervals of the real numbers. Goguen in L-fuzzy set did not discuss the value that satisfied the counter-relation.

Atanassov [1], in 1986, instituted the concept of intuitionistic fuzzy set as a generalization of fuzzy set. An intuitionistic fuzzy set is a pair of mappings, one of them connect the element of intuitionistic fuzzy set with degree of membership in [0,1] and any other element in intuitionistic fuzzy set has degree of non-membership in [0,1] such that the sum of the degree of membership and the degree of non-membership for the same element equal 1. Recently, the intuitionistic fuzzy set has been discussed in [19, 24].

Zhang [28], in 1994, introduced an extension of fuzzy sets, the new concept called bipolar fuzzy sets. A bipolar fuzzy set is also a pair of mappings, a positive membership degree means the element satisfy the property corresponding the bipolar fuzzy set while the negative membership degree indicate that the element satisfy some implicit counter property corresponding the bipolar fuzzy set. In recent years, many researchers have studied bipolar fuzzy sets (See [29, 30, 31]).

In the present paper, we introduced a new extension of the concept of fuzzy set that will be given the name $\beta-$fuzzy set. A $\beta-$fuzzy set K of an universal set H is a map which connect every element in the set H with a membership degree belong to the interval $[-1, 1]$. An element x with a positive value of $\beta_K(x)$ indicates that it satisfies the property of $\beta-$fuzzy set. On the other hand, an element x with a negative value of $\beta_K(x)$ satisfies a counter-property and value means the element is not relevant to the corresponding property. $\beta-$fuzzy t-norm, a $\beta-$fuzzy t-conorm and $\beta-$fuzzy relations are defined. Some characteristics of $\beta-$fuzzy relations like symmetry, transitive and reflexive are intensively studied. Finally, the results on these characteristics are deduced.

2. Preliminaries

In this section, we briefly recall some concepts on fuzzy set, fuzzy relation, intuitionistic fuzzy set and bipolar fuzzy set that are necessary for this paper.

Definition 2.1 ([27]). Let H be a universal set. Then a fuzzy set K in H (i.e., a fuzzy subset K of H) is characterized by a function of the form $\beta_K{:}H{\to}[0,1]$ such a functioni β_K is called the membership function and for each $x{\in}H$, $\beta_K(x)$ is the degree of membership of x (membership grade of x) in the fuzzy set K.

Definition 2.2 ([17]). An L-fuzzy set A on a set X is a function $A{:}X{\to}L$, where L might be a semigroup, a poset, a lattice, a Boolean σ-ring, or subintervals of the real numbers.

Definition 2.3 ([1]). An intuitionistic fuzzy set K in a set H is defined as an object of the form $K{=}\{(x,\beta_K(x),\gamma_K(x)){:}x{\in}H\}$, where $\beta_K{:}H{\to}[0,1]$ and $\gamma_K{:}H{\to}[0,1]$ define the degree of membership and the degree of non-membership of the element $x{\in}H$, respectively and for every $x{\in}H$, $0{\le}\beta_K(x){+}\gamma_K(x){\le}1$.

Definition 2.4 ([28]). Let H be a nonempty set. A bipolar fuzzy set K in H is an object having the form

$$K{=}\left\{\left(x,\beta_K^P(x),\beta_K^N(x)\right){:}x{\in}H\right\},$$

where $\beta_K^P{:}H{\to}[0,1]$ and $\beta_K^N{:}H{\to}[-1,0]$ are two mappings. We use the positive membership degree $\beta_K^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set K, and the negative membership degree $\beta_K^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set K. If $\beta_K^P(x){\neq}0$ and $\beta_K^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for K. If $\beta_K^P(x) = 0$ and $\beta_K^N(x){\neq}0$, it is the situation that x does not satisfy the property of K but somewhat satisfies the counter property of K. It is possible for an element K to be such that $\beta_K^P(x){\neq}0$ and $\beta_K^N(x){\neq}0$ when the membership function of the property overlaps that of its counter property over some portion of H.

Definition 2.5 ([2]). A map $\tau : H \times H \to [0, 1]$ is called a fuzzy relation on H, if

$$\tau(x, y) \le K(x) \wedge K(y),\ \forall x, y \in H.$$

(i) τ is symmetric, if $\tau(x,y)=\tau(y,x)$ for all $x,y\in H$.
(ii) τ is transitive, if $\tau^2\subseteq\tau$.
(iii) τ is said to be reflexive on K, if $\tau(x,x)=K(x)$ for all $x\in H$.

For more details and background see [3, 11, 12, 13, 18, 20, 21, 22, 23, 26].

3. Main results

Firstly, we introduce a new extension of fuzzy set as follows:

Definition 3.1. β-fuzzy set K on the universal set H is a set of ordered pairs $K=\{(x.\beta_K(x)) : x\in H\}$ in which $\beta_K(x)$ is the grade of membership function of the element x in H , $\beta_K(x) : H\to[-1,1]$. We can say, a β-fuzzy set K (shortly, β - FS) is a map that connect every element in the set H with a membership degree in $[-1,1]$. Therefore, when $0<\beta_K(x)$ means x satisfies the fuzziness property of a β-fuzzy set and if $\beta_K(x)<0$ this means x satisfies a counter of fuzziness property. On the other hand, 0 value means the element x is not relevant to the corresponding property.

Remark 3.2. If $\beta_K(x)=0$ and there exist an element y such that $x<y$ and $0<\beta_K(y)$, then we write $\beta_K(x)=0^+$. In the same way, $\beta_K(x)=0^-$ means $\beta_K(x)=0$ and there exist an element z,such that $z<x$ and $\beta_K(z)<0$.

Definition 3.3. If K_1 and K_2 are two β-FSs of H, then the intersection of K_1 and K_2 is defined as

$$(K_1\cap K_2)(x)=K_1(x)\wedge K_2(x)=\min\{K_1(x),K_2(x)\}.$$

Definition 3.4. The union of any two β - FSs K_1 and K_2 is β -fuzzy set given by

$$(K_1\cup K_2)(x)=K_1(x)\vee K_2(x)=\max\{K_1(x),K_2(x)\}.$$

Example 3.5. Consider K_1 and K_2 are two β- FSs of a set $H=\{1,2,3,4,5\}$ defined as follows:

$$K_1=\{(1,0.2),(2,-0.5),(3,0),(4,-0.1),(5,0.3)\},$$
$$K_2=\{(1,-0.3),(2,0.4),\ (3,1),\ (4,0),\ (5,-0.2)\}.$$

Then we can easily have

$$K_1\cap K_2=\{(1,-0.3),(2,-0.5),\ (3,0),\ (4,-0.1),\ (5,-0.2)\},$$
$$K_1\cup K_2=\{(1,0.2),\ (2,0.4),\ (3,1),\ (4,0),\ (5,0.3)\}.$$

Next, we define β-fuzzy t-norm and β-fuzzy t-conorm as follows:

Definition 3.6. β-fuzzy t-norm is a function $T:[-1,1]\times[-1,1]\to[-1,1]$ satisfies the following conditions:
(i) $T(1,x)=x$, $\forall x\in[-1,1]$,
(ii) $T(x,y)=T(y,x)$, $\forall x,y\in[-1,1]$,
(iii) $T(x,T(y,z))=T(T(x,y),z)$, $\forall x,y,z\in[-1,1]$,
(iv) $\forall x,y,z,w\in[-1,1]$, if $x\leq y$ and $z\leq w$, then $T(x,z)\leq T(y,w)$.

Example 3.7. Consider the function T: $[-1,1]\times[-1,1]\to[-1,1]$ defined as:

$$T(x,y)=x\wedge y \text{ for all } x\ y\in[-1,1].$$

Then we prove that T will satisfies the conditions of Definition 3.6.

(i) Let $x\in[-1,1]$. Then we have easily

$$T(1,x)=1\wedge x=\min\{1,x\}=x.$$

(ii) Let $x,\ y\in[-1,1]$. Then we have easily

$$T(x,y)=x\wedge y=\min\{x,y\}=\min\{y,x\}=y\wedge x=T(y,x).$$

(iii) Let $x,\ y,\ z\in[-1,1]$. Then we have

$$\begin{aligned}T(x,T(y,z))&=x\wedge T(y,z)=x\wedge(y\wedge z)\\&=(x\wedge y)\wedge z=T(x,y)\wedge z\\&=T(T(x,y),z).\end{aligned}$$

(iv) Let $x,\ y,\ z,\ w\in[-1,1]$ such that $x\ \leq y,\ z\leq w$. Then we have

$$T(x,z)=x\wedge z=\min\{x,z\}\leq\min\{y,w\}=\ y\wedge w=T(y,w).$$

Thus $T(x,z)\leq T(y,w)$. So from (i)–(iv), T is a β-fuzzy t-norm.

Definition 3.8. A function T^*: $[-1,1]\times[-1,1]\to[-1,1]$ is β-fuzzy t-conorm, if satisfies the following conditions:

(i) $T^*(-1,x)=x\ ,\ \forall x\in[-1,1]$,
(ii) $T^*(x,y)=T^*(y,x)\ ,\ \forall x,y\in[-1,1]$,
(iii) $T^*(x,T^*(y,z))=T^*(T^*(x,y),z)\ ,\ \forall x,y,z\in[-1,1]$,
(iv) $\forall x,y,z,w\in[-1,1]$, if $x\leq y$ and $z\leq w$, then $T^*(x,z)\leq T^*(y,w)$.

Example 3.9. Consider the function T^*: $[-1,1]\times[-1,1]\to[-1,1]$ given by:

$$T^*(x,y)=x\vee y\ .$$

Then we check that T will satisfies the conditions of Definition 3.8.

(i) Let $x\in[-1,1]$. Then we have easily

$$T^*(-1,x)=-1\vee x=\max\{-1,x\}=x.$$

(ii) Let $x,\ y\in[-1,1]$. Then we have easily

$$T^*(x,y)=x\vee y=\max\ \{x,y\}=\max\{y,x\}=y\vee x=T^*(y,x).$$

(iii) Let $x,\ y,\ z\in[-1,1]$. Then we have

$$\begin{aligned}T^*(x,T^*(y,z))&=x\vee T^*(y,z)=x\vee(y\vee z)\\&=(x\vee y)\vee z=T^*(x,y)\vee z\\&=T^*(T^*(x,y),z).\end{aligned}$$

(iv) Let $x,\ y,\ z,\ w\in[-1,1]$ such that $x\ \leq y,\ z\leq w$. Then we have

$$T^*(x,z)=x\vee z=\max\{x,z\}\leq\max\{y,w\}=\ y\vee w=T^*(y,w).$$

Thus $T(x,z)\leq T(y,w)$. So So from (i)–(iv), T^* is a β-fuzzy t-conorm.

Definition 3.10. $\beta-$fuzzy complement is a function C: $[-1,1]\to[-1,1]$ satisfies the following conditions:

(i) $\mathrm{C}(-1)=1,\ \mathrm{C}(1)=-1$,
(ii) if $x\leq y$, then $C(x)\geq C(y)\ \forall x,y\in[-1,1]$.

Example 3.11. Consider the function $C\colon[-1,1]\to[-1,1]$ defined by:

$$C\,(x)=-x,\ \forall x\in[-1,1].$$

Then we show that C will satisfies the conditions of Definition 3.10.

(i) Clearly, $C\,(-1)=-\,(-1)=1,\ \ C\,(1)=-1.$

(ii) Let $x,\ y\in[-1,1]$ such that $x\leq y$. Then $C\,(x)=-x\geq -y=C(y)$. Thus $C(x)\geq C(y)$. So from (i) and (ii), C is $\beta-$fuzzy complement.

Definition 3.12. Let K be a β -FS on a set H and let τ be the relation defined by:

$$\tau : H\times H\to[-1,1]\,.$$

Then τ is a β-fuzzy relation on K, if $\tau\,(x,y)\leq K\,(x)\wedge K\,(y)\,,\ \forall x,y\in H.$

Example 3.13. Given a universal set $H=\{x,y,z\}$, a β-PFS is given by

$$K\,(x)=-0.1,K\,(y)=0.7,\ K\,(z)=0,$$

Define a relation $\tau : H\times H\to[-1,1]$ as follows:

$\tau\,(x,x)=-0.2$	$\tau\,(y,x)=-\ 0.3,$	$\tau\,(z,x)=-0.5,$
$\tau\,(x,y)=-0.1$	$\tau\,(y,y)=0.5,$	$\tau\,(z,y)=-0.2,$
$\tau\,(x,z)=-0.5$	$\tau(y,z)=0.2,$	$\tau\,(z,z)=-0.9.$

Since the given relation τ satisfy the condition $\tau\ (x,y)\leq K\,(x)\wedge K\ (y)\,,\ \ \forall x,y\in H$. Then τ is a β-fuzzy relation (shortly, β-FR) on K.

Definition 3.14. Let K_1 and K_2 be β-FSs on sets H_1 and H_2, respectively. Then a β-FR from the β-FS K_1 into the β-FS K_2 is a β-FS τ of $H_1\times H_2$ defined by $\tau\ : H_1\times H_2\to\ [-1,1]$ such that $\tau\,(x,y)\leq\ K_1\,(x)\wedge K_2\,(y)\,,\ \forall x\in H_1,\ y\ \in H_2.$

Definition 3.15. A strong β-FR τ_K on a β-FS K of a set H is a β -FR defined by $\tau\,(x,y)=K\,(x)\wedge\ K\,(y)\,,\ \ \forall x,y\ \in H.$

Example 3.16. Let H be a universal set given by $H=\{x,y,z\}$, a β-FS K is defined as:

$$K\,(x)=-0.2,K\,(y)=0.6,,K\,(z)=0.$$

A relation $\tau : H\times H\to[-1,1]$ is given by:

τ (x, x) =−0.2,	τ (y, x) = −0.2,	τ (z, x) = −0.2,
τ (x, y) = −0.2,	τ (y, y) =0.6,	τ (z, y) =0,
τ (x, z) = −0.2,	τ (y, z) =0,	τ (z, z) =0,

Then it is clear that $\tau\,(x,y)\ =\ K\,(x)\wedge K\,(y)\,,\forall x,y\in H$. Thus τ is a strong β-FR on K, i.e., $\tau_K=\tau$.

Definition 3.17. Let $\tau_1\ : H_1\times H_2\to[-1,1]$ be a β -FR from a β -fuzzy set K_1 of H_1 into a β -FS K_2 of H_2 and $\tau_2\ : H_2\times H_3\to[-1,1]$ is a β -FR from a β -FS K_2 of H_2 into a β -FS K_3 of H_3. Then, a composition of τ_1 and τ_2, $\tau_1\circ\tau_2 : H_1\times H_3\to[-1,1]$, is defined by $\tau_1\circ\tau_2\,(x,z)=\vee\{\tau_1\,(x,y)\wedge\tau_2\,(y,z) : y\in H_2\}$ for all $x\in H_1$ and $z\in H_3$.

In the following we denote by $\beta-$FR(H) the family of all β-FRs on a β-FS K of a universal set H through our study of operations on β-FRs.

Proposition 3.18. *Let $\tau_1, \tau_2 \in \beta - FR(H)$. Then a composition $\tau_1 \circ \tau_2$ is also belong to β −FR (H) .*

Proof. Let $x,\ y,\ z \in H$. Since $\tau_1,\ \ \tau_2 \in\ \beta$ −FR(H), we have

$$\tau_1\,(x,y) \leq K(x) \wedge K(y) \text{ and } \ \tau_2\,(y,z) \leq K(y) \wedge K(z).$$

Thus $\tau_1\,(x,y) \wedge \tau_2\,(y,z) \leq K(x) \wedge K(y) \wedge K(z) \leq K(x) \wedge K(z)$. So we have

$$\tau_1 \circ \tau_2\ \ (x,z)\ =\ \vee\ \{\tau_1\,(x,y) \wedge \tau_2\,(y,z) : y \in H\} \leq K(x) \wedge K(z).$$

Hence the composition of τ_1 and τ_2 is a β −FR on K. □

Proposition 3.19. *If $\tau_1,\ \tau_2,\ \tau_3 \in\ \beta$ −FR(H), then $(\tau_1 \circ \tau_2) \circ \tau_3$ and $\tau_1 \circ (\tau_2 \circ \tau_3)$ are also β −FRs on K.*

Proof. Since $\tau_1,\ \tau_2,\ \tau_3 \in \beta$ −FR(H), from Proposition 3.18, $\tau_2 \circ \tau_3 \in\ \beta$ −FR(H). Since τ_1 and $\tau_2 \circ \tau_3$ are two β -FRs on K, from Proposition 3.18, the relation $\tau_1 \circ (\tau_2 \circ \tau_3)$ is also β -FR on K. By the same manner, $(\tau_1 \circ \tau_2) \circ \tau_3$ is a β -FR on K. □

Proposition 3.20. *If $\tau_1,\ \tau_2,\ \tau_3 \in\ \beta$ −FR(H), then the following relation holds:*

$$(\tau_1 \circ \tau_2) \circ \tau_3 = \tau_1 \circ (\tau_2 \circ \tau_3).$$

Proof. Consider $\tau_1,\ \tau_2,\ \tau_3 \in\ \ \beta$ −FR(H). Then the β-FRs $\tau_1 \circ \tau_2$ and $\tau_2 \circ \tau_3$ are defined by: for all $x,\ z \in H$ and $y,\ w \in H$,

$$\tau_1 \circ \tau_2\,(x,z) = \vee\,\{\tau_1\,(x,y) \wedge \tau_2\,(y,z) : y \in H\}$$

and

$$\tau_2 \circ \tau_3\,(y,w) = \vee\,\{\tau_2\,(y,z) \wedge \tau_3\,(z,w) : z \in H\}\,.$$

Thus, a β−FR $(\tau_1 \circ \tau_2) \circ \tau_3$ is defined as: for all $x,\ w \in H$,

$$\begin{aligned} [(\tau_1 \circ \tau_2) \circ \tau_3](x,w) &= \vee\,\{\tau_1 \circ \tau_2\,(x,z) \wedge \tau_3\,(z,w)\ : z \in H\} \\ &= \vee\,\{\vee\,\{\tau_1\,(x,y) \wedge \tau_2\,(y,z) : y \in H\} \wedge \tau_3\,(z,w) : z \in H\} \\ &= \vee\,\{\tau_1\,(x,y) \wedge \tau_2\,(y,z) \wedge \tau_3\,(z,w) : y,z \in H\ \}\,. \end{aligned} \tag{3.1}$$

Also $\tau_1 \circ (\tau_2 \circ \tau_3)$ is defined by: for all $x,\ w \in H$,

$$\begin{aligned} [\tau_1 \circ (\tau_2 \circ \tau_3)](x,w) &= \vee\,\{\tau_1\,(x,y) \wedge \tau_2 \circ \tau_3\,(y,w) : y \in H\} \\ &= \vee\,\{\vee\,\{\tau_2\,(y,z) \wedge \tau_3\,(z,w) : z \in H\} \wedge \tau_1\,(x,y) : y \in H\} \\ &= \vee\,\{\tau_1\,(x,y) \wedge \tau_2\,(y,z) \wedge \tau_3\,(z,w) : y,z \in H\ \}\,. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we find $(\tau_1 \circ \tau_2) \circ \tau_3 = \tau_1 \circ (\tau_2 \circ \tau_3)$. □

Definition 3.21. A β-FR τ on a β-FS K of a set H is said to be a reflexive, if $\tau\ \ (x,x) = K(x)$ for all $x \in H$.

Theorem 3.22. *For any two B-FRs τ_1 and τ_2 on β −FR(H), if τ_1 is reflexive on K, then $\tau_2 \subseteq \tau_1 \circ \tau_2$ and $\tau_2 \subseteq \tau_2 \circ \tau_1$.*

Proof. Since $\tau_1,\ \tau_2 \in \beta$ −FR(H) and τ_1 is reflexive, $\tau_1(x,x) = K(x)$ for all $x \in H$ and $\tau_2(x,z) \leq K(x) \wedge K(z)$. Then from Definition 3.17, the composition of τ_1 and τ_2 is given by: for all $x,\ z \in H$,

$$\begin{aligned} \tau_1 \circ \tau_2(x,z) &= \vee\{\tau_1(x,y) \wedge \tau_2(y,z) : y \in H\} \\ &\geq \tau_1(x,x) \wedge \tau_2(x,z) \\ &= K(x) \wedge \tau_2(x,z) = \tau_2(x,z). \end{aligned}$$

Thus $\tau_2 \subseteq \tau_1 \circ \tau_2$. By the same way, we can prove that $\tau_2 \subseteq \tau_2 \circ \tau_1$. □

Corollary 3.23. *If τ is a reflexive β-FR, then $\tau \subseteq \tau^2$.*

Proof. Since τ is a reflexive β-FR, from Theorem 3.22, $\tau \subseteq \tau \circ \tau = \tau^2$. □

Theorem 3.24. *Let τ be a reflexive β-FR on K. Then*

$$\tau(x,x) = \tau^2(x,x) = \cdots = \tau^\infty(x,x) = k(x)$$

for all $x \in H$.

Proof. To prove this theorem, we will use mathematical induction as follows.

Letτ be a reflexive β-FR on a β-FS K of a set H. Then $\tau(x,x) = K(x)$ for all $x \in H$. Assume that the relation $\tau^n(x,x) = K(x)$ is true at $m = n$ for all $x \in H$. Now, we will prove that the relation $\tau^{n+1}(x,x) = K(x)$ is true at $m = n+1$ for all $x \in H$ as follows:

$$\begin{aligned} \tau^{n+1}(x,x) &= \tau \circ \tau^n(x,x) \\ &= \vee\{\tau(x,y) \wedge \tau^n(y,x) : y \in H\} \\ &\leq \vee\{K(x) \wedge K(x) : y \in H\}. \end{aligned}$$

Then we have

$$\tau^{n+1}(x,x) \leq K(x). \tag{3.3}$$

Also,

$$\begin{aligned} \tau^{n+1}(x,x) &= \tau \circ \tau^n(x,x) \\ &= \vee\{\tau(x,y) \wedge \tau^n(y,x) : y \in H\} \\ &\geq \tau(x,x) \wedge \tau^n(x,x) \\ &= K(x) \wedge K(x) = K(x). \end{aligned}$$

Thus we have

$$\tau^{n+1}(x,x) \geq K(x). \tag{3.4}$$

So from (3.3) and (3.4), we have $\tau^{n+1}(x,x) = K(x)$ for all $x \in H$. □

Theorem 3.25. *Let τ_1 and τ_2 be reflexive β-FRs on a β-FS K. Then $\tau_1 \circ \tau_2$ and $\tau_2 \circ \tau_1$ are reflexive.*

Proof. Suppose τ_1 and τ_2 are reflexive β-FRs. Then for all $x \in H$,

$$\begin{aligned} \tau_1 \circ \tau_2(x,x) &= \vee\{\tau_1(x,y) \wedge \tau_2(y,x) : y \in H\} \\ &\leq \vee\{K(x) \wedge K(x) : y \in H\} = K(x). \end{aligned}$$

Thus we have

$$\tau_1 \circ \tau_2 (x, x) \leq K(x). \tag{3.5}$$

Also, for all $x \in H$,

$$\begin{aligned} \tau_1 \circ \tau_2 (x, x) &= \vee \{\tau_1 (x, y) \wedge \tau_2 (y, x) : y \in H \} \\ &\geq \tau_1 (x, x) \wedge \tau_2 (x, x) \\ &= K(x) \wedge K(x) = K(x). \end{aligned}$$

So we have

$$\tau_1 \circ \tau_2 (x, x) \geq K(x). \tag{3.6}$$

Hence, from (3.5) and (3.6), we have $\tau_1 \circ \tau_2 (x, x) = K(x)$ for all $x \in H$. This means that $\tau_1 \circ \tau_2$ is reflexive.

Similarly, we can prove $\tau_2 \circ \tau_1$ is also reflexive. □

Definition 3.26. A β-FR τ on a $\beta-$fuzzy set K of a set H is symmetric, if $\tau(x, y) = \tau(y, x)$ for all $x, y \in H$.

Theorem 3.27. *Let τ_1 and τ_2 be a symmetric β -FRs on K, then $\tau_1 \circ \tau_2$ is symmetric if and only if $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$.*

Proof. Suppose $\tau_1 \circ \tau_2$ is symmetric and let $x,\ z \in H$. Then clearly, $\tau_1 \circ \tau_2 (x, z) = \tau_1 \circ \tau_2 (z, x)$. Thus we have

$$\begin{aligned} \tau_1 \circ \tau_2 (x, z) &= \vee \{\tau_1 (x, y) \wedge \tau_2 (y, z) : y \in H\} \\ &= \vee \{\tau_1 (y, x) \wedge \tau_2 (z, y) : y \in H\} \\ &= \vee \{\tau_2 (z, y) \wedge \tau_1 (y, x) : y \in H\} \\ &= \tau_2 \circ \tau_1 (z, x). \end{aligned}$$

Since $\tau_1 \circ \tau_2 (x, z) = \tau_1 \circ \tau_2 (z, x)$, $\tau_1 \circ \tau_2 (z, x) = \tau_2 \circ \tau_1 (z, x)$. So $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$.

Conversely, suppose $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$ and let $x,\ z \in H$. Then $\tau_1 \circ \tau_2 (z, x)\, \tau_2 \circ \tau_1 (z, x)$. Thus we have

$$\begin{aligned} \tau_1 \circ \tau_2 (x, z) &= \vee \{\tau_1 (x, y) \wedge \tau_2 (y, z) : y \in H\} \\ &= \vee \{\tau_1 (y, x) \wedge \tau_2 (z, y) : y \in H\} \\ &= \vee \{\tau_2 (z, y) \wedge \tau_1 (y, x) : y \in H\} \\ &= \tau_2 \circ \tau_1 (z, x) \\ &= \tau_1 \circ \tau_2 (z, x). \end{aligned}$$

So $\tau_1 \circ \tau_2 (x, z)$ is symmetric. □

Theorem 3.28. *Let τ be a symmetric β-FR on a β-fuzzy set K of a set H. Then every power of τ is also symmetric.*

Proof. If τ is a symmetric β-FR, then from Theorem 3.27, $\tau^2 = \tau \circ \tau$ is symmetric. Assume that τ^n is symmetric at $m = n$ and let $x,\ y \in H$. Then we have

$$\begin{aligned} \tau^{n+1}(x,z) &= \tau \circ \tau^n(x,z) \\ &= \vee\{\tau(x,y) \wedge \tau^n(y,z) : y \in H\} \\ &= \vee\{\tau^n(z,y) \wedge \tau(y,x) : y \in H\} \\ &= \tau^n \circ \tau(z,x) \\ &= \tau^{n+1}(z,x). \end{aligned}$$

Thus τ^{n+1} is symmetric. □

Definition 3.29. A β-FR τ on a β-FS K of a set H is transitive, if $\tau^2 \subseteq \tau$.

Theorem 3.30. *If* $\tau_1,\ \tau_2,\ \tau_3, \tau_4\ \in \beta - FR(H)$, $\tau_1 \subseteq \tau_2$ *and* $\tau_3 \subseteq \tau_4$. *Then* $\tau_1 \circ \tau_3 \subseteq \tau_2 \circ \tau_4$.

Proof. Let $\tau_1 \subseteq \tau_2$ and $\tau_3 \subseteq \tau_4$, and let $x,\ y,\ z \in H$. Then $\tau_1(x,y) \leq \tau_2(x,y)$ and $\tau_3(y,z) \leq \tau_4(y,z)$. Thus we have

$$\begin{aligned} \tau_1 \circ \tau_3(x,z) &= \vee\{\tau_1(x,y) \wedge \tau_3(y,z) : y \in H\} \\ &\leq \vee\{\tau_2(x,y) \wedge \tau_4(y,z) : y \in H\} \\ &= \tau_2 \circ \tau_4(x,z). \end{aligned}$$

So $\tau_1 \circ \tau_3 \subseteq \tau_2 \circ \tau_4$. □

Theorem 3.31. *Let* τ_1 *and* τ_2 *be transitive* β *-FRs on a* β*-fuzzy set* K *of a set* H *and* $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1$. *Then* $\tau_1 \circ \tau_2$ *is transitive.*

Proof. Since τ_1 and τ_2 are transitive**,** ${\tau_1}^2 \subseteq \tau_1$ and ${\tau_2}^2 \subseteq \tau_2$. Then form Theorem 3.30, we have ${\tau_1}^2 \circ {\tau_2}^2 \subseteq \tau_1 \circ \tau_2$. Thus

$$\begin{aligned} (\tau_1 \circ \tau_2)^2 &= (\tau_1 \circ \tau_2) \circ (\tau_1 \circ \tau_2). \\ &= \tau_1 \circ (\tau_2 \circ \tau_1) \circ \tau_2 \\ &= \tau_1 \circ (\tau_1 \circ \tau_2) \circ \tau_2 \\ &= (\tau_1 \circ \tau_1) \circ (\tau_2 \circ \tau_2) \\ &= {\tau_1}^2 \circ {\tau_2}^2 \subseteq \tau_1 \circ \tau_2. \end{aligned}$$

So $\tau_1 \circ \tau_2$ is transitive. □

Theorem 3.32. *Let* τ *be a symmetric and a transitive* β*-FR on* K*, where* K *is a* β*-FS of a set* H. *Then* $\tau(x,y) \leq \tau(x,x)$ *and* $\tau(y,x) \leq \tau(x,x)$ *for all* $x, y \in H$.

Proof. Consider τ is a symmetric and transitive β-FR on K. Then $\tau \circ \tau \subseteq \tau$, i.e., $\tau \circ \tau(x,x) \leq \tau(x,x)$ for all $x \in H$. Thus

$$\begin{aligned} \tau \circ \tau(x,x) &= \vee\{\tau(x,y) \wedge \tau(y,x) : y \in H\} \\ &= \vee\{\tau(x,y) \wedge \tau(x,y) : y\text{i} \in H\} \\ &= \tau(x,y). \end{aligned}$$

So $\tau \circ \tau(x,x) = \tau(x,y) \leq \tau(x,x)$. Hence $\tau(x,y) \leq \tau(x,x)$ for all $x, y \in H$. Since τ is symmetric, $\tau(y,x) \leq \tau(x,x)$ for all $x, y \in H$. □

Theorem 3.33. *Let $\tau_1,\ \tau_2,\ \tau_3\ \in\ \beta - FR(H)$. If τ_1 is transitive and $\tau_2 \subseteq \tau_1$, $\tau_3 \subseteq \tau_1$, then $\tau_2 \circ \tau_3 \subseteq \tau_1$.*

Proof. Since $\tau_2 \subseteq \tau_1$ and $\tau_3 \subseteq \tau_1$, from Theorem 3.30, $\tau_2 \circ \tau_3 \subseteq {\tau_1}^2$. Since τ_1 is transitive, ${\tau_1}^2 \subseteq \tau_1$. Then $\tau_2 \circ \tau_3 \subseteq \tau_1$. □

Theorem 3.34. *If τ is transitive β-FR on a β-FS K of a set H, then every power of τ is also transitive.*

Theorem 3.35. *Let τ_1 be a transitive β-FR on a β -FS of a set H, let τ_2 be a reflexive β-fuzzy relations on K and let $\tau_2 \subseteq \tau_1$. Then $\tau_1 \circ \tau_2 = \tau_2 \circ \tau_1 = \tau_1$.*

Proof. Since τ_1 is a transitive and $\tau_2 \subseteq \tau_1$, from Theorem 3.33, $\tau_1 \circ \tau_2 \subseteq \tau_1$. Then for all $x, z \in H$,

$$\begin{aligned} \tau_1 \circ \tau_2 (x, z) &= \vee \{\tau_1 (x, y) \wedge \tau_2 (y, z) : y \in H\} \\ &\geq \tau_1 (x, z) \wedge \tau_2 (z, z) \\ &= \tau_1 (x, z) \wedge K (z) = \tau_1 (x, z) \end{aligned}$$

Thus $\tau_1 \subseteq \tau_1 \circ \tau_2$. So $\tau_1 \circ \tau_2 = \tau_1$.

By the same way, we can prove that $\tau_2 \circ \tau_1 = \tau_1$. □

In the following, we introduce special type of intersection and of β-fuzzy set.

Definition 3.36. For any two β-FSs K_1 and K_2 of a set H, an F-intersection of K_1 and K_2, denoted by $K_1 \cap^F K_2$, is defined as: for each $x \in H$,

$$\left(K_1 \cap^F K_2\right)(x) = K_2 (x) \wedge^F K_2 (x)$$

$$= \begin{cases} \min\{\mathrm{K}_1(\mathrm{x}), \mathrm{K}_2(\mathrm{x})\} & \text{if } \mathrm{K}_1(\mathrm{x}) \text{ and } \mathrm{K}_2(\mathrm{x}) \text{ have the same sign} \\ 0 & \text{if } \mathrm{K}_1(\mathrm{x}) \text{ and } \mathrm{K}_2(\mathrm{x}) \text{ have different sign} \\ 0 & \text{if } \mathrm{K}_1(\mathrm{x}) = 0, \text{or } \mathrm{K}_2(\mathrm{x}) = 0. \end{cases}.$$

Definition 3.37. Let K_1 and K_2 are two β-FSs of H, then an F- union of K_1 and K_2, denoted by $K_1 \cup^F K_2$, is defined as: for each $x \in H$,

$$\left(K_1 \cup^F K_2\right)(x) = K_2 (x) \vee^F K_2 (x).$$

$$= \begin{cases} \max\{K_1(x), K_2(x)\} & if \ \ K_1(x) \ and \ K_2(x) \text{ have the same sign} \\ 0 & if \ \ K_1(x) \ and \ K_2(x) \text{ have different sign} \\ 0 & if \ K_1(x) = 0, or \ K_2(x) = 0. \end{cases}$$

Example 3.38. Let H=$\{x_1,\ x_2, x_3,\ x_4,\ x_5\}$ be a universal set and let K_1 and K_2 be the β-FSs given by:

$$K_1 = \{(x_1, 0.7),\ (x_2, 0.5), (x_3, 0), (x_4, -0.1),\ (x_5, 0.3)\},$$

$$K_2 = \{\ (x_1, -0.3), (x_2, 0.4),\ (x_3, 1),\ (x_4, -0.2),\ (x_5, -0.2)\}.$$

Then clearly,

$$K_1 \cap^F K_2 = \{(x_1, 0), (x_2, 0.4),\ (x_3, 0),\ (x_4, -0.2),\ (x_5, 0)\},$$

$$K_1 \cup^F K_2 = \{(x_1, 0), (x_2, 0.5),\ (x_3, 0),\ (x_4, -0.1),\ (x_5, 0)\}.$$

Now, a new concept of composition of β-FRs is introduced as follows.

Definition 3.39. Let $\tau_1$be a β-FR from a β-FS K_1 of H_1 into a β-FS K_2 of H_2, i.e., τ_1 :$H_1 \times H_2 \to [-1, 1]$ and τ_2 is a β-FR from a β -FS K_2 of H_2 into a β -FS K_3 of H_3, τ_2 :$H_1 \times H_2 \to [-1, 1]$. Then an F-composition of τ_1 and τ_2, $\tau_1 \circ^F \tau_2 : H_1 \times H_3 \to [-1, 1]$ is defined by: for all $x \in H_1$ and $z \in H_3$,

$$\tau_1 \circ^F \tau_2 (x, z) = \vee \left\{ \left| \tau_1 (x, y) \wedge^F \tau_2 (y, z) \right| : y \in H_2 \right\}.$$

Example 3.40. Let P be a set of patients, let $S = \{s_1,\ s_2,\ s_3, s_4,\ s_5\}$ be a set of symptoms andlet N be a set of illnesses. If τ_1 is a β-FR from P to S defined as:

$\tau_1 (p, s) > 0$ is the degree that a symptom$s$ appear on a patient p,

$\tau_1 (p, s) = 0$ is the degree that a symptom$s$ is not appear on a patient p,

$\tau_1 (p, s) < 0$ is the degree that a counter- symptom$s$ appear on a patient p.

Let τ_2 be a β -FR from S to N defined by:

$\tau_2 (s,\ n) > 0$ is the degree that s is a symptom of illness n,

$\tau_2 (s,\ n) = 0$ is the degree that s is not a symptom of illness n ,

$\tau_2 (s,\ n) < 0$ is the degree that s is a counter-symptom of illness n.

Let us consider p is a patient in P, n is illness in N and τ_1, τ_2 are given by:

τ_1	s_1	s_2	s_3	s_4	s_5
p	0	0.4	0.7	-0.6	-0.3

Table 1.

$\boldsymbol{\tau_2}$	$\boldsymbol{n}$
$\boldsymbol{s_1}$	0.7
$\boldsymbol{s_2}$	0.1
$\boldsymbol{s_3}$	-0.5
$\boldsymbol{s_4}$	1
$\boldsymbol{s_5}$	-0.2

Table 2.

Then we have

$$\tau_1 \circ^F \tau_2 (x, z) = \vee \{0,\ 0.1,\ 0,\ 0,\ 0.3\} = 0.3.$$

It is clear from the formula of the F-composition of $\tau_1$and $\tau_2$that two symptoms of illness n shown by patient p The proportion of his suffering from this disease is 8%.

4. Conclusion

We introduced a new extension of the concept of fuzzy set that was given the name $\beta-$fuzzy set. A β-fuzzy set K of H is a map which connect every element in the set H with a membership degree belong to the interval $[-1, 1]$. An element x with a positive value of $\beta_K(x)$ indicates that it satisfies the property of a perfect fuzzy set. On the other hand, an element x with a negative value of $\beta_K(x)$ satisfies a counter-property and 0 value means the element is not relevant to the corresponding property. A β-fuzzy t-norm, a β-fuzzy t-conorm and β-fuzzy relations are defined. Some characteristics of $\beta-$fuzzy relations like symmetry, transitive and reflexive are intensively studied. A β-fuzzy set is an extremely useful tool for solving many problems in different areas of human life.

Declarations:

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