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Interval-valued intuitionistic sets and their application to topology

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Abstract. In this paper, we introduce the new notion of interval-valued intuitionistic sets providing a tool for approximating undefinable or complex concepts. First, we deal with some of its algebraic structures. Also, we define an interval-valued intuitionistic (vanishing) point and obtain some of its properties. Next, we define an interval-valued intuitionistic topology, base (subbase), neighborhood and interior (closure), respectively and study some of each properties, and give some examples.

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Keywords: Interval-valued intutionistic set, Interval-valued intutionistic (vanishing) point, Interval-valued intutionistic topological space, Interval-valued intutionistic base, Interval-valued intutionistic neighborhood, Interval-valued intutionistic closure, Interval-valued intutionistic interior.

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1. Introduction

In 1996, Çoker [6] proposed the concept of an intuitionistic set as the generalzation of an ordinary set and the specialization of an intuitionistic fuzzy set introduced by Atanassove [1]. After then, many researchers [3, 4, 5, 7, 8, 9, 11, 12, 14, 17, 18] applied the notion to topology and category theory. Recently, Kim et al [13] dealt with some properties of interval-valued sets (by introduced bu Yao [19]) as the generalization of classical sets and the special case of interval-valued fuzzy set proposed by Zadeh [21] and applied it to topological structures.

In order to provide a tool for modelling and processing partially known concepts, we propose a new notion of interval-valued intuitionistic sets by combining interval-valued sets with intuitionistic sets. Furthermore, we apply this concept to topology. To accomplish such research, this paper is composed of six sections. In Section 2, we recall some definitions of intuitionistic sets introduced by Çoker [6] and interval-valued sets proposed by Yao [19], and Kim et al. [13]. In Section 3, we introduce the new concept of interval-valued intuitionistic set and obtain some of its algebraic structures. Also, we define interval-valued intuitionistic points of two types and discuss with the characterizations of inclusions, intersections and unions of interval-valued intuitionistic sets. Furthermore, we introduce the concept of interval-valued intuitionistic ideals and obtain some of its properties. In Section 4, we define an interval-valued intuitionistic topology, an interval-valued intuitionistic base and subbase, and study some of their properties. In Section 5, we introduce the notions of interval-valued neighborhoods of two types and find some of their properties. In particular, we show that there is an IVIT under the hypothesis satisfying some properties of interval-valued intuitionistic neighborhoods. In Section 6, we define an interval-valued interior and closure and obtain some of their properties. Also, we prove that there is a unique IVIT for interval-valued intuitionistic interior [resp. closure] operators.

2. Preliminaries

In this section, we recall the concepts of intuitionistic sets and intuitionistc points introduced by [6]. Also, we recall the notions of interval-valued sets and interval-valued points proposed by [13, 19].

Definition 2.1 ([6])**.** Let X be a non-empty set. Then A is called an intuitionistic set (briefly, IS) of X, if it is an object having the form

$$A = (A^{\in}, A^{\notin}),$$

such that $A^{\in} \cap A^{\notin} = \varnothing$, where $A^{\in}$ [resp. $A^{\notin}$] represents the set of memberships [resp. non-memberships] of elements of X to A. In fact, $A^{\in}$ [resp. $A^{\notin}$] is a subset of X agreeing or approving [resp. refusing or opposing] for a certain opinion, suggestion or policy.

The intuitionistic empty set [resp. the intuitionistic whole set] of X, denoted by $\bar{\varnothing}$ [resp. $\bar{X}$], is defined by $\bar{\varnothing} = (\varnothing, X)$ [resp. $\bar{X} = (X, \varnothing)$]. We will denote the set of all ISs of X as $IS(X)$. Note that for each $A \in IS(X)$, $A^{\in} \cup A^{\notin} \neq X$ in general. The inclusion, the equality, the intersection and the union of ISs, the complement of an IS, and the operations intersection [ ] and $< >$ on $IS(X)$ refer to [6].

It is obvious that $A = (A, \varnothing) \in IS(X)$ for each ordinary subset A of X. Then we can consider an IS of X as the generalization of an ordinary subset of X.

Remark 2.2. Let X be a set and let $A \in IS(X)$. Then we can easily see that

$$\chi_A = (\chi_{A^{\in}}, \chi_{A^{\notin}})$$

is an intuitionistic fuzzy set in X introduced by Atanassov [1]. Thus we can consider an intuitionistic set A in X as the specialization of an intuitionistic fuzzy set in X.

Definition 2.3 ([6])**.** Let X be a non-empty set, $a \in X$ and let $A \in IS(X)$.

(i) The form $(\{a\}, \{a\}^c)$ [resp. $(\varnothing, \{a\}^c)$]is called an intuitionistic point [resp. vanishing point] of X and denoted by a_I [resp. a_{IV}].

(ii) We say that a_I [resp. a_{IV}] is contained in A, denoted by $a_I \in A$ [resp. $a_{IV} \in A$], if $a \in A^{\in}$ [resp. $a \notin A^{\notin}$].

We will denote the set of all intuitionistic points and intuitionistic vanishing points in X as $I_P(X)$.

Result 2.4 ([6], Proposition 3.6)**.** *Let $A \in IS(X)$. Then*

$$A = A_I \cup A_{IV},$$

where $A_I = \bigcup_{a_I \in A} a_I$ *and* $A_{IV} = \bigcup_{a_{IV} \in A} a_{IV}$. *In fact,* $A_I = (A^{\in}, A^{\in^c})$ *and* $A_{IV} = (\varnothing, A^{\not\in})$.

Definition 2.5 ([13, 19])**.** Let X be an non-empty set. Then the form

$$[A^-, A^+] = \{B : A^- \subset B \subset A^+\}$$

is called an interval-valued sets (briefly, IVS) in X, if $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. where $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. In this case, A^- [resp. A^+] represents the set of minimum [resp. maximum] memberships of elements of X to A. In fact, A^- [resp. A^+] is a minimum [resp. maximum] subset of X agreeing or approving for a certain opinion, suggestion or policy. $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the interval-valued empty [resp. whole] set in X and denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$]. We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for classical subset A of X. Then we can consider an IVS in X as the generalization of a classical subset of X. Furthermore, if $A = [A^-, A^+] \in [X]$, then

$$\chi_A = [\chi_{A^-}, \chi_{A^+}]$$

is an interval-valued fuzzy set in X introduced by Zadeh [21]. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS. The inclusion, the equality, the intersection and the union of IVSs and the complement of an IVS refer to [13, 19].

Definition 2.6 ([13])**.** Let X be a non-empty set, let $a \in X$ and let $A \in IVS(X)$. Then the form $[\{a\}, \{a\}]$ [resp. $[\varnothing, \{a\}]$] is called an interval-valued [resp. vanishing] point in X and denoted by a_{IVP} [resp. a_{IVVP}]. We will denote the set of all interval-valued points in X as $IV_P(X)$.

(i) We say that a_{IVP} belongs to A, denoted by $a_{IVP} \in A$, if $a \in A^-$.

(ii) We say that a_{IVVP} belongs to A, denoted by $a_{IVVP} \in A$, if $a \in A^+$.

Result 2.7 ([13], Proposition 3.11)**.** *Let X be a non-empty set and let $A \in IVS(X)$. Then*

$$A = A_{IVP} \cup A_{IVVP},$$

where $A_{IVP} = \bigcup_{a_{IVP} \in A} a_{IV}$ *and* $A_{IVVP} = \bigcup_{a_{IVVP} \in A} a_{IVVP}$.

In fact, $A_{IVP} = [A^-, A^-]$ *and* $A_{IVVP} = [\varnothing, A^+]$

3. Interval-valued intuitionistic sets

In this section, we introduce the notion of interval-valued intuitionistic sets and study some of its properties. Also, we define an ideal of interval-valued intuitionistic sets and obtain some of its properties.

Definition 3.1. Let X be a non-empty set. Then the form

$$A = ([A^{\in,-}, A^{\in,+}], [A^{\not\in,-}, A^{\not\in,+}])$$

is called an interval-valued intuitionistic set (briefly, IVIS) in X, if it satisfies the following conditions:

$$[A^{\in,-}, A^{\in,+}],\ [A^{\not\in,-}, A^{\not\in,+}] \in IVS(X) \text{ and } A^{\in,+} \cap A^{\not\in,+} = \varnothing.$$

In this case, $[A^{\in,-}, A^{\in,+}]$ [resp. $[A^{\not\in,-}, A^{\not\in,+}]$] represents the interval-valued set of memberships [resp. non-memberships] of elements of X to A. In fact, $[A^{\in,-}, A^{\in,+}]$ [resp. $[A^{\not\in,-}, A^{\not\in,+}]$] is an interval-valued set in X agreeing or approving [resp. refusing or opposing] for a certain opinion, suggestion or policy. $(\widetilde{\varnothing}, \widetilde{X})$ [resp. $(\widetilde{X}, \widetilde{\varnothing})$] is called the interval-valued intuitionistic empty [resp. whole] set in X and denoted by $\tilde{\tilde{\varnothing}}$ [resp. $\tilde{\tilde{X}}$]. We will denote the set of all IVISs in X as $IVIS(X)$.

It is clear that $A^{\in,-} \cap A^{\not\in,-} = \varnothing$ for each $A \in IVIS(X)$.

It is obvious that $([A, A], [A^c, A^c]) \in IVIS(X)$ for a classical subset A of X. Then we can consider an IVIS in X as the generalization of a classical subset of X. If $A = ([A^{\in,-}, A^{\in,+}], [A^{\not\in,-}, A^{\not\in,+}]) \in IVIS(X)$, then

$$\chi_A = ([\chi_{A^{\in,-}}, \chi_{A^{\in,+}}], [\chi_{A^{\not\in,-}}, \chi_{A^{\not\in,+}}])$$

is an interval-valued intuitionistic fuzzy set in X (See [2]). Thus we can consider an interval-valued intuitionistic fuzzy set as the generalization of an IVIS. Furthermore, for any IS $A = (A^{\in}, A^{\not\in})$ and any IVS $B = [B^-, B^+]$ in a set X, we may write

$$A = ([A^{\in}, A^{\not\in^c}], [A^{\not\in}, A^{\in^c}]) \text{ and } B = ([B^-, B^+], [B^{+^c}, B^{-^c}]).$$

So we can consider an IVIS as the generalization of both an IS and an IVS. Hence we have the following Figure 1:

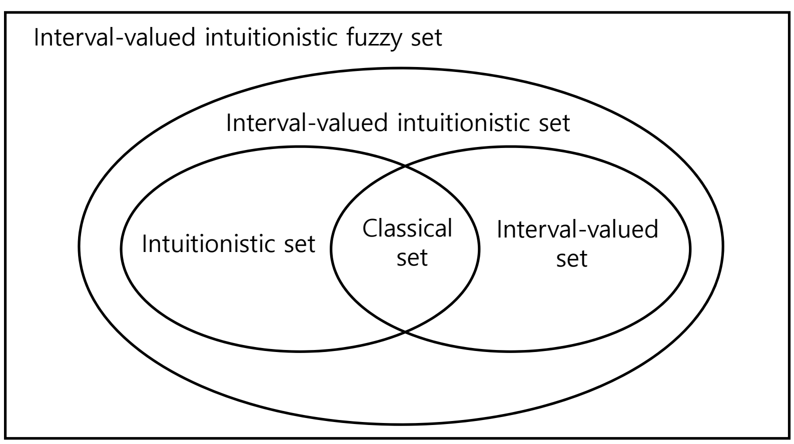

Figure 1.

Example 3.2. Let $X = \{a, b, c\}$. Then we can easily check that

$$([\varnothing, \{a\}], [\varnothing, \{b\}]),\ ([\{a\}, \{a, b\}], [\{c\}, \{c\}]),\ ([\{b\}, \{b\}], [\{c\}, \{a, c\}]) \in IVIS(X).$$

Definition 3.3. Let X be a non-empty set and let $A,\ B \in IVIS(X)$. Then

(i) we say that A contained in B, denoted by $A \subset B$, if it satisfies the following conditions: $A^{\in,-} \subset B^{\in,-}$, $A^{\in,+} \subset B^{\in,+}$, $A^{\not\in,-} \supset B^{\not\in,-}$ and $A^{\not\in,+} \supset B^{\not\in,+}$,

(ii) we say that A equal to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$,

(iii) the complement of A, denoted A^c, is an interval-valued set in X defined by:

$$A^c = ([A^{\not\in,-}, A^{\not\in,+}], [A^{\in,-}, A^{\in,+}]),$$

(iv) the union of A and B, denoted by $A \cup B$, is an interval-valued set in X defined by:

$$A \cup B = ([A^{\in,-} \cup A^{\in,-}, A^{\in,+} \cup A^{\in,+}], [A^{\not\in,-} \cap A^{\not\in,-}, A^{\not\in,+} \cap A^{\not\in,+}]),$$

(v) the intersection of A and B, denoted by $A \cap B$, is an interval-valued set in X defined by:

$$A \cap B = ([A^{\in,-} \cap A^{\in,-}, A^{\in,+} \cap A^{\in,+}], [A^{\not\in,-} \cup A^{\not\in,-}, A^{\not\in,+} \cup A^{\not\in,+}]).$$

(vi) the operations $[\,]$ and $\langle\,\rangle$ on $IVIS(X)$ define as follows: for each $A \in IVS(X)$,

$$[\,]A = ([A^{\in,-}, A^{\in,+}], [A^{\in,+^c}, A^{\in,-^c}]),\ \langle\,\rangle\, A = ([A^{\not\in,+^c}, A^{\not\in,-^c}], [A^{\not\in,-}, A^{\not\in,+}]).$$

Example 3.4. Let $X = \{a, b, c\}$. Consider two IVISs $A = ([\{a\}, \{a, b\}], [\{c\}, \{c\}])$, $B = ([\{b\}, \{b\}], [\{a\}, \{a, c\}])$. Then clearly we have

$A^c = ([\{c\}, \{c\}], [\{a\}, \{a, b\}])$, $A \cup B = ([\{a, b\}, \{a, b\}], [\varnothing, \{c\}])$,
$A \cap B = ([\varnothing, \{b\}], [\{a, c\}, \{a, c\}])$, $[\,]A = ([\{a\}, \{a, b\}], [\{c\}, \{b, c\}])$,
$\langle\,\rangle\, A = ([\{a, b\}, \{a, b\}], [\{c\}, \{c\}])$.

The followings are immediate results of Definition 3.3.

Proposition 3.5 (See [11], Proposition 3.5)**.** *Let X be a non-empty set and let $A,\ B,\ C \in IVIS(X)$. Then*

(1) $\bar{\bar{\varnothing}} \subset A \subset \bar{\bar{X}}$,
(2) *if* $A \subset B$ *and* $B \subset C$, *then* $A \subset C$,
(3) $A \subset A \cup B$ *and* $B \subset A \cup B$,
(4) $A \cap B \subset A$ *and* $A \cap B \subset B$,
(5) $A \subset B$ *if and only if* $A \cap B = A$,
(6) $A \subset B$ *if and only if* $A \cup B = B$.

Proposition 3.6 (See [11], Proposition 3.6)**.** *Let X be a non-empty set and let $A,\ B,\ C \in IVIS(X)$. Then*

(1) (Idempotent laws) $A \cup A = A,\ A \cap A = A$,
(2) (Commutative laws) $A \cup B = B \cup A,\ A \cap B = B \cap A$,
(3) (Associative laws) $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws) $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$,
(6) (DeMorgan's laws) $(A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8_a) $A \cup \bar{\bar{\varnothing}} = A,\ A \cap \bar{\bar{\varnothing}} = \bar{\bar{\varnothing}}$,
(8_b) $A \cup \bar{\bar{X}} = \bar{\bar{X}},\ A \cap \bar{\bar{X}} = A$,
(8_c) $\bar{\bar{X}}^c = \bar{\bar{\varnothing}},\ \bar{\bar{\varnothing}}^c = \bar{\bar{X}}$,

(8_d) $A \cup A^c \neq \tilde{\tilde{X}}$, $A \cap A^c \neq \tilde{\tilde{\varnothing}}$ *in general (See Example 3.7).*

Example 3.7. Let $X = \{a, b, c\}$. Consider an IVIS $A = ([\{a\}, \{a, b\}], [\{c\}, \{c\}]) \in [X]$. Then clearly, $A^c = ([\{c\}, \{c\}], [\{a\}, \{a, b\}])$. Thus we have

$$A \cap A^c = ([\varnothing, \varnothing], [\{a, c\}, X]) \neq \tilde{\tilde{\varnothing}} \text{ and } A \cup A^c = ([\{a, c\}, X], [\varnothing, \varnothing]) \neq \tilde{\tilde{X}}.$$

Definition 3.8. Let $(A_j)_{j\in J}$ be a family of members of $IVIS(X)$. Then

(i) the intersection of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$, is an interval-valued set in X defined by:

$$\bigcap_{j\in J} A_j = ([\bigcap_{j\in J} A_j^{\in,-}, \bigcap_{j\in J} A_j^{\in,+}], [\bigcup_{j\in J} A_j^{\not\in,-}, \bigcup_{j\in J} A_j^{\not\in,+}]),$$

(ii) the union of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} \widetilde{A}_j$, is an interval-valued set in X defined by:

$$\bigcup_{j\in J} A_j = ([\bigcup_{j\in J} A_j^{\in,-}, \bigcup_{j\in J} A_j^{\in,+}], [\bigcap_{j\in J} A_j^{\not\in,-}, \bigcap_{j\in J} A_j^{\not\in,+}]).$$

The following is the immediate result of Definition 3.8.

Proposition 3.9 (See [11], Proposition 3.7)**.** *Let $A \in [X]$ and let $(A_j)_{j\in J}$ be a family of members of $IVIS(X)$. Then*

(1) $(\bigcap_{j\in J} A_j)^c = \bigcup_{j\in J} A_j^c$, $(\bigcup_{j\in J} A_j)^c = \bigcap_{j\in J} A_j^c$,
(2) $A \cap (\bigcup_{j\in J} A_j) = \bigcup_{j\in J}(A \cap A_j)$, $A \cup (\bigcap_{j\in J} A_j) = \bigcap_{j\in J}(A \cup A_j)$.

From Propositions 3.6 and 3.9, we can easily see that $(IVIS(X), \cup, \cap, ^c, \tilde{\tilde{\varnothing}}, \tilde{\tilde{X}})$ forms a Boolian algebra except the property (8_d).

Definition 3.10. Let X be a non-empty set, let $a \in X$ and let $A \in IVIS(X)$. Then the form $([\{a\}, \{a\}], [\{a\}^c, \{a\}^c])$ [resp. $([\varnothing, \{a\}], [\{a\}^c, \{a\}^c])$] is called an interval-valued intuitionistic [resp. vanishing] point in X and denoted by a_{IVI} [resp. a_{IVIV}]. We will denote the set of all interval-valued points in X as $IVIP(X)$.

(i) We say that a_{IVI} belongs to A, denoted by $a_{IVI} \in A$, if $a \in A^{\in,-}$.
(ii) We say that a_{IVIV} belongs to A, denoted by $a_{IVIV} \in A$, if $a \not\in A^{\not\in,+}$.

It is obvious that if $a_{IVIV} \in A$, then $a \not\in A^{\not\in,+}$ and if $a_{IVI} \in A$, then $a \in A^{\in,+}$.

Proposition 3.11. *Let X be a non-empty set and let $A \in IVIS(X)$. Then*

$$A = A_{IVI} \cup A_{IVIV},$$

where $A_{IVI} = \bigcup_{a_{IVI}\in A} a_{IVI}$ and $A_{IVIV} = \bigcup_{a_{IVIV}\in A} a_{IVIV}$.

In fact, $A_{IVI} = ([A^{\in,-}, A^{\in,-}], [A^{\not\in,+}, A^{\not\in,+}])$ and $A_{IVIV} = ([\varnothing, A^{\in,+}], [A^{\not\in,-}, A^{\not\in,+}])$.

Proof. From Definition 3.10 and the definitions of A_{IVI} and A_{IVIV}, we have

$$\begin{aligned} A_{IVI} &= \textstyle\bigcup_{a_{IVI}\in A} a_{IVI} \\ &= ([\textstyle\bigcup_{a_{IVI}\in A}\{a\}, \bigcup_{a_{IVI}\in A}\{a\}], [\bigcap_{a_{IVI}\in A}\{a\}^c, \bigcap_{a_{IVI}\in A}\{a\}^c]) \\ &= ([\textstyle\bigcup_{a\in A^{\in,-}}\{a\}, \bigcup_{a\in A^{\in,-}}\{a\}], [\bigcap_{a\in A^{\in,-}}\{a\}^c, \bigcap_{a\in A^{\in,-}}\{a\}^c]) \\ &= ([\textstyle\bigcup_{a\in A^{\in,-}}\{a\}, \bigcup_{a\in A^{\in,-}}\{a\}], [\bigcap_{a\not\in A^{\not\in,+}}\{a\}^c, \bigcap_{a\not\in A^{\not\in,+}}\{a\}^c]) \\ &= ([A^{\in,-}, A^{\in,-}], [A^{\not\in,+}, A^{\not\in,+}]) \end{aligned}$$

and

$$\begin{aligned} A_{IVIV} &= \bigcup_{a_{IVIV} \in A} a_{IVIV} \\ &= ([\varnothing, \bigcup_{a_{IVIV} \in A}\{a\}], [\bigcap_{a_{IVIV} \in A}\{a\}^c, \bigcap_{a_{IVIV} \in A}\{a\}^c]) \\ &= ([\varnothing, \bigcup_{a \in A^{\in,+}}\{a\}], [\bigcap_{a \notin A^{\notin,-}}\{a\}^c, \bigcap_{a \notin A^{\notin,+}}\{a\}^c]) \\ &= ([\varnothing, A^{\in,+}], [A^{\notin,-}, A^{\notin,+}]). \end{aligned}$$

Then $A = A_{IVI} \cup A_{IVIV}$. □

Example 3.12. Let $X = \{a, b, c, d, e, f, g, h, i\}$. Consider an IVIS

$$A = ([\{a, b, c\}, \{a, b, c, d, e\}], [\{f, g\}, \{f, g, h\}].$$

Then clearly, we have

$$a_{IVI},\ b_{IVI}, c_{IVI} \in A \text{ and } a_{IVIV},\ b_{IVIV},\ c_{IVIV},\ d_{IVIV},\ e_{IVIV},\ i_{IVIV} \in A.$$

Thus we can easily calculate the followings:

$$A_{IVI} = ([\{a, b, c\}, \{a, b, c\}], [\{f, g, h\}, \{f, g, h\}] = ([A^{\in,-}, A^{\in,-}], [A^{\notin,+}, A^{\notin,+}])$$

and

$$A_{IVIV} = ([\varnothing, \{a, b, c, d, e\}], [\{f, g\}, \{f, g, h\}]) = ([\varnothing, A^{\in,+}], [A^{\notin,-}, A^{\notin,+}]).$$

So we can confirm that Proposition 3.11 holds.

Theorem 3.13. *Let $(A_j)_{j \in J} \subset IVIS(X)$ and let $a \in X$.*
(1) $a_{IVI} \in \bigcap_{j \in J} A_j$ [resp. $a_{IVIV} \in \bigcap_{j \in J} A_j$] if and only if $a_{IVI} \in A_j$ [resp. $a_{IVIV} \in A_j$] for each $j \in J$.
(2) $a_{IVI} \in \bigcup_{j \in J} A_j$ [resp. $a_{IVIV} \in \bigcup_{j \in J} A_j$] if and only if there exists $j \in J$ such that $a_{IVI} \in A_j$ [resp. $a_{IVIV} \in A_j$].

Proof. Straightforward. □

Theorem 3.14. *Let $A, B \in IVIS(X)$. Then*
(1) $A \subset B$ if and only if $a_{IVI} \in A \Rightarrow a_{IVI} \in B$ [resp. $a_{IVIV} \in A \Rightarrow a_{IVIV} \in B$] for each $a \in X$.
(2) $A = B$ if and only if $a_{IVI} \in A \Leftrightarrow a_{IVI} \in B$ [resp. $a_{IVIV} \in A \Leftrightarrow a_{IVIV} \in B$] for each $a \in X$.

Proof. Straightforward. □

Definition 3.15. Let X, Y be two non-empty sets, let $f : X \to Y$ be a mapping and let $A \in IVIS(X)$, $B \in IVIS(Y)$.
(i) The image of A under f, denoted by $f(A)$, is an interval set in Y defined as:

$$f(A) = ([f(A^{\in,-}), f(A^{\in,+})], [f(A^{\notin,-}), f(A^{\notin,+})]).$$

(ii) The preimage of B under f, denoted by $f^{-1}(B)$, is an interval set in X defined as:

$$f^{-1}(B) = ([f^{-1}(B^{\in,-}), f^{-1}(B^{\in,+})], [f^{-1}(B^{\notin-}), f^{-1}(B^{\notin,+})]).$$

It is obvious that $f(a_{IVI}) = f(a)_{IVI}$ and $f(a_{IVIV}) = f(a)_{IVIV}$ for each $a \in X$.

Proposition 3.16. *Let X, Y be two non-empty sets, let $f : X \to Y$ be a mapping, let $A,\ A_1,\ A_2 \in IVIS(X)$, $(A_j)_{j\in J} \subset IVIS(X)$ and let $B,\ B_1,\ B_2 \in IVIS(Y)$, $(A_j)_{j\in J} \subset IVIS(Y)$. Then*
(1) *if $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$,*
(2) *if $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_1)$,*
(3) *$A \subset f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A))$,*
(4) *$f(f^{-1}(B)) \subset B$ and if f is surjective, $f(f^{-1}(B)) = B$,*
(5) *$f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$,*
(6) *$f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$,*
(7) *$f(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} f(A_j)$,*
(8) *$f(\bigcap_{j\in J} A_j) \subset \bigcap_{j\in J} f(A_j)$ and if f is injective, then $f(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} f(A_j)$,*
(9) *if f is surjective, then $f(A)^c \subset f(A^c)$.*
(10) *$f^{-1}(B^c) = f^{-1}(B)^c$.*
(11) *$f^{-1}(\tilde{\tilde{\varnothing}}) = \tilde{\tilde{\varnothing}}$, $f^{-1}(\tilde{\tilde{X}}) = \tilde{\tilde{X}}$,*
(12) *$f(\tilde{\tilde{\varnothing}}) = \tilde{\tilde{\varnothing}}$ and if f is surjective, then $f(\tilde{\tilde{X}}) = \tilde{\tilde{X}}$,*
(13) *if $g : Y \to Z$ is a mapping, then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for each $C \in IVIS(Z)$.*

Proof. The proofs are straightforward. □

Definition 3.17. Let X be a non-empty sets and let L be a non-empty family of IVISs in X. Then L^i is called an interval-valued intuitionistic ideal (briefly, IVII) on X, provided that it satisfies the following conditions: for any $A,\ B \in IVIS(X)$,
(i) (Heredity) if$A \in L$ and $B \subset A$, then $B \in L$,
(ii) (Finite additivity) if $A,\ B \in L$, then $A \cup B \in L$.

An IVII L is called a σ-interval-valued intuitionistic ideal (briefly, σ-IVII), provided that it satisfies the following condition:
(Countable additivity) if $(A_n)_{n\in\mathbb{N}} \subset L$, then $\bigcup_{n\in\mathbb{N}} A_n \in L$.

In particular, an IVII L is said to be proper [resp. improper], if $\tilde{\tilde{X}} \notin L$ [resp. $\tilde{\tilde{X}} \in L$].

It is obvious that $\tilde{\tilde{\varnothing}} \in L$ and for each $\tilde{\tilde{\varnothing}} \neq A \in IVIS(X)$,

$$\{B \in IVIS(X) : B \subset A\}$$

is an IVII on X. In this case, we will write $\{B \in IVIS(X) : B \subset A\} = IVII(A)$ and call it as the principal IVII of A, and A is called a base of $IVII(A)$.

We will denote the IVII of IVISs in X having finite [resp. countable] support of X, as $IVII_f$ [resp. $IVII_c$] and the set of all IVIIs on X as $IVII(X)$.

Example 3.18. Let $X = \{a, b, c\}$ and consider the collection of IVISs L in X given by:

$$L = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}\},$$

where $A_1 = ([\{a\}, \{a, b\}], [\{c\}, \{c\}]),\ A_2 = ([\{a\}, \{a, b\}], [\varnothing, \{c\}]),$
$A_3 = ([\{a\}, \{a, b\}], [\varnothing, \varnothing]),\ A_4 = ([\{a\}, \{a\}], [\{c\}, \{c\}]),$
$A_5 = ([\{a\}, \{a\}], [\varnothing, \{c\}]),\ A_6 = ([\{a\}, \{a\}], [\varnothing, \varnothing]),$
$A_7 = ([\varnothing, \{a, b\}], [\{c\}, \{c\}]),\ A_8 = ([\varnothing, \{a, b\}], [\varnothing, \{c\}]),$
$A_9 = ([\varnothing, \{a, b\}], [\varnothing, \varnothing]),\ A_{10} = ([\varnothing, \{a\}], [\{c\}, \{c\}]),$
$A_{11} = ([\varnothing, \{a\}], [\varnothing, \{c\}]),\ A_{12} = ([\varnothing, \{a\}], [\varnothing, \varnothing]),$
$A_{13} = ([\varnothing, \{b\}], [\{c\}, \{c\}]),\ A_{14} = ([\varnothing, \{b\}], [\varnothing, \{c\}]),$
$A_{15} = ([\varnothing, \{b\}], [\varnothing, \varnothing]),\ A_{16} = ([\varnothing, \varnothing], [\{c\}, \{c\}]),$
$A_{17} = ([\varnothing, \varnothing], [\varnothing, \{c\}]),\ A_{18} = ([\varnothing, \varnothing], [\varnothing, \varnothing]) = \tilde{\bar{\bar{\varnothing}}}.$
Then we can easily check that L is an IVII on X.

Definition 3.19. Let $L_1,\ L_2$ be two IVIIs on a non-empty set X. Then
(i) L_2 is said to be finer than L_1 or L_1 is coarser then L_2, if $L_1 \subset L_2$,
(i) L_2 is said to be strictly finer than L_1 or L_1 is strictly coarser then L_2, if $L_1 \subset L_2$ and $L_1 \neq L_2$,
(iii) L_1 and L_2 are said to be comparable, if one is finer than the other.

It is clear that $(IVII(X), \subset)$ is a poset. Furthermore, $\{\tilde{\bar{\bar{\varnothing}}}\}$ [resp. $IVII(X)$] is the smallest [resp. largest] IVII on X.

The following is the immediate result of Definitions 3.3 and 3.17.

Proposition 3.20. *Let X be a non-empty set and let $(L_j)_{j\in J}$ be a non-empty family of IVIIs on X. Then $\bigcap_{j\in J} L_j,\ \bigcup_{j\in J} L_j \in IVII(X)$.*

In fact, $\bigcap_{j\in J} L_j = \inf_{j\in J} L_j$ and $\bigcup_{j\in J} L_j = \sup_{j\in J} L_j$.

The following is the immediate result of Definition 3.17.

Theorem 3.21. *Let X be a non-empty set, $A \in IVIS(X)$ and let $L \in IVII(X)$. Then A is a base of L if and only if $B \subset A$ for each $B \in L$*

Theorem 3.22. *Let X be a non-empty set and $A,\ B \in IVIS(X)$. Let L_1 be an IVII on X with a base A and let L_2 be an IVII on X with a base B. Then L_1 is finer than L_2 if and only if $B \subset A$ for each $C \in IVII(X)$ such that $C \subset B$.*

Proof. The proof is straightforward from Definition 3.19 □

The following is the immediate result of Theorem 3.22.

Corollary 3.23. *Let X be a non-empty set and $A,\ B \in IVIS(X)$. Let L_1 be an IVII on X with a base A and let L_2 be an IVII on X with a base B. Then A and B are equivalent if and only if $C \subset A$ for each $C \in IVIS(X)$ such that $C \subset B$ and $D \subset B$ for each $D \in IVIS(X)$ such that $D \subset A$.*

Proposition 3.24. *Let X be a non-empty set and let $\eta = (A_j)_{j\in J}$ be a non-empty family of IVISs in X. Then there is an IVII $L(\eta)$ on X, where*

$$L(\eta) = \{A \in IVIS(X) : A \subset \bigcup_{j\in J} A_j,\ J \text{ is finite}\}.$$

Proof. The proof is straightforward from Definition 3.19 □

4. Interval-valued intuitionistic topological spaces

In this section, we define an interval-valued intuitionistic topology on a non-empty set X, and study some of its properties, and give some examples. Also, we introduce the concepts of an interval-valued intuitionistic base and subbase, and a family of IVISs obtains the necessary and sufficient conditions to become an IVIB, and gives some examples.

Definition 4.1 ([7, 12])**.** Let X be a non-empty set and let $\tau \subset IS(X)$. Then τ is called an intuitionistic topology (briefly, IT) on X, it satisfies the following axioms:
(IO_1) $\bar{\varnothing}, \bar{X} \in \tau$,
(IO_2) $A \cap B \in \tau$, for any $A, B \in \tau$,
(IO_3) $\bigcup_{j \in J} A_j \in \tau$, for each $(A_j)_{j \in J} \subset \tau$.
In this case, the pair (X, τ) is called an intuitionistic topological space (briefly, ITS) and each member O of τ is called an intuitionistic open set (briefly, IOS) in X. An IS F of X is called an intuitionistic closed set (briefly, ICS) in X, if $F^c \in \tau$.

It is obvious that $\{\phi_I, X_I\}$ is the smallest IT on X and will be called the intuitionistic indiscreet topology and denoted by $\tau_{I,0}$. Also $IS(X)$ is the greatest IT on X and will be called the intuitionistic discreet topology and denoted by $\tau_{I,1}$. The pair $(X, \tau_{I,0})$ [resp. $(X, \tau_{I,1})$] will be called the intuitionistic indiscreet [resp. discreet] space.

We will denote the set of all ITs on X as $IT(X)$. For an ITS X, we will denote the set of all IOSs [resp. ICSs] on X as $IO(X)$ [resp. $IC(X)$].

Definition 4.2 ([13])**.** Let X be a non-empty set and let τ be a non-empty family of IVSs on X. Then τ is called an interval-valued topology (briefly, IVT) on X, if it satisfies the following axioms:
(IVO_1) $\widetilde{\varnothing},\ \widetilde{X} \in \tau$,
(IVO_2) $A \cap B \in\in \tau$ for any $A,\ B \in\in \tau$,
(IVO_3) $\bigcup_{j \in J} A_j \in\in \tau$ for any family $(A_j)_{j \in J}$ of members of τ.
In this case, the pair (X, τ) is called an interval-valued topological space (briefly, IVTS) and each member of τ is called an interval-valued open set (briefly, IVOS) in X. A IVS A is called an interval-valued closed set (briefly, IVCS) in X, if $A^c \in \tau$.

It is obvious that $\{\widetilde{\varnothing}, \widetilde{X}\}$ is an IVT on X, and will be called the interval-valued indiscrete topology on X and denoted by $\tau_{IV,0}$. Also $IVS(X)$ is an IVT on X, and will be called the interval-valued discrete topology on X and denoted by $\tau_{IV,1}$. The pair $(X, \tau_{IV,0})$ [resp. $(X, \tau_{IV,1})$] will be called the interval-valued indiscrete [resp. discrete] space.

We will denote the set of all IVTs on X as $IVT(X)$. for an IVTS X, we will denote the set of all IVOSs [resp. IVCSs] in X as $IVO(X)$ [resp. $IVC(X)$].

Definition 4.3. Let X be a non-empty set and let τ be a non-empty family of IVISs on X. Then τ is called an interval-valued intuitionistic topology (briefly, IVIT) on X, if it satisfies the following axioms:

(IVIO$_1$) $\bar{\tilde{\varnothing}},\ \bar{\tilde{X}} \in \tau$,
(IVIO$_2$) $A \cap B \in\in \tau$ for any $A,\ B \in\in \tau$,
(IVIO$_3$) $\bigcup_{j\in J} A_j \in\in \tau$ for any family $(A_j)_{j\in J}$ of members of τ.

In this case, the pair (X, τ) is called an interval-valued intuitionistic topological space (briefly, IVITS) and each member of τ is called an interval-valued intuitionistic open set (briefly, IVIOS) in X. A IVIS A is called an interval-valued intuitionistic closed set (briefly, IVICS) in X, if $A^c \in \tau$.

It is obvious that $\{\bar{\tilde{\varnothing}}, \bar{\tilde{X}}\}$ is an IVIT on X, and will be called the interval-valued intuitionistic indiscrete topology on X and denoted by $\tau_{IVI,0}$. Also $IVIS(X)$ is an IVIT on X, and will be called the interval-valued intuitionistic discrete topology on X and denoted by $\tau_{IVI,1}$. The pair $(X, \tau_{IVI,0})$ [resp. $(X, \tau_{IVI,1})$] will be called the interval-valued intuitionistic indiscrete [resp. discrete] space.

We will denote the set of all IVITs on X as $IVIT(X)$. For an IVITS X, we will denote the set of all IVIOSs [resp. IVICSs] in X as $IVIO(X)$ [resp. $IVIC(X)$].

We can easily see that for each $\tau \in IVIT(X)$, the family

$$\chi_\tau = \{\chi_A : \chi_A = ([\chi_{A^{\in,-}}, \chi_{A^{\in,+}}], [\chi_{A^{\not\in,-}}, \chi_{A^{\not\in,+}}]),\ A \in \tau\}$$

is an interval-valued intuitionistic fuzzy topology on X introduced by Samanta and Mondal [16].

Remark 4.4. (1) For each $\tau \in IVIT(X)$, consider two families of ISs and two families of IVSs in X, respectively given by:

$$\tau^- = \{(A^{\in,-}, A^{\not\in,-}) \in IS(X) : A \in \tau\},\ \tau^+ = \{(A^{\in,+}, A^{\not\in,+}) \in IS(X) : A \in \tau\}$$

and

$$\tau^\in = \{[A^{\in,-}, A^{\in,+}] \in IVS(X) : A \in \tau\},\ \tau^{\not\in} = \{[A^{\not\in,+^c}, A^{\not\in,-^c}] \in IVS(X) : A \in \tau\}.$$

Then we can easily check that $\tau^-,\ \tau^+ \in IT(X)$ and $\tau^\in,\ \tau^{\not\in} \in IVT(X)$.

In this case, the pair (τ^-, τ^+) [resp. $(\tau^\in, \tau^{\not\in})$] will be called an intuitionistic [resp. interval-valued] bitopology on X (See [10]).

Now let us consider the following families of subsets of X given by:

$$\tau^{\in,-} = \{A^{\in,-} \subset X : A \in \tau\},\ \tau^{\in,+} = \{A^{\in,+} \subset X : A \in \tau\},$$
$$\tau^{\not\in,-} = \{A^{\not\in,-^c} \subset X : A \in \tau\},\ \tau^{\not\in,+} = \{A^{\not\in,+^c} \subset X : A \in \tau\}.$$

Then clearly, $\tau^{\in,-}$ [resp. $\tau^{\in,+}$, $\tau^{\not\in,-}$ and $\tau^{\not\in,+}$] forms an ordinary topology on X.

(2) Let (X, τ_o) be an ordinary topological space. Then clearly,

$$\tau = \{([A, A], [A^c, A^c]) \in IVIS(X) : A \in \tau_o\} \in IVIT(X).$$

(3) Let (X, τ) be an ordinary topological space such that τ is not indiscrete. Then there are two IVITs on X given by:

$$\tau^1 = \{([G, G], [G^c, G^c]) \in IVIS(X) : G \in \tau\},\ \tau^2 = \{([\varnothing, G], [\varnothing, G^c]) : G \in \tau\}.$$

(4) Let τ_I be an intuitionistic topology on a set X in the sense of Coker [7]. Then we can easily see that the following families are IVITs on X:

$$\tau_{I,1} = \{([A^\in, A^\in], [A^{\not\in}, A^{\not\in}]) \in IVIS(X) : A \in \tau_I\},$$

$$\tau_{I,2} = \{([A^{\in}, A^{\not\in c}], [A^{\not\in}, A^{\not\in}]) \in IVIS(X) : A \in \tau_I\},$$
$$\tau_{I,3} = \{([A^{\in}, A^{\not\in c}], [A^{\not\in}, A^{\in c}]) \in IVIS(X) : A \in \tau_I\}.$$

(5) Let τ_{IV} be an interval-valued topology on a set X in the sense of Kim et al. [13]. Then we can easily see that the following families are IVITs on X:

$$\tau_{IV,1} = \{([A^-, A^+], [A^{+c}, A^{-c}]) \in IVIS(X) : A \in \tau_{IV}\},$$
$$\tau_{IV,2} = \{([A^-, A^+], [A^{+c}, A^{+c}]) \in IVIS(X) : A \in \tau_{IV}\},$$
$$\tau_{IV,3} = \{([A^-, A^+], [A^{-c}, A^{-c}]) \in IVIS(X) : A \in \tau_{IV}\}.$$

(6) Let (X, τ) be an IVITS and consider two families of IVISs in given by:

$$[\ ]\tau = \{[\ ]A : A \in \tau\},\ \langle\ \rangle\tau = \{\langle\ \rangle A : A \in \tau\}.$$

Then we can easily check that $[\ ]\tau,\ \langle\ \rangle\tau \in IVIT(X)$.

From Remark 4.4, we have the following Figure 2:

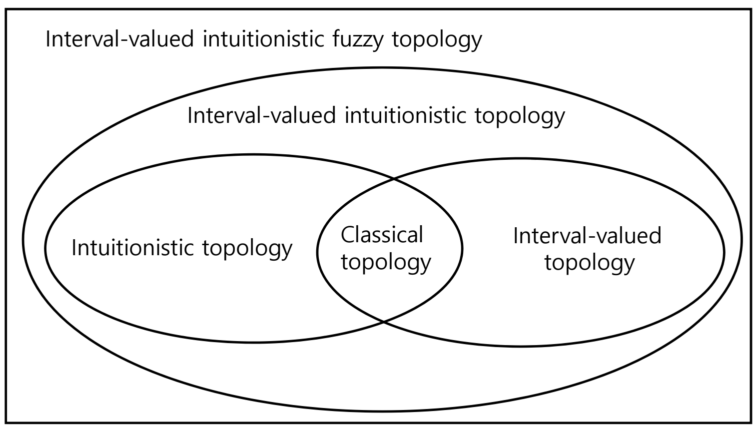

FIGURE 2.

Example 4.5. (1) Let $X = \{a, b\}$. Then clearly, we have

$$\tau_{IVI,1} = \{\tilde{\tilde{\varnothing}}, a_{IVI}, b_{IVI}, a_{IVIV}, b_{IVIV}, ([\{a\}, X], \tilde{\varnothing}), \tilde{\tilde{X}}\}.$$

(2) Let X be a set and let $A \in IVIS(X)$. Then A is said to be finite, if $A^{\in,+}$ is finite. Consider the family $\tau = \{U \in IVIS(X) : U = \tilde{\tilde{\varnothing}}$ or U^c is finite$\}$. Then we can easily check that $\tau \in IVT(X)$. In this case, τ will be called an interval-valued intuitionistic cofinite topology (briefly, IVICFT) on X.

(3) Let X be a set and let $A \in IV(X)$. Then A is said to be countable, if $A^{\in,+}$ is countable. Consider the family $\tau = \{U \in IV(X) : U = \tilde{\tilde{\varnothing}}$ or U^c is countable$\}$. Then we can easily prove that $\tau \in IVIT(X)$. In this case, τ will be called an interval-valued intuitionistic cocountable topology (briefly, IVICCT) on X.

(4) Let $X = \{a, b, c, d, e, f, g, h\}$ and the consider the family τ of IVISs in X given by:

$$\tau = \{\tilde{\tilde{\varnothing}}, A_1, A_2, A_3, A_4, \tilde{\tilde{X}}\},$$

where $A_1 = ([\{a\},\{a,b\}],[\{f\},\{f,g\}])$, $A_2 = ([\{a,c,d\},\{a,b,c,d\}],[\{f,h\},\{f,g,h\}])$,
$A_3 = ([\{a\},\{a,b\}],[\{f,h\},\{f,g,h\}])$, $A_4 = ([\{a,c,d\},\{a,b,c,d\}],[\{f\},\{f,g\}])$.
Then we can easily check that τ is an IVIT on X.

The following is the immediate result of Definition 4.3

Proposition 4.6. *Let X be an IVITS. Then*
(1) $\bar{\tilde{\varnothing}},\ \bar{\tilde{X}} \in IVIC(X)$,
(2) $A \cup B \in IVIC(X)$ *for any* $A,\ B \in IVIC(X)$,
(3) $\bigcap_{j\in J} A_j \in IVIC(X)$ *for any* $(A_j)_{j\in J} \subset IVIC(X)$.

Definition 4.7. Let X be a non-empty set and let $\tau_1,\ \tau_1 \in IVIT(X)$. Then we say that τ_1 is contained in τ_2 or τ_1 is coarser than τ_2 or τ_2 is finer than τ_1, if $\tau_1 \subset \tau_2$, i.e., $A \in \tau_2$ for each $A \in \tau_1$.

It is obvious that $\tau_{IVI,0} \subset \tau \subset \tau_{IVI,1}$ for each $\tau \in IVIT(X)$.
The following is the immediate result of Definitions 3.8 and 4.3.

Proposition 4.8. *Let $(\tau_j)_{j\in J} \subset IVIT(X)$. Then $\bigcap_{j\in J} \tau_j \in IVIT(X)$.*
In fact, $\bigcap_{j\in J} \tau_j$ is the coarsest IVIT on X containing each τ_j.

Proposition 4.9. *Let $\tau,\ \gamma \in IVIT(X)$. We define $\tau \wedge \gamma$ and $\tau \vee \gamma$ as follows:*

$$\tau \wedge \gamma = \{W : W \in \tau,\ W \in \gamma\},$$
$$\tau \vee \gamma = \{W : W = U \cup V,\ U \in \tau,\ V \in \gamma\}.$$

Then we have
(1) *$\tau \wedge \gamma$ is an IVIT on X which is the finest IVIT coarser than both τ and γ,*
(2) *$\tau \vee \gamma$ is an IVIT on X which is the coarsest IVIT finer than both τ and γ,*

Proof. (1) It is clear that $\tau \wedge \gamma \in IVT(X)$. Let η be any IVIT on X which is coarser than both τ and γ, and let $W \in \eta$. Then clearly, $W \in \tau$ and $W \in \gamma$. Thus $W \in \tau \wedge \gamma$. So η is coarser than $\tau \wedge \gamma$.
(2) The proof is similar to (1). □

Definition 4.10. Let (X, τ) be an IVITS.
(i) A subfamily β of τ is called an interval-valued intuitionistic base (briefly, IVIB) for τ, if for each $A \in \tau$, $A = \bar{\tilde{\varnothing}}$ or there is $\beta^{'} \subset \beta$ such that $A = \bigcup \beta^{'}$.
(ii) A subfamily σ of τ is called an interval-valued intuitionistic subbase (briefly, IVISB) for τ, if the family $\beta = \{\bigcap \sigma^{'} : \sigma^{'}$ is a finite subset of $\sigma\}$ is an IVIB for τ.

Remark 4.11. (1) Let β be an IVIB for an IVIT τ on a non-empty set X and consider the families of intuitionistic [resp. interval-valued] sets in X given by:

$$\beta^{-} = \{(A^{\in,-}, A^{\notin,-}) \in IS(X) : A \in \beta\},\ \beta^{+} = \{(A^{\in,+}, A^{\in,+}) \in IS(X) : A \in \beta\}$$

and

$$\beta^{\in} = \{[A^{\in,-}, A^{\in,+}] \in IVS(X) : A \in \beta\},\ \beta^{\notin} = \{[A^{\notin,+^{c}}, A^{\notin,-^{c}}] \in IVS(X) : A \in \beta\}.$$

Then we can easily see that β^{-} [resp. β^{+}] is an IB for τ^{-} [resp. τ^{+}] and $\beta^{\in}$ [resp. $\beta^{\in}$] is an IVB for $\tau^{\in}$ [resp. $\tau^{\notin}$].
Now let us consider the following families of subsets of X given by:

$$\beta^{\in,-} = \{A^{\in,-} \subset X : A \in \beta\},\ \beta^{\in,+} = \{A^{\in,+} \subset X : A \in \beta\},$$

$$\beta^{\notin,-} = \{A^{\notin,-^c} \subset X : A \in \beta\},\ \beta^{\notin,+} = \{A^{\notin,+^c} \subset X : A \in \beta\}.$$

Then clearly, $\beta^{\in,-}$ [resp. $\beta^{\in,+}$, $\beta^{\notin,-}$ and $\beta^{\notin,+}$] is an ordinary base for the ordinary topology $\tau^{\in,-}$ [resp. $\tau^{\in,+}$, $\tau^{\notin,-}$ and $\tau^{\notin,+}$].

(2) Let σ be an IVISB for an IVIT τ on a non-empty set X and consider the families of intuitionistic [resp. interval-valued] sets in X given by:

$$\sigma^{-} = \{(A^{\in,-}, A^{\notin,-}) \in IS(X) : A \in \sigma\},\ \sigma^{+} = \{(A^{\in,+}, A^{\in,+}) \in IS(X) : A \in \sigma\}$$

and

$$\sigma^{\in} = \{[A^{\in,-}, A^{\in,+}] \in IVS(X) : A \in \sigma\},\ \sigma^{\notin} = \{[A^{\notin,+^c}, A^{\notin,-^c}] \in IVS(X) : A \in \sigma\}.$$

Then we can easily see that σ^{-} [resp. σ^{+}] is an ISB for τ^{-} [resp. τ^{+}] and $\sigma^{\in}$ [resp. $\sigma^{\in}$] is an IVSB for $\tau^{\in}$ [resp. $\tau^{\notin}$].

Now let us consider the following families of subsets of X given by:

$$\sigma^{\in,-} = \{A^{\in,-} \subset X : A \in \sigma\},\ \sigma^{\in,+} = \{A^{\in,+} \subset X : A \in \sigma\},$$

$$\sigma^{\notin,-} = \{A^{\notin,-^c} \subset X : A \in \sigma\},\ \sigma^{\notin,+} = \{A^{\notin,+^c} \subset X : A \in \sigma\}.$$

Then clearly, $\sigma^{\in,-}$ [resp. $\sigma^{\in,+}$, $\sigma^{\notin,-}$ and $\sigma^{\notin,+}$] is an ordinary subbase for the ordinary topology $\tau^{\in,-}$ [resp. $\tau^{\in,+}$, $\tau^{\notin,-}$ and $\tau^{\notin,+}$].

Example 4.12. (1) Let $\sigma = \{([(a,b),(a,\infty)],[(-\infty,a],(-\infty,a]]) : a,\ b \in \mathbb{R},\ a \leq b\}$ be the family of IVISs in $\mathbb{R}$. Then σ generates an IVIT τ on $\mathbb{R}$ which will be called the "usual left interval-valued intuitionistic topology (briefly, ULIVIT)" on $\mathbb{R}$. In fact, the IVIB β for τ can be written in the form:

$$\beta = \{\bar{\tilde{\mathbb{R}}}\} \cup \{\cap_{j \in J} S_j : S_j \in \sigma,\ J \text{ is finite}\}$$

and τ consists of the following IVISs in $\mathbb{R}$:

$$\tau = \{\bar{\tilde{\varnothing}}, \bar{\tilde{\mathbb{R}}}, ([\cup(a_j,b_j),(c,\infty)],[(-\infty,c],(-\infty,c]]), ([\cup(a_k,b_k),\mathbb{R}],\widetilde{\mathbb{R}})\},$$

where $a_j, b_j, c \in \mathbb{R}$, $\{a_j : j \in J\}$ is bounded from below, $c < inf\{a_j : j \in J\}$ and $a_k, b_k \in \mathbb{R}$, $\{a_k : k \in K\}$ is not bounded from below.

Similarly, one can define the "usual right interval-valued topology (briefly, URIVT)" on $\mathbb{R}$ using an analogue construction.

(2) Consider the family σ of IVISs in $\mathbb{R}$ given by:

$$\sigma = \{([(a,b),(a_1,\infty) \cap (-\infty,b_1)],[(-\infty,a_1] \cup [b_1,\infty),(-\infty,a_1] \cup [b_1,\infty)]) : a,\ b,\ a_1,\ b_1 \in \mathbb{R},\ a_1 \leq a,\ b_1 \geq b\}.$$

Then σ generates an IVIT τ on $\mathbb{R}$ which will be called the "usual interval-valued intuitionistic topology (briefly, UIVIT)" on $\mathbb{R}$. In fact, the IVIB β for τ can be written in the form:

$$\beta = \{\bar{\tilde{\mathbb{R}}}\} \cup \{\cap_{j \in J} S_j : S_j \in \sigma,\ J \text{ is finite}\}$$

and the elements of τ can be easily written down as in (1).

(3) Consider the family $\sigma_{[0,1]}$ of IVISs in $\mathbb{R}$ given by:

$$\sigma_{[0,1]} = \{([[a,b],[a,\infty) \cap (-\infty,b]],[(-\infty,a) \cup (b,\infty),(-\infty,a) \cup (b,\infty)]) : a,b \in \mathbb{R} \text{ and } 0 \leq a \leq b \leq 1\}.$$

Then $\sigma_{[0,1]}$ generates an IVIT $\tau_{[0,1]}$ on $\mathbb{R}$, which will be called the "usual unit closed interval interval-valued intuitionistic topology" on $\mathbb{R}$. In fact, the IVIB $\beta_{[0,1]}$ for $\tau_{[0,1]}$ can be written in the form:

$$\beta_{[0,1]} = \{\widetilde{\mathbb{R}}\} \cup \{\cap_{j\in J} S_j : S_j \in \sigma_{[0,1]},\ J \text{ is finite}\}$$

and the elements of τ can be easily written down as in (1).

In this case, $([0,1], \tau_{[0,1]})$ is called the "interval-valued intuitionistic usual unit closed interval" and will be denoted by $[0,1]_{IVII}$, where

$$[0,1]_{IVII} = (([0,1],[0,\infty)\cap(-\infty,1]], [(-\infty,0)\cup(1,\infty),(-\infty,0)\cup(1,\infty)])).$$

(4) Let X be a non-empty set and let $\beta = \{a_{_{IVI}} : a \in X\} \cup \{a_{_{IVIV}} : a \in X\}$. Then β is an IVIB for the interval-valued discrete topology τ_1 on X.

(5) Let $X = \{a,b,c\}$ and let $\beta = \{([\{a,b\},X],[\varnothing,\varnothing]),([\{b,c\},X],[\varnothing,\varnothing]),\tilde{\tilde{X}}\}$. Assume that β is an IVIB for an IVIT τ on X. Then by the definition of base, $\beta \subset \tau$. Thus $([\{a,b\},X],[\varnothing,\varnothing])$, $([\{b,c\},X],[\varnothing,\varnothing]) \in \tau$. So $[\{a,b\},X]\cap[\{b,c\},X] = ([\{b\},X],[\varnothing,\varnothing]) \in \tau$. But for any $\beta^{'} \subset \beta$, $([\{b\},X],[\varnothing,\varnothing]) \neq \bigcup\beta^{'}$. Hence β is not an IVIB for an IVIT on X.

From (1), (2) and (3) in Example 4.12, we can define interval-valued intervals as following.

Definition 4.13. Let $a,\ b \in \mathbb{R}$ such that $a \leq b$. Then

(i) (the closed interval)

$$[a,b]_{IVII} = ([[a,b],[a,\infty)\cap(-\infty,b]],[(-\infty,a)\cup(b,\infty),(-\infty,a)\cup(b,\infty)]),$$

(ii) (the open interval)

$$(a,b)_{IVII} = ([(a,b),(a,-\infty)\cap(-\infty,b)],[(-\infty,a]\cup[b,\infty),(-\infty,a]\cup[b,\infty)]),$$

(iii) (the half open interval or the half closed interval)

$$(a,b]_{IVII} = ([(a,b],(a,-\infty)\cap(-\infty,b]],[(-\infty,a]\cup(b,\infty),(-\infty,a]\cup(b,\infty)]),$$

$$[a,b)_{IVII} = ([[a,b),[a,-\infty)\cap(-\infty,b)],[(-\infty,a)\cup[b,\infty),(-\infty,a)\cup[b,\infty)]),$$

(iv) (the half interval-valued real line)

$$(-\infty,a]_{IVII} = ([(-\infty,a],(-\infty,a]],[(a,\infty),(a,\infty)]),$$

$$(-\infty,a)_{IVII} = [(-\infty,a),(-\infty,a)],[[a,\infty),[a,\infty)]),$$

$$[a,\infty)_{IVII} = [[a,\infty),[a,\infty)],[(-\infty,a),(-\infty,a)]),$$

$$(a,\infty)_{IVII} = [(a,\infty),(a,\infty)],[(-\infty,a],(-\infty,a]]),$$

(v) (the interval-valued real line)

$$(-\infty,\infty)_{IVI} = ([(-\infty,\infty),(-\infty,\infty)],[\varnothing,\varnothing]) = \tilde{\tilde{\mathbb{R}}}.$$

Theorem 4.14. *Let X be a non-empty set and let $\beta \subset IVS(X)$. Then β is an IVIB for an IVIT τ on X if and only if it satisfies the followings:*

(1) $\tilde{\tilde{X}} = \bigcup\beta$,

(2) if $B_1, B_2 \in \beta$ and $a_{IVI} \in B_1 \cap B_2$ [resp. $a_{IVIV} \in B_1 \cap B_2$], then there exists $B \in \beta$ such that $a_{IVI} \in B \subset B_1 \cap B_2$ [resp. $a_{IVIV} \in B \subset B_1 \cap B_2$].

Proof. The proof is the same as one in ordinary topological spaces. □

Example 4.15. Let $X = \{a, b, c\}$ and consider the family β of IVISs in X given by:
$\beta = \{([\{a\}, \{a\}], [\{b, c\}, \{b, c\}]), ([\{a, b\}, \{a, b\}], [\{c\}, \{c\}]), ([\{a, c\}, \{a, c\}], [\{b\}, \{b\}])\}$.
Then clearly, β satisfies two conditions of Theorem 4.14. Thus β is an IVIB for an IVIT τ on X. Furthermore, we can easily check that τ is the family of IVISs in X given by:

$$\tau = \{\tilde{\tilde{\varnothing}}, ([\{a\}, \{a\}], [\{b, c\}, \{b, c\}]), ([\{a, b\}, \{a, b\}], [\{c\}, \{c\}]), \\ ([\{a, c\}, \{a, c\}], [\{b\}, \{b\}]), \tilde{\tilde{X}}\}.$$

Proposition 4.16. *Let X be a non-empty set and let $\sigma \subset IVIS(X)$ such that $\tilde{\tilde{X}} = \bigcup \sigma$. Then there exists a unique IVIT τ on X such that σ is an IVISB for τ.*

Proof. Let $\beta = \{B \in IVIS(X) : B = \bigcup_{i=1}^{n} S_i$ and $S_i \in \sigma\}$. Let $\tau = \{U \in IVIS(X) : U = \tilde{\tilde{\varnothing}}$ or there is a subcollection β' of β such that $U = \bigcup \beta'\}$. Then we can prove that τ is the unique IVIT on X such that σ is an IVISB for τ. □

In Proposition 4.16, τ is called the IVIT on X generated by σ.

Example 4.17. Let $X = \{a, b, c, d, e\}$ and let us consider the family of IVISs in X given by:

$$\sigma = \{([\{a\}, \{a\}], [\{b, c, d, e\}, \{b, c, d, e\}]), ([\{a, b, c\}, \{a, b, c\}], [\{d, e\}, \{d, e\}]), \\ ([\{b, c, e\}, \{b, c, e\}], [\{a, d\}, \{a, d\}]), ([\{c, d\}, \{c, d\}], [\{a, b, e\}, \{a, b, e\}])\}.$$

Then clearly, $\bigcup \sigma = \tilde{\tilde{X}}$. Let β be the collection of all finite intersections of members of σ. Then

$$\beta = \{\tilde{\tilde{\varnothing}}, ([\{a\}, \{a\}], [\{b, c, d, e\}, \{b, c, d, e\}]), ([\{c\}, \{c\}], [\{a, b, d, e\}, \{a, b, d, e\}]), \\ ([\{b, c\}, \{b, c\}], [\{a, d, e\}, \{a, d, e\}]), ([\{a, b, c\}, \{a, b, c\}], [\{d, e\}, \{d, e\}]), \\ ([\{b, c, e\}, \{b, c, e\}], [\{a, d\}, \{a, d\}]), ([\{c, d\}, \{c, d\}], [\{a, b, e\}, \{a, b, e\}])\}.$$

Thus the generated IVIT τ by σ is

$$\tau = \{\tilde{\tilde{\varnothing}}, ([\{a\}, \{a\}], [\{b, c, d, e\}, \{b, c, d, e\}]), ([\{c\}, \{c\}], [\{a, b, d, e\}, \{a, b, d, e\}]), \\ ([\{a, c\}, \{a, c\}], [\{b, d, e\}, \{b, d, e\}]), ([\{b, c\}, \{b, c\}], [\{a, d, e\}, \{a, d, e\}]), \\ ([\{c, d\}, \{c, d\}], \{a, b, e\}, \{a, b, e\}]), ([\{a, b, c\}, \{a, b, c\}], [\{d, e\}, \{d, e\}]), \\ ([\{b, c, d\}, \{b, c, d\}], [\{a, e\}, \{a, e\}]), ([\{b, c, e\}, \{b, c, e\}], [\{a, d\}, \{a, d\}]), \\ ([\{a, b, c, e\}, \{a, b, c, e\}], [\{d\}, \{d\}]), \tilde{\tilde{X}}\}.$$

5. Interval-valued intuitionistic neighborhoods

In this section, we introduce the concept of interval-valued intuitionistic neighborhoods of IVIPs of two types, and find their various properties and give some examples.

Definition 5.1 ([8])**.** Let X be an ITS, $p \in X$ and let $N \in IS(X)$. Then
(i) N is called a neighborhood of p_I, if there exists an IOS G in X such that

$$p_I \in G \subset N, \text{ i.e., } p \in G^{\in} \subset N^{\in} \text{ and } G^{\notin} \supset N^{\notin},$$

(ii) N is called a neighborhood of p_{IV}, if there exists an IOS G in X such that

$$p_{IV} \in G \subset N, \text{ i.e., } G^{\in} \subset N^{\in} \text{ and } p \notin N^{\notin} \subset G^{\notin}.$$

We will denote the set of all neighborhoods of p_I [resp. p_{IV}] by $N(p_I)$ [resp. $N(p_{IV})$].

Definition 5.2 ([13])**.** Let X be an IVTS, $a \in X$ and let $N \in IVS(X)$. Then

(i) N is called an interval-valued neighborhood (briefly, IVN) of $a_{_{IVP}}$, if there exists a $U \in IVO(X)$ such that

$$a_{_{IVP}} \in U \subset N, \text{ i.e., } a \in U^- \subset N^-,$$

(ii) N is called an interval-valued vanishing neighborhood (briefly, IVVN) of $a_{_{IVVP}}$, if there exists a $U \in IVO(X)$ such that

$$a_{_{IVVP}} \in U \subset N, \text{ i.e., } a \in U^+ \subset N^+.$$

We will denote the set of all IVNs [resp. IVVNs] of $a_{_{IVP}}$ [resp. $a_{_{IVVP}}$] by $N(a_{_{IVP}})$ [resp. $N(a_{_{IVVP}})$].

Definition 5.3. Let X be an IVITS, $a \in X$ and let $N \in IVIS(X)$. Then

(i) N is called an interval-valued intuitionistic neighborhood (briefly, IVIN) of $a_{_{IVI}}$, if there exists a $U \in IVIO(X)$ such that

$$a_{_{IVI}} \in U \subset N, \text{ i.e., } a \in U^{\in,-} \subset N^{\in,-},$$

(ii) N is called an interval-valued intuitionistic vanishing neighborhood (briefly, IVIVN) of $a_{_{IVIV}}$, if there exists a $U \in IVIO(X)$ such that

$$a_{_{IVIV}} \in U \subset N, \text{ i.e., } a \notin N^{\notin,+} \subset U^{\notin,+}.$$

We will denote the set of all IVINs [resp. IVIVNs] of $a_{_{IVI}}$ [resp. $a_{_{IVIV}}$] by $N(a_{_{IVI}})$ [resp. $N(a_{_{IVIV}})$].

Remark 5.4. (1) Let (X, τ) be an IVITS and let $N \in N(a_{IVI})$ [resp. $N(a_{IVIV})$. Consider two ISs and two IVSs in X, respectively given by:

$$N^- = (A^{\in,-}, A^{\notin,-}),\ N^+ = (A^{\in,+}, A^{\notin,+})$$

and

$$N^{\in} = [A^{\in,-}, A^{\in,+}],\ N^{\notin} = [A^{\notin,+^c}, A^{\notin,-^c}].$$

Then we can easily check that $N^- \in N(a_{_I})$ [resp. $N(a_{_{IV}})$] in the ITS (X, τ^-), $N^+ \in N(a_{_I})$ [resp. $N(a_{_{IV}})$] in the ITS (X, τ^+) and $N^{\in} \in N(a_{_{IVP}})$ [resp. $N(a_{_{IVVP}})$] in the IVTS $(X, \tau^{\in})$, $N^{\notin} \in N(a_{_{IVP}})$ [resp. $N(a_{_{IVVP}})$] in the IVTS $(X, \tau^{\notin})$.

(2) Let (X, τ) be an IVITS and and let $N \in N(a_{_{IVI}})$ [resp. $N(a_{_{IVIV}})$]. Then clearly, $[\ ]N \in N(a_{_{IVI}})$ [resp. $N(a_{_{IVIV}})$] in IVITS $(X, [\ ]\tau)$ and $\langle\ \rangle N \in N(a_{_{IVI}})$ [resp. $N(a_{_{IVIV}})$] in IVITS $(X, \langle\ \rangle \tau)$.

Example 5.5. Let $X = \{a, b, c, d\}$ and let τ be the IVIT on X given by:

$$\tau = \{\tilde{\bar{\bar{\varnothing}}}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, \tilde{\bar{\bar{X}}}\},$$

where $A_1 = ([\varnothing, \{a\}], [\{c\}, \{c, d\}]),\ A_2 = ([\{a\}, \{a\}], [\{c\}, \{c, d\}]),$
$A_3 = ([\{b\}, \{b\}], [\{c\}, \{a, c, d\}]),\ A_4 = ([\{b, c\}, \{b, c, d\}], [\varnothing, \{a\}]),$
$A_5 = ([\{b, c\}, X], [\varnothing, \varnothing]),\ A_6 = ([\{a, b, c\}, X], [\varnothing, \varnothing]),$
$A_7 = ([\{b, c\}, \{b, c, d\}], [\varnothing, \varnothing]),\ A_8 = ([\varnothing, \varnothing], [\{a, c\}, \{a, c, d\}]),$
$A_9 = ([\varnothing, \varnothing], [\{c\}, \{a, c, d\}]).$

Let $N = ([\{a, b\}, \{a, b, d\}], [\{c\}, \{c\}])$. Then we can easily see that

$$N \in N(a_{_{IVI}}) \cap N(a_{_{IVIV}}),\ N \in N(b_{_{IVI}}) \cap N(b_{_{IVIV}}).$$

Proposition 5.6. *Let X be an IVITS and let $a \in X$.*
[IVIN1] *If $N \in N(a_{IVI})$, then $a_{IVI} \in N$.*
[IVIN2] *If $N \in N(a_{IVI})$ and $N \subset M$, then $M \in N(a_{IVI})$.*
[IVIN3] *If $N, M \in N(a_{IVI})$, then $N \cap M \in N(a_{IVI})$.*
[IVIN4] *If $N \in N(a_{IVI})$, then there exists $M \in N(a_{IVI})$ such that $N \in N(b_{IVI})$ for each $b_{IVI} \in M$.*

Proof. The proofs of [IVIN1], [IVIN2] and [IVIN4] are easy.
[IVIN3] Suppose $N, M \in N(a_{IVI})$. Then there are $U,\ V \in IVIO(X)$ such that

$$a_{IVI} \in U \subset N \text{ and } a_{IVI} \in V \subset M.$$

Let $W = U \cap V$. Then clearly, $W \in IVIO(X)$ and $a_{IVI} \in W \subset N \cap M$. Thus $N \cap M \in N(a_{IVI})$. □

Proposition 5.7. *Let X be an IVITS and let $a \in X$.*
[IVIVN1] *If $N \in N(a_{IVIV})$, then $a_{IVIV} \in N$.*
[IVIVN2] *If $N \in N(a_{IVIV})$ and $N \subset M$, then $M \in N(a_{IVIV})$.*
[IVIVN3] *If $N, M \in N(a_{IVIV})$, then $N \cap M \in N(a_{IVIV})$.*
[IVIVN4] *If $N \in N(a_{IVIV})$, then there exists $M \in N(a_{IVIV})$ such that $N \in N(b_{IVIV})$ for each $b_{IVIV} \in M$.*

Proof. The proofs are similar to these of Proposition 5.6. □

Proposition 5.8. *Let (X, τ) be an IVITS and let us define two families:*

$$\tau_{IVI} = \{U \in IVIS(X) : U \in N(a_{IVI}) \text{ for each } a_{IVI} \in U\}$$

and

$$\tau_{IVIV} = \{U \in IVS(X) : U \in N(a_{IVIV}) \text{ for each } a_{IVIV} \in U\}.$$

Then we have
(1) *$\tau_{IVI},\ \tau_{IVIV} \in IVIT(X)$,*
(2) *$\tau \subset \tau_{IVI}$ and $\tau \subset \tau_{IVIV}$.*

Proof. (1) We only prove that $\tau_{IVIV} \in IVIT(X)$.
(IVIO_1) From the definition of τ_{IVIV}, we have $\tilde{\tilde{\varnothing}},\ \tilde{\tilde{X}} \in \tau_{IVIV}$.
(IVIO_2) Let $U\ , V \in IVIS(X)$ such that $U\ , V \in \tau_{IVIV}$ and let $a_{IVIV} \in U \cap V$. Then clearly, $U,\ V \in N(a_{IVIV})$. Thus by [IVIVN3], $U \cap V \in N(a_{IVIV})$. So $U \cap V \in \tau_{IVIV}$.
(IVIO_3) Let $(U_j)_{j \in J}$ be any family of IVISs in τ_{IVIV}, let $U = \bigcup_{j \in J} U_j$ and let $a_{IVIV} \in U$. Then by Theorem 3.13 (2), there is $j_0 \in J$ such that $a_{IVIV} \in U_{j_0}$. Since $U_{j_0} \in \tau_{IVIV}$, $U_{j_0} \in N(a_{IVIV})$ by the definition of τ_{IVIV}. Since $U_{j_0} \subset U$, $U \in N(a_{IVIV})$ by [IVIVN2]. So by the definition of τ_{IVIV}, $U \in \tau_{IVIV}$.
(2) Let $U \in \tau$. Then clearly, $U \in N(a_{IVI})$ and $U \in N(a_{IVIV})$ for each $a_{IVI} \in G$ and $a_{IVIV} \in G$, respectively. Thus $U \in \tau_{IVI}$ and $U \in \tau_{IVIV}$. So the results hold. □

Remark 5.9. (1) From the definitions of τ_{IVI} and τ_{IVIV}, we can easily have:

$$\tau_{IVI} = \tau \cup \{U \in IVIS(X) : U = ([V^{\in,-}, S], [V^{\notin,-}, V^{\notin,+}]),\ V^{\in,-} \neq \varnothing,$$
$$V^{\in,+} \subset S \subset X,\ S \cap V^{\notin,+} = \varnothing \text{ for some } V \in \tau\}$$

and

$$\tau_{IVIV} = \tau \cup \{U \in IVIS(X) : U = ([V^{\in,-}, V^{\in,+}], [U^{\notin,-}, U^{\notin,+}]),$$

$$[U^{\not\in,-}, U^{\not\in,+}] \subset [V^{\not\in,-}, V^{\not\in,+}] \text{ for some } V \in \tau\}.$$

In fact, it is clear that if $V^{\in,-} = \varnothing$ for each $V \in \tau$, then $\tau_{IVI} = \tau$.

(2) For any IVIT τ on a set X, we can have eight ordinary topologies on X given by:

$$\tau^{\in,-}_{IVI} = \{U^- \subset X : U \in \tau_{IVI}\},\ \tau^{\in,+}_{IVI} = \{U^+ \subset X : U \in \tau_{IVI}\},$$

$$\tau^{\not\in,-}_{IVI} = \{U^{\not\in,-^c} \subset X : U \in \tau_{IVI}\},\ \tau^{\not\in,+}_{IVI} = \{U^{\not\in,+^c} \subset X : U \in \tau_{IVI}\},$$

and

$$\tau^{\in,-}_{IVIV} = \{U^- \subset X : U \in \tau_{IVIV}\},\ \tau^{\in,+}_{IVIV} = \{U^+ \subset X : U \in \tau_{IVIV}\},$$

$$\tau^{\not\in,-}_{IVIV} = \{U^{\not\in,-} \subset X : U \in \tau_{IVIV}\},\ \tau^{\not\in,+}_{IVIV} = \{U^{\not\in,+^c} \subset X : U \in \tau_{IVIV}\}.$$

From Remark 4.4 (1) and the above (1), we can see that

$$\tau^{\not\in,-}_{IVI} = \tau^{\not\in,-},\ \tau^{\not\in,+}_{IVI} = \tau^{\not\in,+},\ \tau^{\in,-}_{IVIV} = \tau^{\in,-},\ \tau^{\in,+}_{IVIV} = \tau^{\in,+}.$$

Example 5.10. Let $X = \{a, b, c, d\}$ and consider the family τ of IVISs in X given by:

$$\tau = \{\widetilde{\varnothing}, \widetilde{X}, A_1, A_2, A_3, A_4\},$$

where $A_1 = ([\{a\}, \{a, b\}], [\{c\}, \{c\}]),\ A_2 = ([\{b\}, \{b\}], [\{a\}, \{a, c\}]),$
$A_3 = ([\varnothing, \{b\}], [\{a, c\}, \{a, c\}]),\ A_4 = ([\{a, b\}, \{a, b\}], [\varnothing, \{c\}]).$

Then we can easily check that (X, τ) is an IVITS. Thus we have:

$$\tau_{IVI} = \tau \cup \{A_5, A_6, A_7, A_8\}$$

and

$$\tau_{IVIV} = \tau \cup \{A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{19}, A_{20}, A_{21}, A_{22}\},$$

where $A_5 = ([\{a\}, \{a, b, d\}], [\{c\}, \{c\}]),\ A_6 = ([\{b\}, \{b, d\}], [\{a\}, \{a, c\}]),$
$A_7 = ([\{a, b\}, \{a, b, d\}], [\varnothing, \{c\}]),\ A_8 = ([\{a, b, d\}, \{a, b, d\}], [\varnothing, \{c\}]),$
$A_9 = ([\{a\}, \{a, b\}], [\varnothing, \{c\}]),\ A_{10} = ([\{a\}, \{a, b\}], [\varnothing, \varnothing]),$
$A_{11} = ([\{b\}, \{b\}], [\{a\}, \{a\}]),\ A_{12} = ([\{b\}, \{b\}], [\varnothing, \{a\}]),$
$A_{13} = ([\{b\}, \{b\}], [\varnothing, \{c\}]),\ A_{14} = ([\{b\}, \{b\}], [\varnothing, \{a, c\}]),$
$A_{15} = ([\{b\}, \{b\}], [\varnothing, \varnothing]),\ A_{16} = ([\varnothing, \{b\}], [\{a\}, \{a, c\}]),$
$A_{17} = ([\varnothing, \{b\}], [\{c\}, \{a, c\}]),\ A_{18} = ([\varnothing, \{b\}], [\{a\}, \{a\}]),$
$A_{19} = ([\varnothing, \{b\}], [\varnothing, \{a\}]),\ A_{20} = ([\varnothing, \{b\}], [\varnothing, \{c\}]),$
$A_{21} = ([\varnothing, \{b\}], [\varnothing, \{a, c\}]),\ A_{22} = ([\{a, b\}, \{a, b\}], [\varnothing, \varnothing]).$

So we can confirm that Proposition 5.8 holds.

Furthermore, we obtain six ordinary topologies on X for the IVT τ:

$$\tau^{\in,-} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\},$$

$$\tau^{\in,+} = \{\varnothing, X, \{b\}, \{a, b\}\},$$

$$\tau^{\not\in,-} = \{\varnothing, X, \{a, b, d\}, \{b, c, d\}, \{b, d\}\},$$

$$\tau^{\not\in,+} = \{\varnothing, X, \{a, b, d\}, \{b, d\}\},$$

$$\tau^{\in,-}_{IVI} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\},$$

$$\tau^{\in,+}_{IVI} = \{\varnothing, X, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}\},$$

$$\tau^{\not\in,-}_{IVIV} = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\},$$

$$\tau^{\not\in,+}_{IVIV} = \{\varnothing, X, \{c\}, \{a, c\}\}.$$

The following is the immediate result of Proposition 5.8 (2).

Corollary 5.11. *Let (X,τ) be an IVITS and let $IVIC_\tau$ [resp. $IVIC_{\tau_{IVI}}$ and $IVIC_{\tau_{IVIV}}$] be the set of all IVICSs w.r.t. τ [resp. τ_{IVI} and τ_{IVIV}]. Then*

$$IVIC_\tau \subset IVIC_{\tau_{IVI}}, \text{ and } IVIC_\tau \subset IVIC_{\tau_{IVIV}}.$$

Example 5.12. Let (X,τ) be the IVITS given in Example 5.10. Then we have:

$IVIC_\tau = \{\tilde{\tilde{\varnothing}}, \tilde{\tilde{X}}, A_1^c, A_2^c, A_3^c, A_4^c\}$,
$IVIC_{\tau_{IVI}} = IVIC_\tau \cup \{A_5^c, A_6^c, A_7^c, A_8^c\}$,
$IVC_{\tau_{IVV}} = IVC_\tau \cup \{A_9^c, A_{10}^c, A_{11}^c, A_{12}^c, A_{13}^c, A_{14}^c, A_{15}^c, A_{16}^c, A_{17}^c, A_{18}^c, A_{19}^c, A_{20}^c, A_{21}^c, A_{22}^c\}$,

where $A_1^c = ([\{c\},\{c\}],[\{a\},\{a,b\}]),\ A_2^c = ([\{a\},\{a,c\}],[\{b\},\{b\}])$,
$A_3^c = ([\{a,c\},\{a,c\}],[\varnothing,\{b\}]),\ A_4^c = ([\varnothing,\{c\}],[\{a,b\},\{a,b\}])$
$A_5^c = ([\{c\},\{c\}],[\{a\},\{a,b,d\}]),\ A_6^c = ([\{a\},\{a,c\}],[\{b\},\{b,d\}])$,
$A_7^c = ([\varnothing,\{c\}],[\{a,b\},\{a,b,d\}]),\ A_8^c = ([\varnothing,\{c\}],[\{a,b,d\},\{a,b,d\}])$,
$A_9^c = ([\varnothing,\{c\}],[\{a\},\{a,b\}]),\ A_{10}^c = ([\varnothing,\varnothing],[\{a\},\{a,b\}])$,
$A_{11}^c = ([\{a\},\{a\}],[\{b\},\{b\}]),\ A_{12}^c = ([\varnothing,\{a\}],[\{b\},\{b\}])$,
$A_{13}^c = ([\varnothing,\{c\}],[\{b\},\{b\}]),\ A_{14}^c = ([\varnothing,\{a,c\}],[\{b\},\{b\}])$,
$A_{15}^c = ([\varnothing,\varnothing],[\{b\},\{b\}]),\ A_{16}^c = ([\{a\},\{a,c\}],[\varnothing,\{b\}])$,
$A_{17}^c = ([\{c\},\{a,c\}],[\varnothing,\{b\}]),\ A_{18}^c = ([\{a\},\{a\}],[\varnothing,\{b\}])$,
$A_{19}^c = (\varnothing,\{a\}],[\varnothing,\{b\}]),\ A_{20}^c = ([\varnothing,\{c\}],[\varnothing,\{b\}])$,
$A_{21}^c = ([\varnothing,\{a,c\}],[\varnothing,\{b\}]),\ A_{22}^c = ([\varnothing,\varnothing],[\{a,b\},\{a,b\}])$.

Thus we can confirm that Corollary 5.11 holds.

Now let us the converses of Propositions 5.6 and 5.7.

Proposition 5.13. *Let X be a non-empty set. Suppose to each $a \in X$, there corresponds a set $N_*(a_{IVV)})$ of IVSs in X satisfying the conditions [IVIVN1], [IVIVN2], [IVIVN3] and [IVIVN4] in Proposition 5.7. Then there is an IVIT on X such that $N_*(a_{IVIV)})$ is the set of all IVINs of a_{IVIV} in this IVIT for each $a \in X$.*

Proof. Let

$$\tau_{IVIV} = \{U \in IVIS(X) : U \in N(a_{IVIV}) \text{ for each } a_{IVIV} \in U\},$$

where $N(a_{IVIV})$ denotes the set of all IVIVNs of a_{IVIV} in τ.
Then clearly, $\tau_{IVIV} \in IVIT(X)$ by Proposition 5.7. we will prove that $N_*(a_{IVIV)})$ is the set of all IVIVNs of $a_{IVIV)}$ in τ_{IVIV} for each $a \in X$.

Let $V \in IVIS(X)$ such that $V \in N_*(a_{IVIV})$ and let U be the union of all the IVIVPs b_{IVIV} in X such that $U \in N_*(a_{IVIV})$. If we can prove that

$$a_{IVIV} \in U \subset V \text{ and } U \in \tau_{IVIV},$$

then the proof will be complete.

Since $V \in N_*(a_{IVIV})$, $a_{IVIV} \in U$ by the definition of U. Moreover, $U \subset V$. Suppose $b_{IVIV} \in U$. Then by [IVIVN4], there is an IVIS $W \in N_*(b_{IVIV})$ such that $V \in N_*(c_{IVIV)})$ for each $c_{IVIV} \in W$. Thus $c_{IVIV} \in U$. By Proposition **??**, $W \subset U$. So by [IVIVN2], $U \in N_*(_{IVIV})$ for each $b_{IVIV} \in U$. Hence by the definition of τ_{IVIV}, $U \in \tau_{IVIV}$. This completes the proof. □

Proposition 5.14. *Let X be a non-empty set. Suppose to each $a \in X$, there corresponds a set $N_*(a_{IVI)})$ of IVISs in X satisfying the conditions [IVIN1], [IVIN2], [IVIN3] and [IVIN4] in Proposition 5.6. Then there is an IVIT on X such that $N_*(a_{IVI)})$ is the set of all IVINs of a_{IVI} in this IVT for each $a \in X$.*

Proof. The proof is similar to Proposition 5.13. □

Theorem 5.15. *Let (X, τ) be an IVITS and let $A \in IVIS(X)$. Then $A \in \tau$ if and only if $A \in N(a_{_{IVI}})$ and $A \in N(a_{_{IVIV}})$ for each $a_{_{IVI}},\ a_{_{IVIV}} \in A$.*

Proof. Suppose $A \in N(a_{_{IVI}})$ and $A \in N(a_{_{IVIV}})$ for each $a_{_{IVI}},\ a_{_{IVIV}} \in A$. Then there are $U_{a_{_{IVI}}},\ V_{a_{_{IVIV}}} \in \tau$ such that $a_{_{IVI}} \in U_{a_{_{IVI}}} \subset A$ and $a_{_{IVIV}} \in V_{a_{_{IVIV}}} \subset A$. Thus

$$A = (\bigcup_{a_{_{IVI}} \in A} a_{_{IVI}}) \cup (\bigcup_{a_{_{IVIV}} \in A} a_{_{IVIV}}) \subset (\bigcup_{a_{_{IVI}} \in A} U_{a_{_{IVI}}}) \cup (\bigcup_{a_{_{IVIV}} \in A} V_{_{IVIV}}) \subset A.$$

So $A = (\bigcup_{a_{_{IVI}} \in A} U_{a_{_{IVI}}}) \cup (\bigcup_{a_{_{IVIV}} \in A} V_{a_{_{IVIV}}})$. Since $U_{a_{_{IVI}}},\ V_{a_{_{IVIV}}} \in \tau$, $A \in \tau$.

The proof of the necessary condition is easy. □

Now we will give the relation among three IVITs, τ, $\tau_{_{IVI}}$ and $\tau_{_{IVIV}}$.

Proposition 5.16. $\tau = \tau_{_{IVI}} \cap \tau_{_{IVIV}}$.

Proof. From Proposition 5.8 (2), it is clear that $\tau \subset \tau_{_{IVI}} \cap \tau_{_{IVIV}}$.

Conversely, let $U \in \tau_{_{IVI}} \cap \tau_{_{IVIV}}$. Then clearly, $U \in \tau_{_{IVI}}$ and $U \in \tau_{_{IVIV}}$. Thus U is an IVIN of each of its IVIPs $a_{_{IVI}}$ and an IVIVN of each of its IVIVPs $a_{_{IVIV}}$. Thus there are $U_{a_{_{IVI}}},\ U_{a_{_{IVIV}}} \in \tau$ such that $a_{_{IVI}} \in U_{a_{_{IVI}}} \subset U$ and $a_{_{IVIV}} \in U_{a_{_{IVIV}}} \subset U$. So we have

$$U_{IVI} = \bigcup_{a_{_{IVI}} \in U} a_{_{IVI}} \subset \bigcup_{a_{_{IVI}} \in U} U_{a_{_{IVI}}} \subset U$$

and

$$U_{IVIV} = \bigcup_{a_{_{IVIV}} \in U} a_{_{IVIV}} \subset \bigcup_{a_{_{IVIV}} \in U} U_{a_{_{IVIV}}} \subset U.$$

By Proposition 3.11, we get

$$U = U_{IVI} \cup U_{IVIV} \subset (\bigcup_{a_{_{IVI}} \in U} U_{a_{_{IVI}}}) \cup (\bigcup_{a_{_{IVIV}} \in U} U_{a_{_{IVIV}}}) \subset U, \text{ i.e.,}$$

$$U = (\bigcup_{a_{_{IVI}} \in U} U_{a_{_{IVI}}}) \cup (\bigcup_{a_{_{IVIV}} \in U} U_{a_{_{IVIV}}}).$$

It is obvious that $(\bigcup_{a_{_{IVI}} \in U} U_{a_{_{IVI}}}) \cup (\bigcup_{a_{_{IVIV}} \in U} U_{a_{_{IVIV}}}) \in \tau$. Hence $U \in \tau$. Therefore $\tau_{_{IVI}} \cap \tau_{_{IVIV}} \subset \tau$. This completes the proof. □

The following is the immediate result of Proposition 5.16.

Corollary 5.17. *Let (X, τ) be an IVITS. Then*

$$IVIC_{\tau} = IVIC_{\tau_{_{IVI}}} \cap IVIC_{\tau_{_{IVIV}}}.$$

Example 5.18. In Example 5.12, we can easily check that Corollary 5.17 holds.

6. Interiors and closures of IVISs

In this section, we define interval-valued intuitionistic interiors and closures, and study some of their properties and give some examples. In particular, we will show that there is a unique IVIT on a set X from the interval-valued intuitionistic closure [resp. interior] operator.

Definition 6.1. Let (X, τ) be an IVITS and let $A \in IVIS(X)$.

(i) The interval-valued intuitionistic closure of A w.r.t. τ, denoted by $IVIcl(A)$, is an IVIS in X defined as:

$$IVIcl(A) = \bigcap\{K : K^c \in \tau \text{ and } A \subset K\}.$$

(ii) The interval-valued intuitionistic interior of A w.r.t. τ, denoted by $IVIint(A)$, is an IVIS in X defined as:

$$IVIint(A) = \bigcup\{G : G \in \tau \text{ and } G \subset A\}.$$

(iii) The interval-valued intuitionistic closure of A w.r.t. $\tau_{_{IVI}}$, denoted by $cl_{_{IVI}}(A)$, is an IVIS in X defined as:

$$cl_{_{IVI}}(A) = \bigcap\{K : K^c \in \tau_{_{IVI}} \text{ and } A \subset K\}.$$

(iv) The interval-valued intuitionistic interior of A w.r.t. $\tau_{_{IVI}}$, denoted by $int_{_{IVI}}(A)$, is an IVIS in X defined as:

$$int_{_{IVI}}(A) = \bigcup\{G : G \in \tau_{_{IVI}} \text{ and } G \subset A\}.$$

(v) The interval-valued intuitionistic closure of A w.r.t. $\tau_{_{IVIV}}$, denoted by $cl_{_{IVIV}}(A)$, is an IVS in X defined as:

$$cl_{_{IVIV}}(A) = \bigcap\{K : K^c \in \tau_{_{IVIV}} \text{ and } A \subset K\}.$$

(vi) The interval-valued intuitionistic interior of A w.r.t. $\tau_{_{IVIV}}$, denoted by $int_{_{IVIV}}(A)$, is an IVS in X defined as:

$$int_{_{IVIV}}(A = \bigcup\{G : G \in \tau_{_{IVIV}} \text{ and } G \subset A\}.$$

Remark 6.2. From the above definition, it is obvious that the followings hold:

$$IVIint(A) \subset int_{_{IVI}}(A),\ IVIint(A) \subset int_{_{IVIV}}(A)$$

and

$$cl_{_{IVI}}(A) \subset IVIcl(A),\ cl_{_{IVIV}}(A) \subset IVIcl(A).$$

Example 6.3. Let (X, τ) be the IVTS given in Example 5.12. Consider two IVISs $A = ([\{b\}, \{b, d\}], [\varnothing, \{c\}])$ and $B = ([\varnothing, \{c\}], [\{a, b\}, \{a, b, d\}])$ in X. Then

$$\begin{aligned} IVIint(A) &= \bigcup\{G \in \tau : G \subset A\} = A_2 \cup A_3 = A_2, \\ int_{_{IVI}}(A) &= \bigcup\{G \in \tau_{_{IVI}} : G \subset A\} = A_2 \cup A_6 = A_6, \\ int_{_{IVIV}}(A) &= \bigcup\{G \in \tau_{_{IVIV}} : G \subset A\} \\ &= A_2 \cup A_{13} \cup A_{14} \cup A_{16} \cup A_{17} \cup A_{20} \cup A_{21} = A_{13} \end{aligned}$$

and

$$\begin{aligned} IVIcl(B) &= \bigcap\{F : F^c \in \tau,\ B \subset F\} = A_1^c \cap A_2^c \cap A_3^c \cap A_4^c = A_4^c, \\ cl_{_{IVI}}(B) &= \bigcap\{F : F^c \in \tau_{_{IVI}},\ B \subset F\} = A_4^c \cap A_5^c \cap A_6^c \cap A_7^c = A_7^c, \\ cl_{_{IVIV}}(B) &= \bigcap\{F : F^c \in \tau_{_{IVIV}},\ B \subset F\} \end{aligned}$$

$$= A_4^c \cap A_9^c \cap A_{13}^c \cap A_{14}^c \cap A_{16}^c \cap A_{17}^c \cap A_{20}^c \cap A_{21}^c = A_4^c.$$

Thus we can confirm that Remark 6.2 holds.

Proposition 6.4. *Let (X, τ) be an IVITS and let $A \in IVIS(X)$. Then*

$$IVIint(A^c) = (IVIcl(A))^c \text{ and } IVIcl(A^c) = (IVIint(A))^c.$$

Proof. $IVint(A^c)$
$= \bigcup\{U \in \tau : U \subset A^c\}$
$= \bigcup\{U \in \tau : U^{\in,-} \subset A^{\notin,-},\ U^{\in,+} \subset A^{\notin,+},\ U^{\notin,-} \supset A^{\in,-},\ U^{\notin,+} \supset A^{\in,+}\}$
$= \bigcup\{U \in \tau : A^{\in,-} \subset U^{\notin,-},\ A^{\in,+} \subset U^{\notin,+},\ A^{\notin,-} \supset U^{\in,-},\ A^{\notin,+} \supset U^{\notin,-}\}$
$= \bigcap\{U^c : U \in \tau, A \subset U^c\}$
$= IVIcl(A).$
Similarly, we can show that $IVIcl(A^c) = (IVIint(A))^c$. □

Proposition 6.5. *Let (X, τ) be an IVITS and let $A \in IVIS(X)$. Then*

$$IVIint(A) = int_{IVI}(A) \cap int_{IVIV}(A).$$

Proof. The proof is straightforward from Proposition 5.16 and Definition 6.1. □

The following is the immediate result of Definition 6.1, and Propositions 6.4 and 6.5.

Corollary 6.6. *Let (X, τ) be an IVITS and let $A \in IVIS(X)$. Then*

$$IVIcl(A) = cl_{IVI}(A) \cup cl_{IVIV}(A).$$

Example 6.7. Consider two IVISs A and B in X given in Example 6.3:

$$A = ([\{b\}, \{b, d\}], [\varnothing, \{c\}]) \text{ and } B = ([\varnothing, \{c\}], [\{a, b\}, \{a, b, d\}]).$$

Then we have:

$$IVIint(A) = A_2 = ([\{b\}, \{b\}], [\{a\}, \{a, c\}]),$$
$$int_{IVI}(A) = A_6 = ([\{b\}, \{b, d\}], [\{a\}, \{a, c\}]),$$
$$int_{IVIV}(A) = A_{13} = ([\{b\}, \{b\}], [\varnothing, \{c\}])$$

and

$$IVIcl(B) = A_4^c = ([\varnothing, \{c\}], [\{a, b\}, \{a, b\}]),$$
$$cl_{IVI}(B) = A_7^c = [\varnothing, \{c\}], [\{a, b\}, \{a, b, d\}]),$$
$$cl_{IVIV}(B) = A_4^c = ([\varnothing, \{c\}], [\{a, b\}, \{a, b\}]).$$

Thus $int_{IVI}(A) \cap int_{IVIV}(A) = ([\{b\}, \{b\}], [\{a\}, \{a, c\}]) = IVIint(A)$
and
$$cl_{IVI}(B) \cup cl_{IVIV}(B) = ([\varnothing, \{c\}], [\{a, b\}, \{a, b\}]) = IVIcl(B).$$
So we can confirm that Proposition 6.5 and Corollary 6.6 hold.

Theorem 6.8. *Let X be an IVITS and let $A \in IVIS(X)$. Then*
(1) *$A \in IVIC(X)$ if and only if $A = IVIcl(A)$,*
(2) *$A \in IVIO(X)$ if and only if $A = IVIint(A)$.*

Proof. Straightforward. □

Proposition 6.9 (Kuratowski Closure Axioms)**.** *Let X be an IVTIS and let $A, B \in IVIS(X)$. Then*
[IVIK0] *if $A \subset B$, then $IVIcl(A) \subset IVIcl(B)$,*
[IVIK1] $IVIcl(\tilde{\tilde{\varnothing}}) = \tilde{\tilde{\varnothing}}$,
[IVIK2] $A \subset IVIcl(A)$,
[IVIK3] $IVIcl(IVIcl(A)) = IVIcl(A)$,
[IVIK4] $IVIcl(A \cup B) = IVIcl(A) \cup IVIcl(A)$.

Proof. Straightforward. □

Let $IVcl^* : IVIS(X) \to IVIS(X)$ be the mapping satisfying the properties [IVIK1], [IVIK2],[IVIK3] and [IVIK4]. Then we will call the mapping $IVIcl^*$ as the interval-valued intuitionitic closure operator(briefly, IVICO) on X.

Proposition 6.10. *Let $IVIcl^*$ be the IVICO on X. Then there exists a unique IVIT τ on X such that $IVIcl^*(A) = IVIcl(A)$, for each $A \in IVIS(X)$, where $IVIcl(A)$ denotes the interval-valued intuitionistic closure of A in the IVTS (X, τ). In fact,*
$$\tau = \{A^c \in IVIS(X) : IVIcl^*(A) = A\}.$$

Proof. The proof is almost similar to the case of classical topological spaces. □

Proposition 6.11. *Let X be an IVITS and let $A, B \in IVIS(X)$. Then*
[IVII0] *if $A \subset B$, then $IVIint(A) \subset IVIint(B)$,*
[IVII1] $IVIint(\tilde{\tilde{X}}) = \tilde{\tilde{X}}$,
[IVII2] $IVIint(A) \subset A$,
[IVII3] $IVint(IVint(A)) = IVint(A)$,
[IVII4] $IVIint(A \cap B) = IVIint(A) \cap IVIint(A)$.

Proof. Straightforward. □

Let $IVIint^* : IVIS(X) \to IVIS(X)$ be the mapping satisfying the properties [IVII1], [IVII2],[IVII3] and [IVI4]. Then we will call the mapping $IVint^*$ as the interval-valued intuitionistic interior operator (briefly, IVIIO) on X.

Proposition 6.12. *Let $IVIint^*$ be the IVIIO on X. Then there exists a unique IVIT τ on X such that $IVIint^*(A) = IVIint(A)$, for each $A \in IVIS(X)$, where $IVIint(A)$ denotes the interval-valued intuitionistic interior of A in the IVITS (X, τ). In fact,*
$$\tau = \{A \in IVIS(X) : IVIint^*(A) = A\}.$$

Proof. The proof is similar to one of Proposition 6.10. □

Definition 6.13. Let (X, τ) be an IVITS, $a \in X$ and let $A \in IVIS(X)$. Then
(i) $a_{IVI} \in A$ is called a τ_{IVI}-interior point of A, if $A \in N(a_{IVI})$,
(ii) $a_{IVIV} \in A$ is called a τ_{IVIV}-interior point of A, if $A \in N(a_{IVIV})$.
We will denote the union of all τ_{IVI}-interior points [resp. τ_{IVIV}-interior points] of A as $\tau_{IVI} - int(A)$ [resp. $\tau_{IVIV} - int(A)$]. It is clear that
$$\tau_{IVI} - int(A) = \bigcup\{a_{IVI} : A \in N(a_{IVI})\}$$
$$[\text{resp. } \tau_{IVIV} - int(A) = \bigcup\{a_{IVIV} : A \in N(a_{IVIV})\}].$$

Theorem 6.14. *Let* (X, τ) *be an IVITS and let* $A \in IVIS(X)$.

(1) $A \in \tau_{IVI}$ *if and only if* $A_{IVI} = \tau_{IVI} - int(A)$.

(2) $A \in \tau_{IVIV}$ *if and only if* $A_{IVIV} = \tau_{IVIV} - int(A)$.

Proof. (1) Suppose $A \in \tau_{IVI}$ and let $a_{IVI} \in A_{IVI}$. Then by the definition of A_{IVI}, $a_{IVI} \in A$. Thus by the definition of τ_{IVI}, $A \in N(a_{IVI})$. So $a_{IVI} \in \tau_{IVI} - int(A)$, i.e., $A_{IVI} \subset \tau_{IVI} - int(A)$.

Now let $a_{IVI} \in \tau_{IVI} - int(A)$. Then $A \in N(a_{IVI})$. Thus $a_{IVI} \in A$. So $a_{IVI} \in A_{IVI}$, i.e., $\tau_{IVI} - int(A) \subset A_{IVI}$. Hence $A_{IVI} = \tau_{IVI} - int(A)$.

Conversely, suppose the necessary condition holds and let $a_{IVI} \in A$. Then $a_{IVI} \in A_{IVI}$. Thus by the hypothesis, $a_{IVI} \in \tau_{IVI} - int(A)$. So $A \in N(a_{IVI})$. Hence by the definition of τ_{IVI}, $A \in \tau_{IVI}$.

(2) The proof is similar to that of (1). □

Proposition 6.15. *Let* X *be a non-empty set,* $(A_j)_{j \in J} \subset IVIS(X)$ *and let* $A = \bigcup_{j \in J} A_j$. *Then*

(1) $A_{IVI} = \bigcup_{j \in J} A_{j,IVI}$,

(2) $A_{IVIV} = \bigcup_{j \in J} A_{j,IVIV}$.

Proof. (1) For each $j \in J$, let $A_j = ([A_j^{\in,-}, A_j^{\in,+}], [A_j^{\notin,-}, A_j^{\notin,+}])$. Then clearly, we have

$$A = \bigcup_{j \in J} A_j = ([\bigcup_{j \in J} A_j^{\in,-}, \bigcup_{j \in J} A_j^{\in,+}], [\bigcap_{j \in J} A_j^{\notin,-}, \bigcap_{j \in J} A_j^{\notin,+}]).$$

Now let $a_{IVI} \in A$. Then $a_{IVI} \in \bigcup_{j \in J} A_j$. Thus $a \in \bigcup_{j \in J} A_j^{\in,-}$. So there is $j_0 \in J$ such that $a \in A_{j_0}^{\in,-}$. Hence $a_{IVI} \in A_{j_0,IVI}$, i.e., $a_{IVI} \in \bigcup_{j \in J} A_{j,IVI}$.

Conversely, suppose $a_{IVI} \in \bigcup_{j \in J} A_{j,IVI}$. Then there is $j_0 \in J$ such that $a_{IVI} \in A_{j_0,IVI}$. Thus $a \in A_{j_0}^{\in,-}$. So $a \in \bigcup_{j \in J} A_j^{\in,-}$. Hence $a_{IVI} \in A_{IVI}$. Therefore $A_{IVI} = \bigcup_{j \in J} A_{j,IVI}$.

(2) The proof is similar to that of (1). □

Proposition 6.16. *Let* (X, τ) *be an IVITS and let* $A \in IVIS(X)$. *Then*

(1) $\tau_{IVI} - int(A) = \bigcup_{G \subset A,\ G \in \tau_{IVI}} G_{IVI}$,

(2) $\tau_{IVIV} - int(A) = \bigcup_{G \subset A,\ G \in \tau_{IVIV}} G_{IVIV}$.

Proof. Suppose $a_{IVI} \in \bigcup_{G \subset A,\ G \in \tau_{IVI}} G_{IVI}$. Then there is $G \in \tau_{IVI}$ such that

$$G \subset A \text{ and } a_{IVI} \in G_{IVI}.$$

Thus $a_{IVI} \in G$. Since $G \in \tau_{IVI}$, $G \in N(a_{IVI})$. So $A \in N(a_{IVI})$. Hence $a_{IVI} \in \tau_{IVI} - int(A)$.

Conversely, suppose $a_{IVI} \in \tau_{IVI} - int(A)$. Then there is $G \in \tau$ such that

$$a_{IVI} \in G \subset A.$$

Moreover, $a_{IVI} \in G_{IVI}$ and $G \in \tau_{IVI}$. Thus $a_{IVI} \in \bigcup_{G \subset A,\ G \in \tau_{IVI}} G_{IVI}$. So the result holds.

(2) The proof is similar to that of (1). □

Remark 6.17. From Definitions 6.1 and 6.13, we have the following inclusions:

$$\tau_{IVI} - int(A) \subset int_{IVI}(A),\ \tau_{IVIV} - int(A) \subset int_{IVIV}(A).$$

But the reverse inclusions do not hold in general (See Example 6.18).

Example 6.18. Let (X, τ) be the IVITS given in Example 5.10 and consider the IVIS $A = ([\{b\}, \{b, d\}], [\varnothing, \{c\}])$. Then clearly, we have

$$int_{IVI}(A) = A_6 = ([\{b\}, \{b, d\}], [\{a\}, \{a, c\}])$$

and

$$int_{IVIV}(A) = A_2 = ([\{b\}, \{b\}], [\{a\}, \{a, c\}]).$$

On the other hand, by Propositions 3.11 and 6.16, we have

$$\tau_{IVI} - int(A) = ([\{b\}, \{b\}], [\{a, c\}, \{a, c\}]),\ \tau_{IVIV} - int(A) = ([\varnothing, \{b\}], [\{a\}, \{a, c\}]).$$

Thus we can confirm Remark 6.17.

Remark 6.19. From Example 6.18, we have the following strict inclusions:

$$\tau_{IVI} - int(A) \subset int_{IVI}(A),\ \tau_{IVI} - int(A) \neq int_{IVI}(A),$$

$$\tau_{IVIV} - int(A) \subset int_{IVIV}(A),\ \tau_{IVIV} - int(A) \neq int_{IVIV}(A).$$

Proposition 6.20. *Let (X, τ) be an IVITS and let $A,\ B \in IVIS(X)$. Then*
(1) $\tau_{IVI} - int(A) \subset A_{IVI}$, $\tau_{IVIV} - int(A) \subset A_{IVIV}$,
(2) *if* $A \subset B$, *then* $\tau_{IVI} - int(A) \subset \tau_{IVI} - int(B)$, $\tau_{IVIV} - int(A) \subset \tau_{IVIV} - int(B)$,
(3) $\tau_{IVI} - int(A \cap B) = \tau_{IVI} - int(A) \cap \tau_{IVI} - int(B)$,
$\tau_{IVIV} - int(A \cap B) = \tau_{IVIV} - int(A) \cap \tau_{IVIV} - int(B)$,
(4) $\tau_{IVI} - int(\tilde{\tilde{X}}) = \tilde{\tilde{X}}$, $\tau_{IVIV} - int(\tilde{\tilde{X}}) = ([\varnothing, X], \bar{\varnothing})$.

Proof. From Definition 6.13 and Proposition 6.16, the proofs of (1) and (2) are obvious. Also, the proof of (4) is clear from Proposition 6.16. We will prove only (3).

Let $a_{IVI} \in \tau_{IVI} - int(A \cap B)$. Then clearly, $A \cap B \in N(a_{IVI})$. Thus $A \in N(a_{IVI})$ and $B \in N(a_{IVI})$. So $a_{IVI} \in \tau_{IVI} - int(A)$ and $a_{IVI} \in \tau_{IVI} - int(B)$, i.e.,

$$a_{IVI} \in \tau_{IVI} - int(A) \cap \tau_{IVI} - int(B).$$

Hence $\tau_{IVI} - int(A \cap B) \subset \tau_{IVI} - int(A) \cap \tau_{IVI} - int(B)$.

Conversely, suppose $a_{IVI} \in \tau_{IVI} - int(A) \cap \tau_{IVI} - int(B)$. Then $A \in N(a_{IVI})$ and $B \in N(a_{IVI})$. Thus $A \cap B \in N(a_{IVI})$. So a_{IVI} is a τ_{IVI}-interior point of $A \cap B$, i.e.,

$$a_{IVI} \in \tau_{IVI} - int(A \cap B).$$

Hence $\tau_{IVI} - int(A) \cap \tau_{IVI} - int(B) \subset \tau_{IVI} - int(A \cap B)$. Therefore the equality holds.

The proof of the second part is similar to that of the first part. □

Remark 6.21. The equalities $\tau_{IVI} - int(\tau_{IVI} - int(A)) = \tau_{IVI} - int(A)$ and $\tau_{IVIV} - int(\tau_{IVIV} - int(A)) = \tau_{IVIV} - int(A)$ do not hold in general (See Example 6.22)

Example 6.22. Let (X, τ) be the IVITS given in Example 5.10 and let A be the IVIS in X given in Example 6.18. Then we can easily check that

$$\tau_{IVI} - int(A) = ([\{b\}, \{b\}], [\{a, c\}, \{a, c\}])$$

and

$$\tau_{IVI} - int(\tau_{IVI} - int(A)) = (\bar{\varnothing}, [\{a, c\}, \{a, c\}]).$$

Thus $\tau_{IVI} - int(A) \neq \tau_{IVI} - int(\tau_{IVI} - int(A))$.

7. Conclusions

We introduced the new concept of interval-valued intuitinistic sets which are the generalization of classical sets and the special case of interval-valued intuitionistic fuzzy sets, and obtained its various properties. Also, we defined an interval-valued intuitionistic ideal and studied some of its properties. Next, we introduced the notion of interval-valued intuitionistic topological spaces which are considered as a bitopological space proposed by Kelly [10]. Moreover, we defined an interval-valued intuitionistic base and subbase and found the characterization of an interval-valued intuitionistic base. Finally, we introduced the concept of interval-valued intuitionistic neighborhoods and obtained some similar properties to classical neighborhoods. Furthermore, we defined an interval-valued intuitionistic closure and interior and dealt with their some properties. In the future, we expect that one can apply the concept of interval-valued intuitionistic sets to group and ring theory, BCK-algebra and category theory, etc.

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