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Concept Lattice on Cubic Sets

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Abstract. In 2012, Jun et al. introduced the notion of a cubic set, which is a generalization of Zedeh's fuzzy set. In 1999, Ganter and Wille put forward formal concept analysis theory(FCA), formal context, concept lattice are two main notions and tools. In the paper, we define formal context, Galois connection, concept lattice on cubic sets, and generalize formal concept analysis theory on cubic sets structure.

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1. Introduction

In 1965, Zedeh [27] initiated fuzzy sets. As a generalization, in 2012, Jun et al. [14] introduced the notion of a cubic set. After then, Kang and Kim [19] defined a mapping of cubic sets, Kim et al. [20] investigated a cubic relation between cubic sets. The related contents may be referred [1, 2, 7, 12, 13, 15, 16, 17, 18, 21, 22, 23, 24, 25].

Ganter and Wille [9] put forward formal concept analysis theory(FCA), which is an order-theoretical analysis of scientific data. Formal context, concept lattice are two main notions and tools. In [3], from the point of view of graded approach, Bělohlávek established **L**-order, **L**-context and **L**-concept lattice. Djouadi and Prade [8] investigated the interval-valued formal concept analysis and Brito et al. [4] considered fuzzy formal concept analysis. We also discussed approximate concept lattice in fuzzy setting and the relation to fuzzy rough sets [5, 6]. More references see International Conference Homepages: Concept Lattices and Their Applications.

In the paper, we introduce the notions of formal context, concept lattice on cubic sets, combine formal concept analysis and cubic sets together. The contents are arranged into two parts, Section 3: Concept lattice on Cubic sets. In Section 2, we give an overview of concept lattice and cubic sets, which surveys Preliminaries

2. Preliminaries

The section is devoted to some main notions for each area, i.e., formal concept analysis [3, 9] fuzzy sets [27], interval-valued fuzzy sets [11, 26] and cubic sets [14, 20].

2.1. **Formal Concept Analysis.**

FCA is an order-theoretic method for the mathematical analysis of scientific data, pioneered by Wille and others [9], has attracted a growing number of researchers and practitioners. We introduce by formalizing the notion of (formal) context.

A context is a triple (G, M, I) consisting of two sets G and M and a relation I between them. The elements of G are called the objects and the elements of M are called the attributes. We write gIm or $(g, m) \in I$ to show that the object g has the attribute m.

For a set $A \subseteq G$ of objects define $A^{'} = \{m \in M \mid gIm\ for\ all\ g \in A\}$. Correspondingly, for a set $B \subseteq M$ of attributes define $B^{'} = \{g \in G \mid gIm\ for\ all\ m \in B\}$. A concept of the context (G, M, I) is a pair (A, B) where $A \subseteq G, B \subseteq M$, $A^{'} = B$ and $B^{'} = A$. We call A the extent and B the intent of the concept (A, B).

Suppose (A_1, B_1), (A_2, B_2) are two concepts of the context (G, M, I), we define an order $\leq$, where $(A_1, B_1) \leq (A_2, B_2)$ if $A_1 \subseteq A_2$ (which is equivalent to $B_1 \supseteq B_2$).

Let $\mathcal{C}(G, M, I, \leq)$ be the set of all concepts of the context (G, M, I) with the order $\leq$. It forms a complete lattice (Concept Lattice) in which join and meet are given by:

$$\bigvee_{j\in J}(A_j, B_j) = ((\bigcup_{j\in J} A_j)^{''}, \bigcap_{j\in J} B_j), \qquad \bigwedge_{j\in J}(A_j, B_j) = (\bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{''}).$$

2.2. **Fuzzy sets.**

We introduce some basic definitions about fuzzy sets, fuzzy relations [27].

For a set X, a mapping $\lambda : X \to I$ is called a fuzzy set in X, where $I = [0, 1]$. The collection of all fuzzy sets in X is denoted by I^X. In particular, 0 and 1 denote the fuzzy empty set and the fuzzy whole set in X, respectively.

For any $\lambda,\ \mu \in I^X$, the join $(\vee)$ and meet $(\wedge)$ of λ and μ, denoted by $\lambda \vee \mu$ and $\lambda \wedge \mu$, are defined as follows: for each $x \in X$,

$$(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}, \qquad (\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}.$$

For any family $(\lambda_j)_{j\in J}$ of fuzzy sets in X, the join $(\vee)$ and meet $(\wedge)$ of $(\lambda_j)_{j\in J}$, denoted by $\bigvee_{j\in J} \lambda_j$ and $\bigwedge_{j\in J} \lambda_j$, are defined as follows: for each $x \in X$,

$$(\bigvee_{j\in J} \lambda_j)(x) = \sup_{j\in J}\lambda_j(x) \qquad (\bigwedge_{j\in J} \lambda_j)(x) = \inf_{j\in J}\lambda_j(x).$$

For two sets $X,\ Y$, $r \in I^{X\times Y}$ is called a fuzzy relation from X to Y.

2.3. **Interval-valued fuzzy sets.**

In the section, we introduce some basic definitions related to interval-valued fuzzy sets and interval-valued fuzzy relations (See [10, 11]).

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called interval numbers and are denoted by $\widetilde{a},\ \widetilde{b},\ \widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$.

We define relations $\preceq$ and $=$ on $[I]$ as follows:

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} \preceq \widetilde{b} \Longleftrightarrow a^- \leq b^- \text{ and } a^+ \leq b^+),$$

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} = \widetilde{b} \Longleftrightarrow \widetilde{a} \succeq \widetilde{b} \text{ and } \widetilde{a} \preceq \widetilde{b}), i.e.,$$

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} = \widetilde{b} \Longleftrightarrow a^- = b^- \text{ and } a^+ = b^+).$$

To say $\widetilde{a} < \widetilde{b}$, we mean $\widetilde{a} \preceq \widetilde{b}$ and $\widetilde{a} \neq \widetilde{b}$.

For any $\widetilde{a},\ \widetilde{b} \in [I]$, their minimum and maximum, denoted by $\widetilde{a} \wedge \widetilde{b}$ and $\widetilde{a} \vee \widetilde{b}$ are defined as follows:

$$\widetilde{a} \wedge \widetilde{b} = [a^- \wedge b^-, a^+ \wedge b^+], \qquad \widetilde{a} \vee \widetilde{b} = [a^- \vee b^-, a^+ \vee b^+].$$

Let $(\widetilde{a}_j)_{j\in J} \subset [I]$. Then its inf and sup, denoted by $\bigwedge_{j\in J} \widetilde{a}_j$ and $\bigvee_{j\in J} \widetilde{a}_j$, are defined as follows:

$$\bigwedge_{j\in J} \widetilde{a}_j = [\bigwedge_{j\in J} a_j^-, \bigwedge_{j\in J} a_j^+], \qquad \bigvee_{j\in J} \widetilde{a}_j = [\bigvee_{j\in J} a_j^-, \bigvee_{j\in J} a_j^+].$$

For a nonempty set X, a mapping $A : X \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^-, A^+ \in I^X$ are called a lower fuzzy set and an upper fuzzy set in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in X, respectively. We define relations $\subset$ and $=$ on $[I]^X$ as follows:

$$(\forall\ A,\ B \in [I]^X)(A \subset B \Longleftrightarrow (x \in X)(A(x) \preceq B(x)),$$

$$(\forall\ A,\ B \in [I]^X)(A = B \Longleftrightarrow (x \in X)(A(x) = B(x)).$$

For any $(A_j)_{j\in J} \subset [I]^X$, its intersection $\bigcap_{j\in J} A_j$ and union $\bigcup_{j\in J} A_j$ are defined, respectively as follows: for each $x \in X$,

$$(\bigcap_{j\in J} A_j)(x) = \bigwedge_{j\in J} A_j(x), \qquad (\bigcup_{j\in J} A_j)(x) = \bigvee_{j\in J} A_j(x).$$

For two sets $X,\ Y$, $R \in [I]^{X\times Y}$ is called an interval-valued fuzzy relation (briefly, IVF relation) from X to Y.

2.4. **Cubic sets and cubic relations.** As a generalization, the notions of a cubic fuzzy set and a cubic fuzzy relation are introduced in [14, 20].

Definition 2.1 ([14])**.** Let X be a nonempty set. Then a complex mapping $\mathcal{A} =< A, \lambda >: X \to [I] \times I$ is called a cubic set in X.

In special, a cubic set $\mathcal{A} =< A, \lambda >$ in which $A(x) = \mathbf{0}$ and $\lambda(x) = 0$ (resp. $A(x) = \mathbf{1}$ and $\lambda(x) = 1$) for each $x \in X$ is denoted by $\hat{0}$ (resp. $\hat{1}$). In this case, $\hat{0}$ (resp. $\hat{1}$) will be called a cubic empty (resp. whole) set in X.

We denote the set of all cubic sets in X as $([I] \times I)^X$.

Definition 2.2 ([14]). Let $\mathcal{A} =< A, \lambda >$, $\mathcal{B} =< B, \mu >\in ([I] \times I)^X$. Then we define the following relations:
(i) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow A = B$ and $\lambda = \mu$,
(ii) (P-order) $\mathcal{A} \sqsubset \mathcal{B} \Leftrightarrow A \subset B$ and $\lambda \leq \mu$,
(iii) (R-order) $\mathcal{A} \Subset \mathcal{B} \Leftrightarrow A \subset B$ and $\lambda \geq \mu$

Definition 2.3 ([14]). Let $\mathcal{A} =< A, \lambda >$, $\mathcal{B} =< B, \mu >\in ([I] \times I)^X$ and let $(\mathcal{A}_j)_{j\in J} = (< A_j, \lambda_j >)_{j\in J} \subset ([I] \times I)^X$. Then the complement $\mathcal{A}^c$ of $\mathcal{A}$, P-union $\sqcup$, P-intersection $\sqcap$, R-union $\Cup$ and R-intersection $\Cap$ are defined as follows, respectively: for each $x \in X$,
(i) (Complement) $\mathcal{A}^c(x) =< A^c(x), \lambda^c(x) >$,
(ii) (P-union) $(\mathcal{A} \sqcup \mathcal{B})(x) =< (A \cup B)(x), (\lambda \vee \mu)(x) >$,
$(\sqcup_{j\in J}\mathcal{A}_j)(x) =< (\bigcup_{j\in J} A_j)(x), (\bigvee_{j\in J} \lambda_j)(x) >$,
(iii) (P-intersection) $(\mathcal{A} \sqcap \mathcal{B})(x) =< (A \cap B)(x), (\lambda \wedge \mu)(x) >$,
$(\sqcap_{j\in J}\mathcal{A}_j)(x) =< (\bigcap_{j\in J} A_j)(x), (\bigwedge_{j\in J} \lambda_j)(x) >$,
(iv) (R-union) $(\mathcal{A} \Cup \mathcal{B})(x) =< (A \cup B)(x), (\lambda \wedge \mu)(x) >$,
$(\Cup_{j\in J}\mathcal{A}_j)(x) =< (\bigcup_{j\in J} A_j)(x), (\bigwedge_{j\in J} \lambda_j)(x) >$,
(v) (R-intersection) $(\mathcal{A} \Cap \mathcal{B})(x) =< (A \cap B)(x), (\lambda \vee \mu)(x) >$,
$(\Cap_{j\in J}\mathcal{A}_j)(x) =< (\bigcap_{j\in J} A_j)(x), (\bigvee_{j\in J} \lambda_j)(x) >$.

In special, for cubic sets $< \widetilde{a}, \lambda >, < \widetilde{b}, \mu >, < \widetilde{a}_j, \lambda_j >, < \widetilde{b}_j, \mu_j >$ $(j \in J)$, we also adopt the above symbols:

$$< \widetilde{a}, \lambda >\sqsubset< \widetilde{b}, \mu >\Leftrightarrow \widetilde{a} \preceq \widetilde{b} \text{ and } \lambda \leq \mu, \qquad < \widetilde{a}, \lambda >=< \widetilde{b}, \mu >\Leftrightarrow \widetilde{a} = \widetilde{b} \text{ and } \lambda = \mu,$$

$$< \widetilde{a}, \lambda > \sqcup < \widetilde{b}, \mu >=< \widetilde{a} \vee \widetilde{b}, \lambda \vee \mu >, \qquad < \widetilde{a}, \lambda > \sqcap < \widetilde{b}, \mu >=< \widetilde{a} \wedge \widetilde{b}, \lambda \wedge \mu >,$$

$$\bigsqcup_{j\in J} < \widetilde{a}_j, \lambda_j >=< \bigvee_{j\in J} \widetilde{a}_j, \bigvee_{j\in J} \lambda_j >, \qquad \sqcap_{j\in J} < \widetilde{a}_j, \lambda_j >=< \bigwedge_{j\in J} \widetilde{a}_j, \bigwedge_{j\in J} \lambda_j >,$$

$$< \widetilde{a}, \lambda >\Subset< \widetilde{b}, \mu >\Leftrightarrow \widetilde{a} \preceq \widetilde{b} \text{ and } \lambda \geq \mu.$$

Next, we define a cubic point.

Definition 2.4. Let $\mathcal{A} =< A, \lambda >\in ([I] \times I)^X$, let $\widetilde{a} \in [I]$ with $a^+ > 0$ and let $\alpha \in I$ with $\alpha > 0$. Then $\mathcal{A} =< A, \lambda >$ is called a cubic point in X with the support $x \in X$ and the value $< \widetilde{a}, \alpha >$, denoted by $x_{<\widetilde{a},\alpha>}$, if for each $y \in X$,

$$x_{<\widetilde{a},\alpha>} = \begin{cases} < \widetilde{a}, \alpha > & \text{if } y = x \\ < \mathbf{0}, 0 > & \text{otherwise.} \end{cases}$$

The set of all cubic points in X is denoted by M_X.

Definition 2.5. Let $x_{<\widetilde{a},\alpha>} \in M_X$ and let $\mathcal{A} =< A, \lambda >\in ([I] \times I)^X$.
(i) $x_{<\widetilde{a},\alpha>}$ is said to belong to $\mathcal{A}$ by P-order type, denoted by $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$, if
$\widetilde{a} \preceq A(x)$ and $\alpha \leq \lambda(x)$, i.e., $x_{\widetilde{a}} \in A$ and $x_\alpha \in \lambda$.
(ii) $x_{<\widetilde{a},\alpha>}$ is said to belong to $\mathcal{A}$ by R-order type, denoted by $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}$, if
$\widetilde{a} \preceq A(x)$ and $\alpha \geq \lambda(x)$, i.e., $x_{\widetilde{a}} \in A$ and $x_{1-\alpha} \in \lambda^c$.

Proposition 2.6 ([14]). *Let $x_{<\widetilde{a},\alpha>} \in M_X$ and let $\mathcal{A} =< A, \lambda >$, $\mathcal{B} =< B, \mu >\in ([I] \times I)^X$.*
(1) *$\mathcal{A} \sqsubset \mathcal{B}$ if and only if $x_{<\widetilde{a},\alpha>} \in_P \mathcal{B}$, for each $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$.*
(2) *$\mathcal{A} \Subset \mathcal{B}$ if and only if $x_{<\widetilde{a},\alpha>} \in_R \mathcal{B}$, for each $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}$.*

Proposition 2.7 ([14]). *Let* $\mathcal{A} =< A, \lambda >$, $\mathcal{B} =< B, \mu >\in ([I] \times I)^X$, *let* $(\mathcal{A}_j)_{j\in J} = (< A_j, \lambda_j >)_{j\in J} \subset ([I] \times I)^X$ *and let* $x_{<\widetilde{a},\alpha>} \in M_X$*.*

(1) *If* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$ *or* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{B}$*, then* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A} \sqcup \mathcal{B}$*.*

$(1)'$ *If there is* $j \in J$ *such that* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}_j$*, then* $x_{<\widetilde{a},\alpha>} \in_P \sqcup_{j\in J}\mathcal{A}_j$*.*

(2) *If* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}$ *or* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{B}$*, then* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A} \Cup \mathcal{B}$*.*

$(2)'$ *If there is* $j \in J$ *such that* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}_j$*, then* $x_{<\widetilde{a},\alpha>} \in_R \Cup_{j\in J}\mathcal{A}_j$*.*

3. Concept lattice on cubic sets

In the section, we introduce the notions of Galois connection, formal context, concept lattice on cubic sets. First, Galois connection is the basic structure related to concept lattice, closure operator, etc..

Definition 3.1. A Galois connection between two cubic sets X and Y is a pair (f, g) of mappings

$f : ([I] \times I)^X \to ([I] \times I)^Y$, $\quad g : ([I] \times I)^Y \to ([I] \times I)^X$, satisfying

(1) $\mathcal{A}_1 \sqsubset \mathcal{A}_2 \Rightarrow f(\mathcal{A}_2) \sqsubset f(\mathcal{A}_1)$, $\quad$ (2) $\mathcal{B}_1 \sqsubset \mathcal{B}_2 \Rightarrow g(\mathcal{B}_2) \sqsubset g(\mathcal{B}_1)$,

(3) $\mathcal{A} \sqsubset gf(\mathcal{A})$, $\quad$ (4) $\mathcal{B} \sqsubset fg(\mathcal{B})$,

for any $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2 \in ([I] \times I)^X$ and $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2 \in ([I] \times I)^Y$.

In the theory of formal concept analysis, formal context is one of important notions and tools. On cubic sets, a formal context is the triple structure $(X, Y, \mathcal{R})$, where X is the sets of all objects, Y the sets of all attributes, $\mathcal{R}$ is a cubic relation between them. See the follow example.

Example 3.2. Let $X = \{a, b, c\}$, $Y = \{p, q\}$ be two sets, a cubic relation $\mathcal{R}$ between X and Y given by the following table:

$\mathcal{R} =< R, r >$	p	q
a	$< [0, 1], 1 >$	$< [0.4, 0.8], 0.4 >$
b	$< [0.2, 0.6], 0.4 >$	$< [0.2, 1], 1 >$
c	$< [0, 0.5], 0.5 >$	$< [0.5, 1], 0.5 >$

Table 3.1

Suppose $\mathcal{R}$ is a cubic relation on $X \times Y$, we obtain a Galois connection $(^{\uparrow_\mathcal{R}}, ^{\downarrow_\mathcal{R}})$, in simple way, $(^\uparrow, ^\downarrow)$.

Definition 3.3. Suppose $\mathcal{R}$ is a cubic relation on $X \times Y$, for any $\mathcal{A} \in ([I] \times I)^X$, $\mathcal{B} \in ([I] \times I)^Y$, and for every $x \in X$, $y \in Y$, two cubic sets are defined.

$$\mathcal{A}^\uparrow(y) = \sqcap_{x\in X} \bigsqcup_{y_{<\widetilde{b},\mu>}\in M_Y} \{< \widetilde{b}, \mu >|< \widetilde{b}, \mu > \sqcap \mathcal{A}(x) \sqsubset \mathcal{R}(x, y)\}$$

$$\mathcal{B}^\downarrow(x) = \sqcap_{y\in Y} \bigsqcup_{x_{<\widetilde{a},\lambda>}\in M_X} \{< \widetilde{a}, \lambda >|< \widetilde{a}, \lambda > \sqcap \mathcal{B}(y) \sqsubset \mathcal{R}(x, y)\}$$

If $\mathcal{A}^{\uparrow} = \mathcal{B}$, and $\mathcal{A} = \mathcal{B}^{\downarrow}$, then $(\mathcal{A}, \mathcal{B})$ is called a cubic concept of $(X, Y, \mathcal{R})$, $\mathcal{A}$ is the extent and $\mathcal{B}$ the intent of the concept $(\mathcal{A}, \mathcal{B})$. The set of all cubic concepts is denoted by $C(X, Y, \mathcal{R})$. i.e.,

$$C(X, Y, \mathcal{R}) = \{(\mathcal{A}, \mathcal{B}) \mid \mathcal{A}^{\uparrow} = \mathcal{B}, \mathcal{A} = \mathcal{B}^{\downarrow}\}$$

We define a partial order $\sqsubset$ on $C(X, Y, \mathcal{R})$, for $(\mathcal{A}_1, \mathcal{B}_1), (\mathcal{A}_2, \mathcal{B}_2) \in C(X, Y, \mathcal{R})$, $(\mathcal{A}_1, \mathcal{B}_1) \sqsubset (\mathcal{A}_2, \mathcal{B}_2) \Leftrightarrow \mathcal{A}_1 \sqsubset \mathcal{A}_2$ (which is equivalent to $\mathcal{B}_2 \sqsubset \mathcal{B}_1$). Clearly $(\mathcal{A}_1, \mathcal{B}_1) = (\mathcal{A}_2, \mathcal{B}_2) \Leftrightarrow \mathcal{A}_1 = \mathcal{A}_2$ (which is equivalent to $\mathcal{B}_2 = \mathcal{B}_1$).

Next, we introduce a example as follows:

Example 3.4. Following Example 3.2, for a cubic set $\mathcal{A} = \hat{1}_X$ on X, we obtain $\mathcal{A}^{\uparrow}$,

$$\begin{aligned}
\mathcal{A}^{\uparrow}(p) &= \sqcap_{x\in X} \bigsqcup_{p_{<\widetilde{b},\mu>}\in M_Y} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap \mathcal{A}(x) \sqsubset \mathcal{R}(x, p)\} \\
&= (\bigsqcup_{p_{<\widetilde{b},\mu>}\in M_Y} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap <\mathbf{1}, 1> \sqsubset <[0,1], 1>\}) \\
&\quad \sqcap (\bigsqcup_{p_{<\widetilde{b},\mu>}\in M_Y} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap <\mathbf{1}, 1> \sqsubset <[0.2, 0.6], 0.4>\}) \\
&\quad \sqcap (\bigsqcup_{p_{<\widetilde{b},\mu>}\in M_Y} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap <\mathbf{1}, 1> \sqsubset <[0, 0.5], 0.5>) \\
&= <[0,1], 1> \sqcap <[0.2, 0.6], 0.4> \sqcap <[0, 0.5], 0.5> \\
&= <[0, 0.5], 0.4>,
\end{aligned}$$

In the same way, we obtain $\mathcal{A}^{\uparrow}(q) = <[0.2, 0.8], 0.4>$. Then $\mathcal{A}^{\uparrow}$ is a cubic set on Y.

Furthermore, let $\mathcal{B} = \mathcal{A}^{\uparrow}$, we obtain $\mathcal{B}^{\downarrow}$, $\mathcal{B}^{\downarrow}(a) = <\mathbf{1}, 1>$, $\mathcal{B}^{\downarrow}(b) = <\mathbf{1}, 1>$, $\mathcal{B}^{\downarrow}(c) = <\mathbf{1}, 1>$, i.e., $\mathcal{B}^{\downarrow} = \hat{1}_X$. Clearly $\mathcal{B}^{\downarrow}$ is a cubic set on X, and $\mathcal{B}^{\downarrow} = \mathcal{A}$.

By Definition 3.3, we know that $(\mathcal{A}, \mathcal{B})$ is a cubic concept of the cubic formal context $(X, Y, \mathcal{R})$.

Third, we prove the set $C(X, Y, \mathcal{R})$ forms a complete lattice.

Proposition 3.5. *Suppose $x \in X, y \in Y$, we have*

$$(x_{<\mathbf{1},1>})^{\uparrow}(y) = (y_{<\mathbf{1},1>})^{\downarrow}(x)$$

Proof. (1)

$$\begin{aligned}
(x_{<\mathbf{1},1>})^{\uparrow}(y) &= \sqcap_{z\in X} \bigsqcup_{y_{<\widetilde{b},\mu>}} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap x_{<\mathbf{1},1>}(z) \sqsubset \mathcal{R}(z, y)\} \\
&= (\sqcap_{z\neq x} \bigsqcup_{y_{<\widetilde{b},\mu>}} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap x_{<\mathbf{1},1>}(z) \sqsubset \mathcal{R}(z, y)\}) \\
&\quad \sqcap (\bigsqcup_{y_{<\widetilde{b},\mu>}} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap x_{<\mathbf{1},1>}(x) \sqsubset \mathcal{R}(x, y)\}) \\
&= (\sqcap_{z\neq x} \bigsqcup_{y_{<\widetilde{b},\mu>}} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap <\mathbf{0}, 0> \sqsubset \mathcal{R}(z, y)\}) \\
&\quad \sqcap (\bigsqcup_{y_{<\widetilde{b},\mu>}} \{<\widetilde{b}, \mu> \mid <\widetilde{b}, \mu> \sqcap <\mathbf{1}, 1> \sqsubset \mathcal{R}(x, y)\})
\end{aligned}$$

$$= (\sqcap s_{z\neq x}\{<\mathbf{1}, 1>\}) \sqcap (\bigsqcup_{y_{<\widetilde{b},\mu>}} \{<\widetilde{b},\mu>|<\widetilde{b},\mu>\sqsubset \mathcal{R}(x,y)\})$$
$$=<\mathbf{1},1> \sqcap (\bigsqcup_{y_{<\widetilde{b},\mu>}} \{<\widetilde{b},\mu>|<\widetilde{b},\mu>\sqsubset \mathcal{R}(x,y)\})$$
$$= \bigsqcup_{y_{<\widetilde{b},\mu>}} \{<\widetilde{b},\mu>|<\widetilde{b},\mu>\sqsubset \mathcal{R}(x,y)\}$$
$$= \mathcal{R}(x,y),$$

(2) In the similar way, we obtain $(y_{<\mathbf{1},1>})^{\downarrow}(x) = \mathcal{R}(x,y)$.

So $x_{<\mathbf{1},1>}(y) = \mathcal{R}(x,y) = y_{<\mathbf{1},1>}(x)$. □

Now, we consider the surpremum, the infimum of a family cubic concepts of $C(X,Y,\mathcal{R})$.

Proposition 3.6. *Suppose* $(\mathcal{A}_j,\mathcal{B}_j)$ $(j\in J)$ *is a family of cubic concepts of* $C(X,Y,\mathcal{R})$, *then*

$$(\bigsqcup_{j\in J}\mathcal{A}_j)^{\uparrow} = \sqcap_{j\in J}\mathcal{A}_j^{\uparrow}, \qquad (\bigsqcup_{j\in J}\mathcal{B}_j)^{\downarrow} = \sqcap_{j\in J}\mathcal{B}_j^{\downarrow}.$$

Proof. (1) We first to prove $(\bigsqcup_{j\in J}\mathcal{A}_j)^{\uparrow} = \sqcap_{j\in J}\mathcal{A}_j^{\uparrow}$.

For every $y_{<\widetilde{b},\mu>} \in (\bigsqcup_{j\in J}\mathcal{A}_j)^{\uparrow}$

$$\Rightarrow \forall x\in X, \quad <\widetilde{b},\mu> \sqcap (\bigsqcup_{j\in J}\mathcal{A}_j)(x) \sqsubset \mathcal{R}(x,y)$$
$$\Rightarrow \forall x\in X, \quad <\widetilde{b},\mu> \sqcap \bigsqcup_{j\in J}\mathcal{A}_j(x) \sqsubset \mathcal{R}(x,y)$$
$$\Rightarrow \forall j\in J, \ \ \forall x\in X, \quad <\widetilde{b},\mu> \sqcap \mathcal{A}_j(x) \sqsubset \mathcal{R}(x,y)$$
$$\Rightarrow \forall j\in J, \quad y_{<\widetilde{b},\mu>} \in \mathcal{A}_j^{\uparrow}$$
$$\Rightarrow \forall j\in J, \quad y_{<\widetilde{b},\mu>} \in \sqcap_{j\in J}\mathcal{A}_j^{\uparrow}.$$

Then $(\bigsqcup_{j\in J}\mathcal{A}_j)^{\uparrow} \sqsubset \sqcap_{j\in J}\mathcal{A}_j^{\uparrow}$.

On the other hand,

For every $y_{<\widetilde{b},\mu>} \in \sqcap_{j\in J}\mathcal{A}_j^{\uparrow}$

$$\Rightarrow \forall j\in J, \quad y_{<\widetilde{b},\mu>} \in \mathcal{A}_j^{\uparrow}$$
$$\Rightarrow \forall j\in J, \ \ \forall x\in X, \ \ <\widetilde{b},\mu> \sqcap \mathcal{A}_j(x) \sqsubset \mathcal{R}(x,y)$$
$$\Rightarrow \forall x\in X, \ \ <\widetilde{b},\mu> \sqcap \bigsqcup_{j\in J}\mathcal{A}_j(x) \sqsubset \mathcal{R}(x,y)$$
$$\Rightarrow \forall x\in X, \ \ <\widetilde{b},\mu> \sqcap (\bigsqcup_{j\in J}\mathcal{A}_j)(x) \sqsubset \mathcal{R}(x,y)$$
$$\Rightarrow y_{<\widetilde{b},\mu>} \in (\bigsqcup_{j\in J}\mathcal{A}_j)^{\uparrow}.$$

Thus $\sqcap_{j\in J}\mathcal{A}_j^{\uparrow} \sqsubset (\bigsqcup_{j\in J}\mathcal{A}_j)^{\uparrow}$. So the equality holds.

(2) In the similar way.

□

Since $\mathcal{A}_j^{\uparrow} = \mathcal{B}_j$, we obtain $(\bigsqcup_{j\in J} \mathcal{A}_j)^{\uparrow} = \sqcap_{j\in J}\mathcal{A}_j^{\uparrow} = \sqcap_{j\in J}\mathcal{B}_j$.

In the similar way, we prove $(\sqcap_{j\in J}\mathcal{B}_j)^{\downarrow} = \bigsqcup_{j\in J} \mathcal{A}_j$. Then $((\bigsqcup_{j\in J} \mathcal{A}_j)^{\uparrow\downarrow}, \sqcap_{j\in J}\mathcal{B}_j)$ is a cubic concept of $(X, Y, \mathcal{R})$.

Moreover, we also obtain $(\sqcap_{j\in J}\mathcal{A}_j, (\bigsqcup_{j\in J} \mathcal{B}_j)^{\downarrow\uparrow})$ is a cubic concept of $(X, Y, \mathcal{R})$.

In $C(X, Y, \mathcal{R})$, we define $((\bigsqcup_{j\in J} \mathcal{A}_j)^{\uparrow\downarrow}, \sqcap_{j\in J}\mathcal{B}_j)$ as the surpremum of the family $(\mathcal{A}_j, \mathcal{B}_j)$ $(j \in J)$ cubic concepts of $(X, Y, \mathcal{R})$, and $(\sqcap_{j\in J}\mathcal{A}_j, (\bigsqcup_{j\in J} \mathcal{B}_j)^{\downarrow\uparrow})$ as the infimum of the family $(\mathcal{A}_j, \mathcal{B}_j)$ $(j \in J)$ cubic concepts of $(X, Y, \mathcal{R})$. So we obtain,

Proposition 3.7. $(C(X, Y, \mathcal{R}), \sqsubset)$ *forms a complete lattice with respect the operators* $\bigsqcup$ *and* $\sqcap$*, where*

$$\bigsqcup_{j\in J}(\mathcal{A}_j, \mathcal{B}_j) = ((\bigsqcup_{j\in J} \mathcal{A}_j)^{\uparrow\downarrow}, \sqcap_{j\in J}\mathcal{B}_j), \qquad \sqcap_{j\in J}(\mathcal{A}_j, \mathcal{B}_j) = (\sqcap_{j\in J}\mathcal{A}_j, (\bigsqcup_{j\in J} \mathcal{B}_j)^{\downarrow\uparrow}).$$

By Definition 3.3 and the above results, we obtain

Proposition 3.8. *Suppose* $\mathcal{R}$ *is a cubic relation between* X *and* Y*, then* $(^{\uparrow_{\mathcal{R}}}, {}^{\downarrow_{\mathcal{R}}})$ *is a Galois connection induced by* $\mathcal{R}$*.*

Conversely, suppose $(^{\uparrow}, {}^{\downarrow})$ is a Galois connection on cubic sets, we also define a cubic relation $\mathcal{R}$ as: for any $x \in X, y \in Y$, $\mathcal{R}(x, y) = (x_{<\mathbf{1},1>})^{\uparrow}(y) = (y_{<\mathbf{1},1>})^{\downarrow}(x)$, and $(^{\uparrow}, {}^{\downarrow})$ is precisely generated by $\mathcal{R}$, i.e., $(^{\uparrow}, {}^{\downarrow}) = (^{\uparrow_{\mathcal{R}}}, {}^{\downarrow_{\mathcal{R}}})$.

Note: In the paper, we introduce the notions of Galois connection, Concept lattice on cubic sets with respect to the P-order, P-union, P-intersection. The above results hold with respect to the R-order, R-union, R-intersection.

Conclusion

In the paper, we define formal context, Galois connection, concept lattice on cubic sets, thus combine formal concept analysis and cubic sets together, provide a new platform to further study.

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