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Intuitionistic neutrosophic crisp sets and their application to topology

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Abstract. In this paper, we introduce the new notion of intuitionistic neutrosophic crisp sets as a tool for approximating undefinable or complex concepts in real world. First, we deal with some of its algebraic structures. Next, we define an intuitionistic neutrosophic crisp topology, base (subbase) and interior (closure), respectively and investigate some of each properties, and give some examples. Finally, we discussed various intuitionistic neutrosophic crisp continuities.

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1. Introduction

In 2014, Salama et al. [18] proposed the concept of neutrosophic crisp sets as the generalization of classical sets and the special case of neutrosophic sets proposed by Smarandache [19, 20, 21], and studied its some algebraic structures and dealt with topological structures. After then, Hur et al. [7] investigated categorical structures via neutrosophic crisp sets. From now on, the notion of neutrosophic crisp sets has been mainly studied by many researchers [5, 6, 11, 13, 15, 16, 17]. Recently, Kim et al. [9] defined an interval-valued set and applied it to topological structures. Also, Kim et al. [10] introduced the concept of interval-valued neutrosophic crisp sets, and investigated its some algebraic and topological structures.

In order to express mathematically the complex real world, we propose a new concept combined intuitionistic set and neutrosophic crisp set, and apply it to topology. In order to accomplish such research, this paper is composed of six sections. In Section 2, we recall some definitions related to intuitionistic sets and neutrosophic crisp sets. In Section 3, we introduce the new concept of intuitionistic neutrosophic crisp set and obtain some of its algebraic structures, and give some examples. In Section 4, we define an intuitionistic neutrosophic crisp topology, an intuitionistic neutrosophic crisp base and subbase, and study some of their properties. In Section 5, we define an intuitionistic neutrosophic crisp interior and closure and obtain some of their properties. Also, we show that there is a unique INCT for intuitionistic neutrosophic crisp interior [resp. closure] operators. In Section 6, we deal with various properties of intuitionistic neutrosophic crisp continuities.

2. Preliminaries

In this section, we recall the concepts of an intuitionistic set introduced in [3]. Also we recall some concepts proposed in [12] and [9, 22].

Definition 2.1 ([3])**.** Let X be a non-empty set. Then A is called an intuitionistic set (briefly, IS) of X, if it is an object having the form

$$A = (A^{\in}, A^{\notin}),$$

such that $A^{\in} \cap A^{\notin} = \varnothing$, where $A^{\in}$ [resp. $A^{\notin}$] represents the set of memberships [resp. non-memberships] of each element $x \in X$ to A.

In fact, $A^{\in}$ [resp. $A^{\notin}$] is a subset of X agreeing or approving [resp. refusing or opposing] for a certain opinion, view, suggestion or policy.

The intuitionistic empty set [resp. the intuitionistic whole set] of X, denoted by $\bar{\varnothing}$ [resp. $\bar{X}$], is defined by $\bar{\varnothing} = (\varnothing, X)$ [resp. $\bar{X} = (X, \varnothing)$]. We will denote the set of all ISs of X as $IS(X)$. Also, it is clear that for each $A \in IS(X)$, $\chi_A = (\chi_{A^{\in}}, \chi_{A^{\notin}})$ is an intuitionistic fuzzy set in X proposed by Atanassov [1]. Thus we can consider the intuitionistic set A in X as an intuitionistic fuzzy set in X.

For the inclusion, the equality, the union and the intersection of intuionistic sets, and the complement of an intuitionistic set, the operations [ ] and < > on $IS(X)$, refer to [3].

Definition 2.2 ([12, 14])**.** Let X be a non-empty set. Then the form $A = \langle A^T, A^I, A^F \rangle$ is called a neutrosophic crisp set in X, if $A^T,\ A^I,\ A^F \subset X$.

In this case, A^T, A^I and A^F represent the set of memberships, indeterminacies and non-memberships respectively of each element $x \in X$ to A. In particular, a neutrosophic crisp set is defined as three types below.

A neutrosophic crisp set $A = \langle A^T, A^I, A^F \rangle$ in X is said to be of:

(i) Type 1, if it satisfies the following conditions:

$$A^T \cap A^I = \varnothing,\ A^T \cap A^F = \varnothing,\ A^I \cap A^F = \varnothing,$$

(ii) Type 2, if it satisfies the following conditions:

$$A^T \cap A^I = \varnothing,\ A^T \cap A^F = \varnothing,\ A^I \cap A^F = \varnothing,\ A^T \cup A^I \cup A^F = X,$$

(iii) Type 3, if it satisfies the following conditions:

$$A^T \cap A^I \cap A^F = \varnothing,\ A^T \cup A^I \cup A^F = X.$$

We consider neutrosophic crisp empty [resp. whole] sets of two types in X, denoted by $\varnothing_{1,N}$, $\varnothing_{2,N}$ [resp. $X_{1,N}$, $X_{2,N}$] and defined by (See Remark 1.1.1 in [12]):

$$\varnothing_{1,N} = \langle \varnothing, \varnothing, X \rangle\,,\ \varnothing_{2,N} = \langle \varnothing, X, X \rangle\ \ [\text{resp.} X_{1,N} = \langle X, X, \varnothing \rangle\,,\ X_{2,N} = \langle X, \varnothing, \varnothing \rangle].$$

We will denote the set of all neutrosophic crisp sets in X denoted by $N(X)$.

It is obvious that $A = \langle A, \varnothing, A^c \rangle \in N(X)$ for each ordinary subset A of X. Then we can consider a neutrosophic set in X as the generalization of an ordinary subset of X. Also, it is clear that $A = \left\langle A^{\in}, \varnothing, A^{\not\in} \right\rangle$ is an neutrosophic crisp set in X for each $A \in I(X)$. Thus we can consider a neutrosophic crisp set in X as the generalization of an intuitionisatic set in X. Furthermore, we can easily see that for each $A \in N(X)$,

$$\chi_A = \left\langle \chi_{A^T}, \chi_{A^I}, \chi_{A^F} \right\rangle$$

is a neutrosophic set in X introduced by Salama and Smarandache [19, 20, 21]. So we can consider the neutrosophic crisp set as the specialization of a neutrosophic set.

Definition 2.3 ([12]). Let $A \in N(X)$. Then the complement of A, denoted by $A^{i,c}$ $(i = 1,\ 2)$ and defined by:

$$A^{1,c} = \left\langle A^F, A^{I^c}, A^T \right\rangle,\ A^{2,c} = \left\langle A^F, A^I, A^T \right\rangle.$$

Definition 2.4 ([12]). Let X be a non-empty set and let $A,\ B \in N(X)$.

(i) We say that A is a 1-type subset of B, denoted by $A \subset_1 B$, if it satisfies the following conditions:

$$A^T \subset B^T,\ A^I \subset B^I,\ A^F \supset B^F.$$

(ii) We say that A is a 2-type subset of B, denoted by $A \subset_2 B$, if it satisfies the following conditions:

$$A^T \subset B^T,\ A^I \supset B^I,\ A^F \supset B^F.$$

Definition 2.5 ([12]). Let X be a non-empty set and let $A,\ B \in N(X)$.

(i) The i-intersection of A and B, denoted by $A \cap^i B$ $(i = 1,\ 2)$ and defined by:

$$A \cap^1 B = \left\langle A^T \cap B^T, A^I \cap B^I, A^F \cup B^F \right\rangle,\ A \cap^2 B = \left\langle A^T \cap B^T, A^I \cup B^I, A^F \cup B^F \right\rangle.$$

(ii) The i-union of A and B, denoted by $A \cup^i B$ $(i = 1,\ 2)$ and defined by:

$$A \cup^1 B = \left\langle A^T \cup B^T, A^I \cup B^I, A^F \cap B^F \right\rangle,\ A \cup^2 B = \left\langle A^T \cup B^T, A^I \cap B^I, A^F \cap B^F \right\rangle.$$

(iii) $[\ ]A = \left\langle A^T, A^I, A^{T^c} \right\rangle,\ \langle\ \rangle A = \left\langle A^{F^c}, A^I, A^F \right\rangle.$

Definition 2.6 ([12]). Let X be a non-empty set and let $(A_j)_{j \in J}$ be a family of neutrosophic crisp sets in X.

(i) The i-intersection of $(A_j)_{j \in J}$, denoted by $\bigcap^i{}_{j \in J} A_j$ $(i = 1,\ 2)$ and defined as follows:

$$\overset{1}{\bigcap}_{j \in J} A_j = \left\langle \bigcap_{j \in J} A_j^T, \bigcap_{j \in J} A_j^I, \bigcup_{j \in J} A_j^F \right\rangle,\ \overset{2}{\bigcap}_{j \in J} A_j = \left\langle \bigcap_{j \in J} A_j^T, \bigcup_{j \in J} A_j^I, \bigcup_{j \in J} A_j^F \right\rangle.$$

(ii) The i-union of $(A_j)_{j\in J}$, denoted by $\bigcup^i{}_{j\in J}A_j$ $(i = 1,\ 2)$ and defined as follows:

$$\overset{1}{\bigcup}_{j\in J}A_j = \left\langle \bigcup_{j\in J} A_j^T, \bigcup_{j\in J} A_j^I, \bigcap_{j\in J} A_j^F \right\rangle,\ \overset{2}{\bigcup}_{j\in J}A_j = \left\langle \bigcup_{j\in J} A_j^T, \bigcap_{j\in J} A_j^I, \bigcap_{j\in J} A_j^F \right\rangle.$$

Definition 2.7 ([9])**.** Let X be an non-empty set. Then the form

$$[A^-, A^+] = \{B \subset X : A^- \subset B \subset A^+\}$$

is called an interval-valued sets (briefly, IVS) in X, where $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. In particular, $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the interval-valued empty [resp. whole] set in X and denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$].

We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for classical subset A of X. Then we can consider an IVS in X as the generalization of a classical subset of X. Also, if $A = [A^-, A^+] \in IVS(X)$, then $\chi_A = [\chi_{A^-}, \chi_{A^+}]$ is an interval-valued fuzzy set in X introduced by Zadeh [23]. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS.

For the inclusion, the equality, the union and the intersection of interval-valued sets, and the complement of an interval-valued set, refer to [9, 22].

3. Intuitionistic neutrosophic crisp sets

In this section, we introduce the concept of an intuitionistic neutrosophic crisp set combined by a neutrosophic crisp set and an intuitionistic set, and obtain some of its properties.

Definition 3.1. Let X be a non-empty set. Then the form

$$\left\langle (A^{T,\in}, A^{T,\notin}), (A^{I,\in}, A^{I,\notin}), (A^{F,\in}, A^{F,\notin}) \right\rangle$$

is called an intuitionistic neutrosophic crispset (briefly, INCS) in X,
where $(A^{T,\in}, A^{T,\notin}),\ (A^{I,\in}, A^{I,\notin}),\ (A^{F,\in}, A^{F,\notin}) \in IS(X)$ such that $A^{T,\in} \cap A^{F,\in} = \varnothing$.

In this case, $(A^{T,\in}, A^{T,\notin})$, $(A^{I,\in}, A^{I,\notin})$ and $(A^{F,\in}, A^{F,\notin})$ represent the IS of memberships, indeterminacies and non-memberships respectively of each element $x \in X$ to A. In particular, an INCS is defined as three types below.

An INCS $A = \left\langle (A^{T,\in}, A^{T,\notin}), (A^{I,\in}, A^{I,\notin}), (A^{F,\in}, A^{F,\notin}) \right\rangle$ in X is said to be of:

(i) Type 1, if it satisfies the following conditions:

$$A^{T,\in} \cap A^{I,\in} = \varnothing,\ A^{I,\in} \cap A^{F,\in} = \varnothing,\ A^{T,\in} \cap A^{F,\in} = \varnothing,$$

(ii) Type 2, if it satisfies the following conditions:

$$A^{T,\in} \cap A^{I,\in} = \varnothing,\ A^{I,\in} \cap A^{F,\in} = \varnothing,\ A^{T,\in} \cup A^{I,\in} \cup A^{F,\in} = X,$$

(iii) Type 3, if it satisfies the following conditions:

$$A^{T,\in} \cap A^{I,\in} \cap A^{F,\in} = \varnothing,\ A^{T,\in} \cup A^{I,\in} \cup A^{F,\in} = X.$$

We will denote the set of all INCSs of Type 1 [resp. Type 2 and Type 3] in X denoted by $IN_1(X)$ [resp. $IN_2(X)$ and $IN_3(X)$], and $INC(X) = IN_1(X) \cup IN_2(X) \cup IN_3(X)$.

It is obvious that $\langle (A, A^c), \bar{\varnothing}, (A^c, A) \rangle \in INC(X)$ for classical subset A of X. Then we can consider an INCS in X as the generalization of a classical subset of X. Moreover, if $A = \left\langle (A^{T,\in}, A^{T,\not\in}), (A^{I,\in}, A^{I,\not\in}), (A^{F,\in}, A^{F,\not\in}) \right\rangle \in INC(X)$, then $\chi_A = \langle (\chi_{A^{T,\in}}, \chi_{A^{T,\not\in}}), (\chi_{A^{I,\in}}, \chi_{A^{I,+\not\in}}), (\chi_{A^{F,\in}}, \chi_{A^{F,\not\in}}) \rangle\rangle$ is an intuitionistic neutrosophic set in X. Thus we can consider an intuitionistic neutrosophic set as the generalization of an INCS.

Remark 3.2. In general, the followings hold:
(1) $IN_2(X) \subset IN_1(X),\ IN_2(X) \subset IN_3(X)$,
(2) $IN_1(X) \not\subset IN_2(X),\ IN_1(X) \not\subset IN_3(X)$ in general,
(3) $IN_3(X) \not\subset IN_1(X),\ IN_3(X) \not\subset IN_2(X)$ in general.

Example 3.3. Let $X = \{a, b, c, d, e, f, g, h, i\}$. Consider two INCSs in given by:

$$A = \langle (\{a, b, c, d\}, \{e, f\}), (\{e, f\}, \{a\}), (\{g, h\}, \{a, f\}) \rangle,$$

$$B = \langle (\{a, b, c, f\}, \{e, i\}), (\{a, d, e\}, \{f\}), (\{g, h, i\}, \{a, f\}) \rangle.$$

(i) $A^{T,\in} \cap A^{I,\in} = \varnothing,\ A^{T,\in} \cap A^{F,\in} = \varnothing,\ A^{I,\in} \cap A^{F,\in} = \varnothing$. Then $A \in IN_1(X)$. But we have $A^{T,\in} \cup A^{I,\in} \cup A^{F,\in} = [\{a, b, c, d, e, f, g, h\} \neq X$. Thus $A \not\in IN_2(X)$. Moreover, $A^{T,\in} \cap A^{I,\in} \cap A^{T,\in} = \varnothing$. So $A \not\in IN_3(X)$. Hence Remark 3.2 (2) holds.

(ii) $B^{T,\in} \cap B^{I,\in} \cap B^{F,\in} = \varnothing$ and $B^{T,\in} \cup B^{I,\in} \cup B^{F,\in} = X$. Then $B \in IN_3(X)$. But we have $B^{T,\in} \cap B^{I,\in} = \{a\} \neq \varnothing$. Thus $B \not\in IN_1(X),\ B \not\in IN_2(X)$. So Remark 3.2 (3) holds.

Definition 3.4. Let X be a non-empty set. Then we may define the intuitionistic neutrosophic crisp empty sets and the intuitionistic neutrosophic crisp whole sets, denoted by $\varnothing_{i,IN}$ and $X_{i,IN}$ $(i = 1,\ 2,\ 3,\ 4)$, respectively as follows:

(i) $\varnothing_{1,IN} = \left\langle \bar{\varnothing}, \bar{\varnothing}, \bar{X} \right\rangle,\ \varnothing_{2,IN} = \left\langle \bar{\varnothing}, \bar{X}, \bar{X} \right\rangle,$
$\varnothing_{3,IN} = \left\langle \bar{\varnothing}, \bar{X}, \bar{\varnothing} \right\rangle,\ \varnothing_{4,IN} = \langle \bar{\varnothing}, \bar{\varnothing}, \bar{\varnothing} \rangle,$

(ii) $X_{1,IN} = \left\langle \bar{X}, \bar{X}, \bar{\varnothing} \right\rangle,\ X_{2,IN} = \left\langle \bar{X}, \bar{\varnothing}, \bar{\varnothing} \right\rangle,$
$X_{3,IN} = \left\langle \bar{X}, \bar{\varnothing}, \bar{X} \right\rangle,\ X_{4,IN} = \left\langle \bar{X}, \bar{X}, \bar{X} \right\rangle.$

Definition 3.5. Let X be a non-empty set and let $A \in INC(X)$. Then the complements of A, denoted by $A^{i,c}$ $(i = 1,\ 2,\ 3)$, is an INCS in X, respectively as follows:

$$A^{1,c} = \left\langle (A^{T,\in}, A^{T,\not\in})^c, (A^{I,\in}, A^{I,\not\in})^c, (A^{F,\in}, A^{F,\not\in})^c \right\rangle,$$

$$A^{2,c} = \left\langle (A^{F,\in}, A^{F,\not\in}), (A^{I,\in}, A^{I,\not\in}), (A^{T,\in}, A^{T,\not\in}) \right\rangle,$$

$$A^{3,c} = \left\langle (A^{F,\in}, A^{F,\not\in}), (A^{I,\in}, A^{I,\not\in})^c, (A^{T,\in}, A^{T,\not\in}) \right\rangle.$$

Example 3.6. Let $A = \langle (\{a, b, c, d\}, \{e, f\}), (\{e, f\}, \{a\}), (\{g, h\}, \{a, f\}) \rangle$ be the INCS in X given in Example 3.3. Then we can easily check that

$$A^{1,c} = \langle (\{e, f\}, \{a, b, c, d\}), (\{a\}, \{e, f\}), (\{a, f\}, \{g, h\}) \rangle,$$

$$A^{2,c} = \langle (\{a, f\}, \{g, h\}, (\{e, f\}, \{a\}), (\{e, f\}, \{a, b, c, d\}) \rangle,$$

$$A^{3,c} = \langle (\{a, f\}, \{g, h\}, (\{a\}, \{e, f\}), (\{e, f\}, \{a, b, c, d\}) \rangle.$$

Definition 3.7. Let X be a non-empty set and let $A,\ B \in INC(X)$. Then we may define the inclusions between A and B, denoted by $A \subset_i B$ $(i = 1,\ 2)$, as follows:

$$A \subset_1 B \text{ iff } (A^{T,\in}, A^{T,\notin}) \subset (B^{T,\in}, B^{T,\notin}),\ (A^{I,\in}, A^{I,\notin}) \subset (B^{I,-}, B^{I,+}),\ (A^{F,\in}, A^{F,\notin}) \supset (B^{F,\in}, B^{F,\notin}),$$

$$A \subset_2 B \text{ iff } (A^{T,\in}, A^{T,\notin}) \subset (B^{T,\in}, B^{T,\notin}),\ (A^{I,\in}, A^{I,\notin}) \supset (B^{I,-}, B^{I,\notin}),\ (A^{F,\in}, A^{F,\notin}) \supset (B^{F,\in}, B^{F,\notin}).$$

Proposition 3.8. *For any $A \in INC(X)$, the followings hold:*

(1) $\varnothing_{1,IN} \subset_1 A \subset_1 X_{1,IN},\ \varnothing_{2,IN} \subset_2 A \subset_2 X_{2,IN}$,

(2) $\varnothing_{i,IN} \subset_j \varnothing_{i,IN},\ X_{i,IN} \subset_j X_{i,IN},\ (i = 1,\ 2,\ 3,\ 4,\ j = 1,\ 2)$.

Proof. Straightforward. □

Definition 3.9. Let X be a non-empty set and let $A,\ B \in INC(X)$.

(i) $A \cap B$ may be defined as two types:

$$A \cap^1 B =< (A^{T,\in}, A^{T,\notin}) \cap (B^{T,\in}, B^{T,\notin}), (A^{I,\in}, A^{I,\notin}) \cap (B^{I,\in}, B^{I,\notin}), (A^{F,\in}, A^{F,\notin}) \cup (B^{F,\in}, B^{F,\notin}) >,$$

$$A \cap^2 B =< (A^{T,\in}, A^{T,\notin}) \cap (B^{T,\in}, B^{T,\notin}), (A^{I,\in}, A^{I,\notin}) \cup [B^{I,\in}, B^{I,\notin}], (A^{F,\in}, A^{F,\notin}) \cup (B^{F,\in}, B^{F,\notin}) > .$$

(ii) $A \cup B$ may be defined as two types:

$$A \cup^1 B =< (A^{T,\in}, A^{T,\notin}) \cup (B^{T,\in}, B^{T,\notin}), (A^{I,\in}, A^{I,\notin}) \cup (B^{I,\in}, B^{I,\notin}), (A^{F,\in}, A^{F,\notin}) \cap (B^{F,\in}, B^{F,\notin}) >,$$

$$A \cap^2 B =< (A^{T,\in}, A^{T,\notin}) \cup (B^{T,\in}, B^{T,\notin}), (A^{I,\in}, A^{I,\notin}) \cap (B^{I,\in}, B^{I,\notin}), (A^{F,\in}, A^{F,\notin}) \cap (B^{F,\in}, B^{F,\notin}) > .$$

(iii) $[\ ]A = \langle [(A^{T,\in}, A^{T,\notin}), (A^{I,\in}, A^{I,\notin}), (A^{T,\in}, A^{T,\notin})^c \rangle$.

(iv) $<> A = \langle (A^{F,\in}, A^{F,\notin})^c, (A^{I,\in}, A^{I,\notin}), (A^{F,\in}, A^{F,\notin}) \rangle$.

From Definitions 3.4, 3.5, 3.7 and 3.9, we get the similar results of Propositions 3.5 and 3.6 in [9].

Proposition 3.10. *Let X be a non-empty set, let $A,\ B,\ C \in INC(X)$ and let $i = 1,\ 2$. Then*

(1) *if $A \subset_i B$ and $B \subset_i C$, then $A \subset_i C$,*

(2) *$A \subset_i A \cup^i B$ and $B \subset_i A \cup^i B$,*

(3) *$A \cap^i B \subset_i A$ and $A \cap^i B \subset_i B$,*

(4) *$A \subset_i B$ if and only if $A \cap^i B = A$,*

(5) *$A \subset_i B$ if and only if $A \cup^i B = B$.*

Proposition 3.11. *Let X be a non-empty set, let $A,\ B,\ C \in INC(X)$and let $i = 1,\ 2,\ j = 1,\ 2,\ 3,\ k = 1,\ 2,\ 3,\ 4$. Then*

(1) (Idempotent laws) $A \cup^i A = A,\ A \cap^i A = A$,

(2) (Commutative laws) $A \cup^i B = B \cup^i A,\ A \cap^i B = B \cap^i A$,

(3) (Associative laws) $A \cup^i (B \cup^i C) = (A \cup^i B) \cup^i C,\ A \cap^i (B \cap^i C) = (A \cap^i B) \cap^i C$,

(4) (Distributive laws) $A \cup^i (B \cap^i C) = (A \cup^i B) \cap^i (A \cup^i C)$,
$A \cap^i (B \cup^i C) = (A \cap^i B) \cup^i (A \cap^i C)$,

(5) (Absorption laws) $A \cup^i (A \cap^i B) = A,\ A \cap^i (A \cup^i B) = A$,

(6) (DeMorgan's laws) $(A \cup^1 B)^{1,c} = A^{1,c} \cap^1 B^{1,c},\ (A \cap^1 B)^{1,c} = A^{1,c} \cup^1 B^{1,c}$,
$(A \cup^1 B)^{2,c} = A^{2,c} \cap^2 B^{2,c},\ (A \cap^1 B)^{2,c} = A^{2,c} \cup^2 B^{2,c}$,
$(A \cup^1 B)^{3,c} = A^{3,c} \cap^1 B^{3,c},\ (A \cap^1 B)^{3,c} = A^{3,c} \cup^1 B^{3,c}$,

$$(A \cup^2 B)^{1,c} = A^{1,c} \cap^2 B^{1,c},\ (A \cap^2 B)^{1,c} = A^{1,c} \cup^2 B^{1,c},$$
$$(A \cup^2 B)^{2,c} = A^{2,c} \cap^1 B^{2,c},\ (A \cap^2 B)^{2,c} = A^{2,c} \cup^1 B^{2,c},$$
$$(A \cup^2 B)^{3,c} = A^{3,c} \cap^2 B^{3,c},\ (A \cap^2 B)^{3,c} = A^{3,c} \cup^2 B^{3,c},$$

(7) $(A^{j,c})^{j,c} = A$,

(8) (8_a) $A \cup^i \varnothing_{i,IN} = A,\ A \cap^i \varnothing_{i,IN} = \varnothing_{i,IN}$,

(8_b) $A \cup^i X_{i,IN} = X_{i,IN},\ A \cap^i X_{i,IN} = A$,

(8_c) $X_{1,IN}{}^{1,c} = \varnothing_{1,IN},\ X_{1,IN}{}^{2,c} = \varnothing_{2,IN},\ X_{1,IN}{}^{3,c} = \varnothing_{1,IN}$,
$X_{2,IN}{}^{1,c} = \varnothing_{2,IN},\ X_{2,IN}{}^{2,c} = \varnothing_{1,IN},\ X_{2,IN}{}^{3,c} = \varnothing_{2,IN}$,
$X_{3,IN}{}^{1,c} = \varnothing_{3,IN},\ X_{3,IN}{}^{2,c} = X_{3,IN},\ X_{3,IN}{}^{3,c} = X_{4,IN}$,
$X_{4,IN}{}^{1,c} = \varnothing_{4,IN},\ X_{4,IN}{}^{2,c} = X_{4,IN},\ X_{4,IN}{}^{3,c} = X_{3,IN}$,
$\varnothing_{1,IN}{}^{1,c} = X_{1,IN},\ \varnothing_{1,IN}{}^{2,c} = X_{2,IN},\ \varnothing_{1,IN}{}^{3,c} = X_{1,IVN}$,
$\varnothing_{2,IN}{}^{1,c} = X_{2,IN},\ \varnothing_{2,IN}{}^{2,c} = X_{1,IN},\ \varnothing_{2,IN}{}^{3,c} = X_{2,IN}$,
$\varnothing_{3,IN}{}^{1,c} = X_{3,IN},\ \varnothing_{3,IN}{}^{2,c} = \varnothing_{3,IN},\ \varnothing_{3,IN}{}^{3,c} = \varnothing_{4,IN}$,
$\varnothing_{4,IN}{}^{1,c} = X_{4,IN},\ \varnothing_{4,IN}{}^{2,c} = \varnothing_{4,IN},\ \varnothing_{4,IN}{}^{3,c} = \varnothing_{3,IN}$,

(8_d) $A \cup^i A^{j,c} \neq X_{k,IN},\ A \cap^i A^{j,c} \neq \varnothing_{k,IN}$ *in general* (See Example 3.12).

Example 3.12. Consider the IVNCS A in X given in Example 3.6. Then

$$\begin{aligned}&A \cap^1 A^{3,c}\\ &= \langle (\{a,b,c,d\},\{e,f\}),(\{e,f\},\{a\}),(\{g,h\},\{a,f\})\rangle\\ &\quad \cap^1 \langle (\{g,h\},\{a,f\}),(\{a\},\{e,f\}),(\{a,b,c,d\},\{e,f\})\rangle\\ &= \langle [\varnothing,\{a,e,f\}],[\varnothing,\{a,e,f\}],[\{a,b,c,d,g,h\},\{f\}]\rangle\\ &\neq \varnothing_{k,IN}.\end{aligned}$$

Similarly, we can check that

$$A \cup^1 A^{3,c} \neq X_{k,IN},\ A \cap^1 A^{1,c} \neq \varnothing_{k,IN},\ A \cup^1 A^{1,c} \neq X_{k,IN},$$

$$A \cap^1 A^{2,c} \neq \varnothing_{k,IN},\ A \cup^1 A^{2,c} \neq X_{k,IN}.$$

Also, we can easily check the remainders.

Definition 3.13. Let $(A_j)_{j \in J}$ be a family of INCSs in X. Then

(i) the intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$, is an INCS in X defined by:

$$\bigcap^{1}_{j \in J} A_j = \left\langle (\bigcap_{j \in J} A_j^{T,\in}, \bigcup_{j \in J} A_j^{T,\notin}), (\bigcap_{j \in J} A_j^{I,\in}, \bigcup_{j \in J} A_j^{I,\notin}), (\bigcup_{j \in J} A_j^{F,\in}, \bigcap_{j \in J} A_j^{F,\notin}] \right\rangle,$$

$$\bigcap^{2}_{j \in J} A_j = \left\langle (\bigcap_{j \in J} A_j^{T,\in}, \bigcup_{j \in J} A_j^{T,\notin}), (\bigcup_{j \in J} A_j^{I,\in}, \bigcap_{j \in J} A_j^{I,\notin}), (\bigcup_{j \in J} A_j^{F,\in}, \bigcap_{j \in J} A_j^{F,\notin}) \right\rangle,$$

(ii) the union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} \widetilde{A}_j$, is an INCS in X in X defined by:

$$\bigcup^{1}_{j \in J} A_j = \left\langle (\bigcup_{j \in J} A_j^{T,\in}, \bigcap_{j \in J} A_j^{T,\notin}), (\bigcup_{j \in J} A_j^{I,\in}, \bigcap_{j \in J} A_j^{I,\notin}), (\bigcap_{j \in J} A_j^{F,\in}, \bigcup_{j \in J} A_j^{F,\notin}) \right\rangle,$$

$$\bigcup^{2}_{j \in J} A_j = \left\langle (\bigcup_{j \in J} A_j^{T,\in}, \bigcap_{j \in J} A_j^{T,\notin}), (\bigcap_{j \in J} A_j^{I,\in}, \bigcup_{j \in J} A_j^{I,\notin}), (\bigcap_{j \in J} A_j^{F,\in}, \bigcup_{j \in J} A_j^{F,\notin}) \right\rangle,$$

From Definition 3.13, we get the similar result of Proposition 3.11 (6).

Proposition 3.14. *Let $A \in INC(X)$ and let $(A_j)_{j\in J}$ be a family of INCSs in X. Then*

(1) $(\bigcap^1_{j\in J} A_j)^{1,c} = \bigcup^1_{j\in J} A^{1,c}_j,\ (\bigcup^1_{j\in J} A_j)^{1,c} = \bigcap^1_{j\in J} A^{1,c}_j,$
$(\bigcap^1_{j\in J} A_j)^{2,c} = \bigcup^2_{j\in J} A^{2,c}_j,\ (\bigcup^1_{j\in J} A_j)^{2,c} = \bigcap^2_{j\in J} A^{2,c}_j,$
$(\bigcap^1_{j\in J} A_j)^{3,c} = \bigcup^1_{j\in J} A^{3,c}_j,\ (\bigcup^1_{j\in J} A_j)^{3,c} = \bigcap^1_{j\in J} A^{3,c}_j,$
$(\bigcap^2_{j\in J} A_j)^{1,c} = \bigcup^2_{j\in J} A^{1,c}_j,\ (\bigcup^2_{j\in J} A_j)^{1,c} = \bigcap^1_{j\in J} A^{1,c}_j,$
$(\bigcap^2_{j\in J} A_j)^{2,c} = \bigcup^1_{j\in J} A^{2,c}_j,\ (\bigcup^2_{j\in J} A_j)^{2,c} = \bigcap^1_{j\in J} A^{2,c}_j,$
$(\bigcap^2_{j\in J} A_j)^{3,c} = \bigcup^2_{j\in J} A^{3,c}_j,\ (\bigcup^2_{j\in J} A_j)^{3,c} = \bigcap^2_{j\in J} A^{3,c}_j,$

(2) $A \cap^i (\bigcup^i_{j\in J} A_j) = \bigcup^i_{j\in J}(A \cap^i A_j),\ A \cup^i (\bigcap^i_{j\in J} A_j) = \bigcap^i_{j\in J}(A \cup^i A_j)$ *$(i = 1,\ 2)$.*

From Propositions 3.11 and 3.14, we can easily see that $(INC(X), \cup^i, \cap^i, ^{j,c})$ is a boolean algebra except the condition (8_d) with the least element $\varnothing_{i,IN}$ and the largest $X_{i,IN}$, where $i = 1,\ 2$ and $j = 1,\ 2,\ 3$.

Definition 3.15. Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping and let $A \in INC(X),\ B \in INC(Y)$.

(i) The image of A under f, denoted by $f(A)$, is an INCS in Y defined as:

$$f(A) = \left\langle (f(A^{T,\in}), f(A^{T,\notin})), (f(A^{I,\in}), f(A^{I,\notin})), (f(A^{F,\in}), f(A^{F,\notin}))\right\rangle.$$

(ii) The preimage of B under f, denoted by $f^{-1}(B)$, is an interval set in X defined as:

$$f^{-1}(B) = \left\langle (f^{-1}(B^{T,\in}), f^{-1}(B^{T,\notin})), (f^{-1}(B^{I,\in}), f^{-1}(B^{I,\notin})), (f^{-1}(B^{F,\in}), f^{-1}(B^{F,\notin}))\right\rangle.$$

Proposition 3.16. *Let $X,\ Y$ be two non-empty sets, let $f : X \to Y$ be a mapping, let $A,\ A_1,\ A_2 \in INC(X),\ (A_j)_{j\in J} \subset INC(X)$ and let $B,\ B_1,\ B_2 \in INC(Y),\ (A_j)_{j\in J} \subset INC(Y)$. Let $i = 1,\ 2,\ j = 1,\ 2,\ 3,\ k = 1,\ 2,\ 3,\ 4$. Then*

(1) *if $A_1 \subset_i A_2$, then $f(A_1) \subset_i f(A_2)$,*
(2) *if $B_1 \subset_i B_2$, then $f^{-1}(B_1) \subset_i f^{-1}(B_1)$,*
(3) *$A \subset_i f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A))$,*
(4) *$f(f^{-1}(B)) \subset_i B$ and if f is surjective, $f(f^{-1}(B)) = B$,*
(5) *$f^{-1}(\bigcup^i_{j\in J} B_j) = \bigcup^i_{j\in J} f^{-1}(B_j)$,*
(6) *$f^{-1}(\bigcap^i_{j\in J} B_j) = \bigcap^i_{j\in J} f^{-1}(B_j)$,*
(7) *$f(\bigcup^i_{j\in J} A_j)_i \subset_i \bigcup^i_{j\in J} f(A_j)$ and if f is surjective, then $f(\bigcup^i_{j\in J} A_j)_i = \bigcup^i_{j\in J} f(A_j)$,*
(8) *$f(\bigcap^i_{j\in J} A_j) \subset_i \bigcap^i_{j\in J} f(A_j)$ and if f is injective, then $f(\bigcap^i_{j\in J} A_j) = \bigcap^i_{j\in J} f(A_j)$,*
(9) *if f is surjective, then $f(A)^{j,c} \subset_i f(A^{j,c})$.*
(10) *$f^{-1}(B^{j,c}) = f^{-1}(B)^{j,c}$.*
(11) *$f^{-1}(\varnothing_{k,IN}) = \varnothing_{k,IN},\ f^{-1}(X_{k,IN}) = X_{k,IN}$,*
(12) *$f(\varnothing_{k,IN}) = \varnothing_{k,IN}$ and if f is surjective, then $f(X_{k,IN}) = X_{k,IN}$,*
(13) *if $g : Y \to Z$ is a mapping, then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for each $C \in [Z]$.*

Proof. The proofs are straightforward. □

4. Intuitionistic topological spaces

In this section, we define an intuitionistic neutrosophic crisp topology on a non-empty set X and study some of its properties, and give some examples. Also, we introduce the concepts of an intuitionistic neutrosophic crisp base and subbase, and a family of INCSs gets the necessary and sufficient conditions to become INCB and gives some examples.

From this section to the rest sections, we will denote $\subset_1$, $\cup^1$, $\cap^1$, ^{3,c}, $\varnothing_{1,IN}$ and $X_{1,IN}$ by $\subset$, $\cap$, $\cup$, ^{c}, $\varnothing_{IN}$ and X_{IN}, respectively.

Definition 4.1. Let X be a non-empty set and let τ be a non-empty family of INCSs on X, i.e., $\varnothing \neq \tau \subset INC(X)$. Then τ is called an intuitionistic neutrosophic crisp topology (briefly, INCT) on X, if it satisfies the following axioms:
(INCO$_1$) $\varnothing_{IN},\ X_{IN} \in \tau$,
(INCO$_2$) $A \cap B \in \tau$ for any $A,\ B \in \tau$,
(INCO$_3$) $\bigcup_{j\in J} A_j \in \tau$ for any family $(A_j)_{j\in J}$ of members of τ.
In this case, the pair (X, τ) is called an intuitionistic neutrosophic crisp topological space (briefly, INCTS) and each member of τ is called an intuitionistic neutrosophic crisp open set (briefly, INCOS) in X. An INCS A is called an intuitionistic neutrosophic crisp closed set (briefly, INCCS) in X, if $A^c \in \tau$.

It is obvious that $\{\varnothing_{IN}, X_{IN}\}$ is an INCT on X, and will be called the intuitionistic neutrosophic crisp indiscrete topology (briefly, INCIT) on X and denoted by $\tau_{IN,0}$. Also $IN_1(X)$ is an INCT on X, and will be called the intuitionistic neutrosophic crisp discrete topology (briefly, INCDT) on X and denoted by $\tau_{IN,1}$. The pair $(X, \tau_{IN,0})$ [resp. $(X, \tau_{IN,1})$] will be called an intuitionistic neutrosophic crisp indiscrete [resp. discrete] space (briefly, INCITS) [resp. (briefly, INCDTS)].

We will denote the set of all INCTs on X as $INCT(X)$. For an INCTS X, we will denote the set of all INCOs [resp. INCCSs] in X as $INCO(X)$ [resp. $INCC(X)$].

Remark 4.2. (1) For each $\tau \in INCT(X)$, consider three families of ISs in X:
$$\tau^T = \{(A^{T,\in}, A^{T,\notin}) \in IS(X) : A \in \tau\},\ \tau^I = \{(A^{I,\in}, A^{I,\notin}) \in IS(X) : A \in \tau\},$$
$$\tau^F = \{(A^{F,\notin}, A^{F,\in}) \in IS(X) : A \in \tau\}.$$
Then we can easily check that τ^T, τ^I and τ^F are ITs on X proposed by Çoker [4].
In this case, τ^T [resp. τ^I and τ^F] will be called the membership [resp. indeterminacy and non-membership] topology of τ and we will write $\tau = \langle \tau^T, \tau^I, \tau^F \rangle$. In fact, we can consider $(X, \tau^T, \tau^I, \tau^F)$ as an intuitionistic tri-topological space on X (See the concept of bitopology introduced by Kelly [8]).
Also, let us consider six families of ordinary subsets of X:
$$\tau^{T,\in} = \{A^{T,\in} \subset X : A \in \tau\},\ \tau^{T,\notin} = \{A^{T,\notin^c} \subset X : A \in \tau\},$$
$$\tau^{I,\in} = \{A^{I,\in} \subset X : A \in \tau\},\ \tau^{I,\notin} = \{A^{I,\notin^c} \subset X : A \in \tau\},$$
$$\tau^{F,\in} = \{A^{F,\in^c} \subset X : A \in \tau\},\ \tau^{F,\notin} = \{A^{F,\notin} \subset X : A \in \tau\}.$$
Then clearly, $\tau^{T,\in}$, $\tau^{T,\notin}$, $\tau^{I,\in}$, $\tau^{I,\notin}$, $\tau^{F,\in}$, $\tau^{F,\notin}$ are ordinary topologies on X.
(2) Let (X, τ_o) be an ordinary topological space. Then there are three INCTs on X given by: for each $G \in \tau_o$,
$$\tau^1 = \begin{cases} \{\langle (G, G^c), \bar{\varnothing}, (G^c, G)\rangle : G \in \tau_o\} & \text{if } G \neq X \\ X_{IN} & \text{if } G = X, \end{cases}$$

$$\tau^2 = \begin{cases} \{\langle (G, \varnothing), \bar{X}, (\varnothing, G) \rangle : G \in \tau_o\} & \text{if } G \neq \varnothing \\ \varnothing_{IN} & \text{if } G = \varnothing, \end{cases}$$

$$\tau^3 = \begin{cases} \{\langle (\varnothing, G^c), \bar{\varnothing}, (G^c, \varnothing) \rangle : G \in \tau_o\} & \text{if } G \neq X \\ X_{IN} & \text{if } G = X. \end{cases}$$

(3) Let (X, τ_{IV}) be an IVTS introduced by Kim et al. [9]. Consider the family τ of intuitionistic neutrosophic sets in a set X given by:

$$\tau = \{\langle (A^-, A^{+c}), \bar{\varnothing}, (A^{-c}, A^+) \rangle \in INC(X) : A \in \tau_{IV}\}.$$

Then clearly, $\tau \in INCT(X)$.

(4) Let (X, τ_I) be an ITS introduced by Çoker [4]. Consider the family τ of intuitionistic neutrosophic sets in a set X given by:

$$\tau = \{\langle (A^{\in}, A^{\notin}), \bar{\varnothing}, (A^{\in c}, A^{\notin}) \rangle \in INC(X) : A \in \tau_I\}.$$

Then clearly, $\tau \in INCT(X)$.

(5) Let (X, τ_{NC}) be a neutrosophic crisp topological space introduced by Salama and Smarandache [12]. Then clearly,

$$\tau = \{\left\langle (A^T, A^{Tc}), (A^I, A^{Ic}), A^{Fc}, A^F) \right\rangle \in INC(X) : A \in \tau_{NC}\} \in INCT(X).$$

From Remark 4.2, we can easily see that an INCT is a generalization of a classical topology, an interval-valued intuitionistic topology (briefly, IVT) proposed by Cha et al. [2], an intuitionistic topology (briefly, IT) defined by Çoker [4] and a neutrosophic crisp topology introduced by Salama et al. [13]. Then we have the following Figure 1:

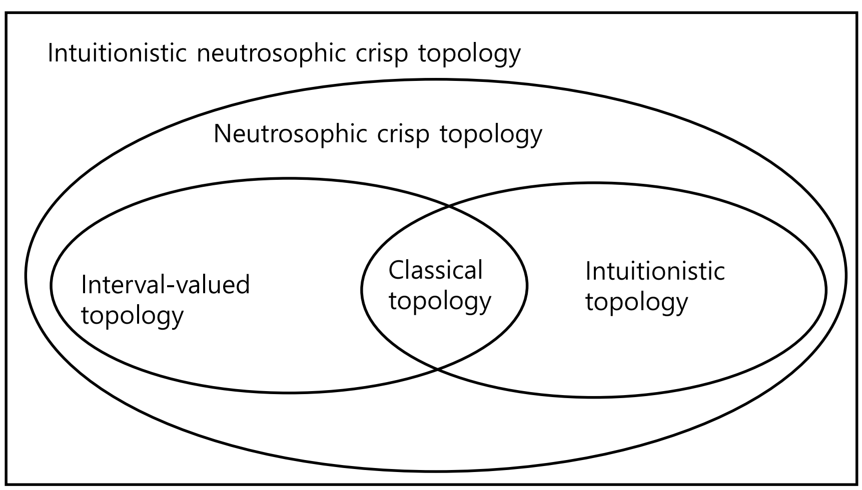

FIGURE 1.

Example 4.3. (1) Let X be a set and let $A \in INC(X)$. Then A is said to be finite, if $A^{T,\notin}$, $A^{I,\notin}$ and $A^{F,\notin}$ are finite. Consider the family

$$\tau = \{U \in INC(X) : U = \varnothing_{IN} \text{ or } U^c \text{ is finite}\}.$$

Then we can easily check that $\tau \in INCT(X)$.

In this case, τ will be called an intuitionistic neutrosiophic crisp cofinite topology (briefly, INCCFT) on X.

(2) Let X be a set and let $A \in INC(X)$. Then A is said to be countable, if $A^{T,\not\in}$, $A^{I,\not\in}$ and $A^{F,\not\in}$ are countable. Consider the family

$$\tau = \{U \in INC(X) : U = \varnothing_{IN} \text{ or } U^c \text{ is countable}\}.$$

Then we can easily prove that $\tau \in INCT(X)$.

In this case, τ will be called an intuitionistic neutrosiophic crisp cocountable topology (briefly, INCCCT) on X.

(3) Let $X = \{a, b, c, d, e, f, g, h, i\}$ and the family τ of IVNCSs on X given by:

$$\tau = \{\varnothing_{IN}, A_1, A_2, A_3, A_4, X_{IN}\},$$

where $A_1 = \langle(\{a, b, c\}, \{d, e\}), (\{e, f\}, \{g\}), (\{g, h\}, \{b, i\})\rangle$,
$A_2 = \langle(\{a, c, d\}, \{e, i\}), (\{e, g\}, \{h\}), (\{h, i\}, \{a\})\rangle$,
$A_3 = \langle(\{a, c\}, \{d, e, i\}), (\{e\}, \{g, h\}), (\{g, h, i\}, \varnothing)\rangle$,
$A_4 = \langle(\{a, b, c, d\}, \{e\}), (\{e, f, g\}, \varnothing), (\{h\}, \{a, b, i\})\rangle$.
Then we can easily check that $\tau \in INCT(X)$.

(4) Let $X = \{0, 1\}$. Consider the family τ of INCSs on X given by:

$$\tau = \{\varnothing_{IN}, \langle(\{0\}, \{1\}), \bar{\varnothing}, (\{1\}, \{0\})\rangle, X_{IN}\}.$$

Then we can easily check that $\tau \in INCT(X)$. In this case, (X, τ) will be called the intuitionistic neutrosophic crisp Sierpin$'$ski space.

The following is the immediate result of Definition 4.1

Proposition 4.4. *Let X be an IVNCTS. Then*
(1) $\varnothing_{IN}$, $X_{IN} \in IVNCC(X)$,
(2) $A \cup B \in INCC(X)$ *for any* A, $B \in INCC(X)$,
(3) $\bigcap_{j \in J} A_j \in INCC(X)$ *for any* $(A_j)_{j \in J} \subset INCC(X)$.

Definition 4.5. Let X be a non-empty set and let τ_1, $\tau_2 \in INCT(X)$. Then we say that τ_1 is contained in τ_2 or τ_1 is coarser than τ_2 or τ_2 is finer than τ_1, if $\tau_1 \subset \tau_2$, i.e., $A \in \tau_2$ for each $A \in \tau_1$.

It is obvious that $\tau_{IN,0} \subset \tau \subset \tau_{IN,1}$ for each $\tau \in INCT(X)$.

The following is the immediate result of Definitions 3.13 and 4.1.

Proposition 4.6. *Let $(\tau_j)_{j \in J} \subset INCT(X)$. Then $\bigcap_{j \in J} \tau_j \in INCT(X)$.*
In fact, $\bigcap_{j \in J} \tau_j$ is the coarsest INCT on X containing each τ_j.

Proposition 4.7. *Let τ, $\gamma \in INCT(X)$. We define $\tau \wedge \gamma$ and $\tau \vee \gamma$ as follows:*

$$\tau \wedge \gamma = \{W : W \in \tau, \ W \in \gamma\},$$

$$\tau \vee \gamma = \{W : W = U \cup V, \ U \in \tau, \ V \in \gamma\}.$$

Then we have
(1) $\tau \wedge \gamma$ *is an INCT on X which is the finest INCT coarser than both τ and γ,*
(2) $\tau \vee \gamma$ *is an INCT on X which is the coarsest INCT finer than both τ and γ,*

Proof. (1) It is clear that $\tau \wedge \gamma \in INCT(X)$. Let η be any INCT on X which is coarser than both τ and γ, and let $W \in \eta$. Then clearly, $W \in \tau$ and $W \in \gamma$. Thus $W \in \tau \wedge \gamma$. So η is coarser than $\tau \wedge \gamma$.

(2) The proof is similar to (1). □

Definition 4.8. Let (X, τ) be an IVNCTS.

(i) A subfamily β of τ is called an intuitionistic neutrosophic crisp base (briefly, INCB) for τ, if for each $A \in \tau$, $A = \varnothing_{IN}$ or there is $\beta^{'} \subset \beta$ such that $A = \bigcup \beta^{'}$.

(ii) A subfamily σ of τ is called an intuitionistic neutrosophic crisp subbase (briefly, INCSB) for τ, if the family $\beta = \{\bigcap \sigma^{'} : \sigma^{'}$ is a finite subset of $\sigma\}$ is an INCB for τ.

Remark 4.9. (1) Let β be an INCB for an INCT τ on a non-empty set X and consider three families of ISs in X:

$$\beta^T = \{(A^{T,\in}, A^{T,\not\in}) \in IS(X) : A \in \beta\},\ \beta^I = \{(A^{I,\in}, A^{I,\not\in}) \in IS(X) : A \in \beta\},$$

$$\beta^F = \{A^{F,\in^c}, A^{F,\not\in^c}) \in IS(X) : A \in \beta\}.$$

Then we can easily see that β^T, β^I and β^F are an intuitionistic base (See [4]) for τ^T, τ^I and τ^F, respectively.

Also, let us consider six families of ordinary subsets of X:

$$\beta^{T,\in} = \{A^{T,\in} \subset X : A \in \beta\},\ \beta^{T,\not\in} = \{A^{T,\not\in^c} \subset X : A \in \beta\},$$

$$\beta^{I,\in} = \{A^{I,\in} \subset X : A \in \beta\},\ \beta^{I,\not\in} = \{A^{I,\not\in^c} \subset X : A \in \beta\},$$

$$\beta^{F,\in} = \{A^{F,\in^c} \subset X : A \in \beta\},\ \beta^{F,\not\in} = \{A^{I,\not\in} \subset X : A \in \beta\}.$$

Then clearly, $\beta^{T,\in}$, $\beta^{T,\not\in}$, $\beta^{I,\in}$, $\beta^{I,\not\in}$, $\beta^{F,\in}$, $\beta^{F,\not\in}$ are ordinary bases for ordinary topologies $\tau^{T,\in}$, $\tau^{T,\not\in}$, $\tau^{I,\in}$, $\tau^{I,\not\in}$, $\tau^{F,\in}$, $\tau^{F,\not\in}$ on X, respectively.

(2) Let σ be an INCSB for an INCT τ on a non-empty set X and consider three families of ISs in X:

$$\sigma^T = \{(A^{T,\in}, A^{T,\not\in}) \in IS(X) : A \in \sigma\},\ \sigma^I = \{[A^{I,\in}, A^{I,\not\in}) \in IS(X) : A \in \sigma\},$$

$$\sigma^F = \{(A^{F,\in^c}, A^{F,\not\in^c}) \in IS(X) : A \in \sigma\}.$$

Then we can easily see that σ^T, σ^I and σ^F are an intuitionistic subbases (See [4]) for τ^T, τ^I and τ^F, respectively.

Also, let us consider six families of ordinary subsets of X:

$$\sigma^{T,\in} = \{A^{T,-\in} \subset X : A \in \sigma\},\ \sigma^{T,\not\in} = \{A^{T,\not\in^c} \subset X : A \in \sigma\},$$

$$\sigma^{I,\in} = \{A^{I,\in} \subset X : A \in \sigma\},\ \sigma^{I,\not\in} = \{A^{I,\not\in^c} \subset X : A \in \sigma\},$$

$$\sigma^{F,\in} = \{A^{F,\in^c} \subset X : A \in \sigma\},\ \sigma^{F,\not\in} = \{A^{F,\not\in} \subset X : A \in \sigma\}.$$

Then clearly, $\sigma^{T,\in}$, $\sigma^{T,\not\in}$, $\sigma^{I,\in}$, $\sigma^{I,\not\in}$, $\sigma^{F,\in}$, $\sigma^{F,\not\in}$ are ordinary subbases for ordinary topologies $\tau^{T,\in}$, $\tau^{T,\not\in}$, $\tau^{I,\in}$, $\tau^{I,\not\in}$, $\tau^{F,\in}$, $\tau^{F,\not\in}$ on X, respectively.

Example 4.10. (1) Let us consider the family of INCs σ in $\mathbb{R}$ given by:

$$\sigma = \{\big\langle((a,b), (-\infty, a]), \bar{\mathbb{R}}, \bar{\varnothing}\big\rangle : a,\ b \in \mathbb{R}\}.$$

Then σ generates an INCT τ on $\mathbb{R}$ which will be called the "usual left intuitionistic neutrosophic crisp topology (briefly, ULINCT)" on $\mathbb{R}$. In fact, the INCB β for τ can be written in the form:

$$\beta = \{\mathbb{R}_{IN}\} \cup \{B \in INC(\mathbb{R}) : B \text{ is a finite intersection of members of } \sigma\}$$

and τ consists of the following INCSs in $\mathbb{R}$:

$$\tau = \{\varnothing_{IN}, \mathbb{R}_{IN}\} \cup \{\left\langle(\cup(a_j, b_j), (-\infty, c]), \bar{\mathbb{R}}, \bar{\varnothing}\right\rangle\}$$

or

$$\tau = \{\varnothing_{IN}, \mathbb{R}_{IN}\} \cup \{\left\langle(\cup(a_k, b_k), \varnothing), \bar{\mathbb{R}}, \bar{\varnothing}\right\rangle\}$$

where $a_j, b_j, c, d \in \mathbb{R}$, $c < inf\{a_j : j \in J\}$ and $a_k, b_k \in \mathbb{R}$, $\{a_k : k \in K\}$ is not bounded from below. Similarly, one can define the "usual right intuitionistic neutrosophic crisp topology (briefly, URINCT)" on $\mathbb{R}$ using an analogue construction.

(2) Consider the family σ of INCSs in $\mathbb{R}$:

$$\sigma = \{\left\langle((a, b), (-\infty, a_1] \cup [b_1, \infty), \bar{\mathbb{R}}, \bar{\varnothing}\right\rangle : a,\ b,\ a_1,\ b_1 \in \mathbb{R},\ a_1 \leq a,\ b_1 \geq b\}.$$

Then σ generates an INCT τ on $\mathbb{R}$ which will be called the "usual intuitionistic neutrosophic crisp topology (briefly, UINCT)" on $\mathbb{R}$. In fact, the INCB β for τ can be written in the form:

$$\beta = \{\mathbb{R}_{IN}\} \cup \{B \in INC(\mathbb{R}) : B \text{ is a finite intersection of members of } \sigma\}$$

and the elements of τ can be easily written down as in (1).

(3) Consider the family $\sigma_{[0,1]}$ of INCSs in $\mathbb{R}$:

$$\sigma_{[0,1]} = \{\left\langle([a, b], (-\infty, a) \cup (b, \infty)), \bar{\mathbb{R}}, \bar{\varnothing}\right\rangle : a, b \in \mathbb{R} \text{ and } 0 \leq a \leq b \leq 1\}.$$

Then $\sigma_{[0,1]}$ generates an INCT $\tau_{[0,1]}$ on $\mathbb{R}$ which will be called the "usual unit closed intuitionistic neutrosophic crisp topology" on $\mathbb{R}$. In fact, the INCB $\beta_{[0,1]}$ for $\tau_{[0,1]}$ can be written in the form:

$$\beta_{[0,1]} = \{\mathbb{R}_{IN}\} \cup \{B \in INC(\mathbb{R}) : B \text{ is a finite intersection of members of } \sigma_{[0,1]}\}$$

and the elements of τ can be easily written down as in (1).

In this case, $([0, 1], \tau_{[0,1]})$ is called the "intuitionistic neutrosophic crisp usual unit closed interval" and will be denoted by $[0, 1]_{INCI}$. In fact,

$$[0, 1]_{INCI} = \left\langle([0, 1], (-\infty, 0) \cup (1, \infty)), \bar{\mathbb{R}}, \bar{\varnothing}\right\rangle.$$

(4) Let $X = \{a, b, c, d, e, f, g, h, i\}$ and consider the family β of INCSs in X given by:

$$\beta = \{A, B, X_{IVN}\},$$

where $A = \langle(\{a, b, c\}, \{f, g\}), (\{e, f\}, \{h\}), (\{g, i\}, \{d\})\rangle$,
$B = \langle(\{a, c, d\}, \{f, h\}), (\{e, g\}, \{f\}), (\{f, h, i\}, \{a, g\})\rangle$.

Assume that β is an INCB for an INCT τ on X. Then by the definition of base, $\beta \subset \tau$. Thus $A,\ B \in \tau$. So $A \cap B = \langle(\{a, c\}, \{f, g, h\}), (\{e\}, \{f, h\}), (\{f, g, h, i\}, \varnothing)\rangle \in \tau$. But for any $\beta' \subset \beta$, $A \cap B \neq \bigcup \beta'$. Hence β is not an INCB for an INCT on X.

From (1), (2) and (3) in Example 4.10, we can define intuitionistic neutrosophic crisp intervals as following.

Definition 4.11. Let $a,\ b \in \mathbb{R}$ such that $a \leq b$. Then

(i) (the closed interval)

$$[a,b]_{INCI} = \left\langle ([a,b], (-\infty,a) \cup (b,\infty,b)), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle,$$

(ii) (the open interval)

$$(a,b)_{INCI} = \left\langle ((a,b), (-\infty,a] \cap [b,\infty,b)), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle,$$

(iii) (the half open interval or the half closed interval)

$$(a,b]_{INCI} = \left\langle ((a,b], (-\infty,a] \cup (b,\infty)), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle,$$
$$[a,b)_{INCI} = \left\langle ([a,b), (-\infty,a) \cup [b,\infty)), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle,$$

(iv) (the half interval-valued real line)

$$(-\infty,a]_{INCI} = \left\langle ((-\infty,a], (a,\infty)), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle,$$
$$(-\infty,a)_{INCI} = \left\langle ((-\infty,a), [a,\infty)), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle,$$
$$[a,\infty)_{INCI} = \left\langle ([a,\infty), (-\infty,a)), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle,$$
$$(a,\infty)_{INCI} = \left\langle ((a,\infty), (-\infty,a]), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle,$$

(v) (the interval-valued real line)

$$(-\infty,\infty)_{INCI} = \left\langle ((-\infty,\infty), \varnothing), \bar{\mathbb{R}}, \bar{\varnothing} \right\rangle = \mathbb{R}_{IN}.$$

The following provides the sufficient conditions for a family of intuitionistic neutrosophic crisp sets to be an INCSB of an INCT.

Proposition 4.12. *Let X be a non-empty set and let $\sigma \subset INC(X)$ such that $X_{IN} = \bigcup \sigma$.*
Then there exists a unique INCT τ on X such that σ is an INCSB for τ.

Proof. Let $\beta = \{B \in IN_1(X) : B = \bigcup_{i=1}^{n} S_i$ and $S_i \in \sigma\}$. Let $\tau = \{U \in IN_1(X) : U = \widetilde{\varnothing}$ or there is a subcollection $\beta^{'}$ of β such that $U = \bigcup \beta^{'}\}$. Then we can show that τ is the unique INCT on X such that σ is an INCSB for τ. □

In Proposition 4.12, τ is called the INCT on X generated by σ.

Example 4.13. Let $X = \{a,b,c,d,e\}$ and consider the family σ of INCSs in X given by:

$$\sigma = \{A_1, A_2, A_3, A_4\},$$

where
$A_1 = \langle (\{a\},\{b\}), (\{c,d\},\varnothing), (\{b\},\{a\}) \rangle$,
$A_2 = \langle (\{a,b,c\},\{d\}), (\{e\},\varnothing), (\{d\},\{a,b,c\}) \rangle$,
$A_3 = \langle (\{b,c,e\},\{a\}), (\{d\},\varnothing), (\{a\},\{b,e\}) \rangle$,
$A_4 = \langle (\{c,d\},\{e\}], (\{a,b\},\{e\}), (\{e\},\{d\}) \rangle$.

Then clearly, $\bigcup \sigma = X_{IN}$. Let β be the collection of all finite intersections of members of σ. Then we have

$$\beta = \{A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}\},$$

where
$A_5 = \langle (\{a\},\{b,d\}), \bar{\varnothing}, (\{b,d\},\{a\}) \rangle$,
$A_6 = \langle (\varnothing,\{a,b\}), (\{d\},\varnothing), (\{a,b\},\varnothing) \rangle$,
$A_7 = \langle (\varnothing,\{b,e\}), (\varnothing,\{e\}), (\{b,e\},\varnothing) \rangle$,
$A_8 = \langle (\{b,c\},\{a,d\}), \bar{\varnothing}, (\{a,d\},\{b\}) \rangle$,

$A_9 = \langle (\{c\},\{d,e\}),(\varnothing,\{e\}),(\{d,e\},\varnothing)\rangle$,
$A_{10} = \langle (\{c\},\{a,e\}),(\varnothing,\{e\}),(\{a,e\},\varnothing)\rangle$,
$A_{11} = \langle (\varnothing,\{a,b,d\}),\bar{\varnothing},(\{a,b,d\},\varnothing)\rangle$,
$A_{12} = \langle (\{c\},\{a,d,e\}),\bar{\varnothing},(\{b,d,e\},\varnothing)\rangle$,
$A_{13} = \langle (\varnothing,\{b,d,e\}),\bar{\varnothing},(\{a,d,e\},\varnothing)\rangle$.

Thus we have the generated INCT τ by σ:

$$\tau = \{\varnothing_{IN}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{19}, A_{20}, A_{21}, A_{22}, X_{IN}\},$$

where $A_{14} = \langle (\{a,b,c\},\varnothing),(\{c,d,e\},\varnothing),(\varnothing,\{a,b,c\})\rangle$,
$A_{15} = \langle (\{a,b,c,e\},\varnothing),(\{c,d\},\varnothing),(\varnothing,\{a,b,e\})\rangle$,
$A_{16} = \langle (\{a,c,d\},\varnothing),(\{a,b,c,d\},\varnothing),(\varnothing,\{a,d\})\rangle$,
$A_{17} = \langle (\{a,b,c,e\},\varnothing),(\{c,d\},\varnothing),(\varnothing,\{a,b,c,e\})\rangle$,
$A_{18} = \langle (\{a,b,c,d\},\varnothing),(\{a,b,e\},\varnothing),(\varnothing,\{a,b,c,d\})\rangle$,
$A_{19} = \langle (\{b,c,d,e\},\varnothing),(\{a,b,d\},\varnothing),(\varnothing,\{b,d,e\})\rangle$,
$A_{20} = \langle (\{a,b,c,e\},\varnothing),(\{c,d,e\},\varnothing),(\varnothing,\{a,b,c,e\})\rangle$,
$A_{21} = \left\langle \bar{X},(\{a,c,d\},\varnothing),\bar{\varnothing}\right\rangle$,
$A_{22} = \langle (\{b,c,d,e\},\varnothing),(\{a,b,d\},\varnothing),(\varnothing,\{b,d,e\})\rangle$.

5. Interiors and closures of INCSs

In this section, we define intuitionistic neutrosophic crisp interiors and closures, and investigate some of their properties and give some examples. In particular, we will show that there is a unique INCT on a set X from the intuitionistic neutrosophic crisp closure [resp. interior] operator.

Definition 5.1. Let (X,τ) be an INCTS and let $A \in INC(X)$.

(i) The intuitionistic neutrosophic crisp closure of A w.r.t. τ, denoted by $INcl(A)$, is an INCS in X defined as:

$$IVNcl(A) = \bigcap \{K : K^c \in \tau \text{ and } A \subset K\}.$$

(ii) The intuitionistic neutrosophic crisp interior of A w.r.t. τ, denoted by $IVNint(A)$, is an IVS in X defined as:

$$IVNint(A) = \bigcup \{G : G \in \tau \text{ and } G \subset A\}.$$

Example 5.2. Let $X = \{a,b,c,d,e,f,g,h,i\}$ and consider INCT τ on X given in Example 4.3 (3):

$$\tau = \{\varnothing_{IN}, A_1, A_2, A_3, A_4, X_{IN}\},$$

where $A_1 = \langle (\{a,b,c\},\{d,e\}),(\{e,f\},\{g\}),(\{g,h\},\{b,i\})\rangle$,
$A_2 = \langle (\{a,c,d\},\{e,i\}),(\{e,g\},\{h\}),(\{h,i\},\{a\})\rangle$,
$A_3 = \langle (\{a,c\},\{d,e,i\}),(\{e\},\{g,h\}),(\{g,h,i\},\varnothing)\rangle$,
$A_4 = \langle (\{a,b,c,d\},\{e\}),(\{e,f,g\},\varnothing),(\{h\},\{a,b,i\})\rangle$.

Then clearly, we have

$$INCC(X) = \{\varnothing_{IN}, A_1^c, A_2^c, A_3^c, A_4^c, X_{IN}\},$$

where, $A_1^c = \langle (\{g,h\},\{b,i\}),(\{g\},\{e,f\}),\{a,b,c\},\{d,e\})\rangle$,
$A_2^c = \langle (\{h,i\},\{a\}),(\{h\},\{e,g\}),(\{a,c,d\},\{e,i\})\rangle$,
$A_3^c = \langle (\{g,h,i\},\varnothing),(\{g,h\},\{e\}),(\{a,c\},\{d,e,i\}))\rangle$,

$A_4^c = \langle(\{h\},\{a,b,i\}),(\varnothing,\{e,f,g\}),(\{a,b,c,d\},\{e\})\rangle$.
Consider two INCSs in X given by:

$$A = \langle(\{a,b,c\},\{e\}),(\{e,f,g\},\varnothing),(\{h\},\{a,b,i\}]\rangle,$$

$$B = \langle(\{h\},\{a,b,i\}),(\varnothing,\{e,f,g\}),(\{a,b,c,d\},\varnothing)\rangle.$$

Then we have

$$\begin{aligned} IVNint(A) &= \bigcup\{G \in \tau : G \subset A\} = A_1 \cup A_3 \\ &= \langle(\{a,b,c\},\{d,e\}),(\{e,f\},\{g\}),(\{g,h\},\{a,b,i\}]\rangle, \\ IVNcl(B) &= \bigcap\{F : F^c \in \tau,\ B \subset F\} = A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \\ &= \langle(\{h\},\{a,b,i\}),(\varnothing,\{e,f,g\}),(\{a,c\},\{d,e,i\})\rangle. \end{aligned}$$

Proposition 5.3. *Let (X,τ) be an INCTS and let $A \in INC(X)$. Then*

$$INint(A^c) = (INcl(A))^c \text{ and } INcl(A^c) = (INint(A))^c.$$

Proof.

$$\begin{aligned} INint(A^c) &= \bigcup\{U \in \tau : U \subset A^c\} = \bigcup\{U \in \tau : U \subset \langle A^F, A^{I^c}, A^T\rangle\} \\ &= \bigcup\{U \in \tau : U^T \subset A^F,\ U^I \subset A^{I^c},\ U^F \supset A^T\} \\ &= \bigcup\{U \in \tau : U^F \supset A^T,\ U^{I^c} \supset A^I,\ U^T \subset A^F\} \\ &= (\bigcap\{U^c : U \in \tau, A \subset U^c\})^c \\ &= (INcl(A))^c. \end{aligned}$$

Similarly, we can show that $INcl(A^c) = (INint(A))^c$. □

Theorem 5.4. *Let X be an INCTS and let $A \in IVNC(X)$. Then*
(1) *$A \in INCC(X)$ if and only if $A = INcl(A)$,*
(2) *$A \in INCO(X)$ if and only if $A = INint(A)$.*

Proof. Straightforward. □

Proposition 5.5 (Kuratowski Closure Axioms)**.** *Let X be an INCTS and let $A, B \in IVNC(X)$. Then*
[INCK0] *if $A \subset B$, then $INcl(A) \subset INcl(B)$,*
[INCK1] *$INcl(\varnothing_{IN}) = \varnothing_{IN}$,*
[INCK2] *$A \subset INcl(A)$,*
[INCK3] *$INcl(INcl(A)) = INcl(A)$,*
[INCK4] *$INcl(A \cup B) = INcl(A) \cup INcl(A)$.*

Proof. Straightforward. □

Let $INcl^* : INC(X) \to INC(X)$ be the mapping satisfying the properties [INCK1], [INCK2],[INCK3] and [INCK4]. Then we will call the mapping $INcl^*$ as the intuitionistic neutrosophic crisp closure operator(briefly, INCCO) on X.

Proposition 5.6. *Let $INcl^*$ be the INCCO on X. Then there exists a unique INCT τ on X such that $INcl^*(A) = INcl(A)$, for each $A \in INC(X)$, where $INcl(A)$ denotes the intuitionistic neutrosophic crisp closure of A in the INCTS (X,τ). In fact,*

$$\tau = \{A^c \in INC(X) : INcl^*(A) = A\}.$$

Proof. The proof is almost similar to the case of ordinary topological spaces. □

Proposition 5.7. *Let X be an INCTS and let $A, B \in INC(X)$. Then*
[INCI0] *if $A \subset B$, then $INint(A) \subset INint(B)$,*
[INCI1] $INint(X_{IN}) = X_{IN}$,
[INCI2] $INint(A) \subset A$,
[INCI3] $INint(INint(A)) = INint(A)$,
[INCI4] $INint(A \cap B) = INint(A) \cap INint(A)$.

Proof. Straightforward. □

Let $INint^* : INC(X) \to INC(X)$ be the mapping satisfying the properties [INCI1], [INCI2],[INCI3] and [INCI4]. Then we will call the mapping $INint^*$ as the intuitionistic neutrosophic crisp interior operator (briefly, INCIO) on X.

Proposition 5.8. *Let $IVNint^*$ be the IVNCIO on X. Then there exists a unique IVNCT τ on X such that $IVNint^*(A) = IVNint(A)$ for each $A \in IVN_1(X)$, where $IVNint(A)$ denotes the interval-valued neutrosophic crisp interior of A in the IVNCTS (X, τ). In fact,*

$$\tau = \{A \in IVN_1(X) : IVNint^*(A) = A\}.$$

Proof. The proof is similar to one of Proposition 5.6. □

Remark 5.9. By using "$\subset_2$, $\cup_2$, $\cap_2$, ${}^{i,c}(i = 1,\ 2,\ 3)$, $\varnothing_{2,IN}$, $X_{2,IN}$ and $INC(X)$, we can have the definitions corresponding to Definitions 4.1, 4.8 and 5.1, respectively.

6. Intuitionistic neutrosophic crisp continuous mappings

In this section, we define intuitionistic neutrosophic crisp continuous mappings and study some of their properties.

Definition 6.1 ([4]). Let (X, τ), (Y, δ) be two ITSs. Then a mapping $f : X \to Y$ is said to be intuitionistic continuous, if $f^{-1}(V) \in \tau$ for each $v \in \delta$.

Definition 6.2. Let (X, τ), (Y, δ) be two INCTSs. Then a mapping $f : X \to Y$ is said to be intuitionistic neutrosophic crisp continuous, if $f^{-1}(V) \in \tau$ for each $V \in \delta$.

From Remark 4.2 (1), and Definitions 6.1 and 6.2, we can easily have the following.

Theorem 6.3. *Let (X, τ), (Y, δ) be two INCTSs and let $f : X \to Y$ be a mapping. Then f is intuitionistic neutrosophic crisp continuous if and only if $f : (X, \tau^T) \to (Y, \delta^T)$, $f : (X, \tau^I) \to (Y, \delta^I)$ and $f : (X, \tau^F) \to (Y, \delta^F)$ are intuitionistic continuous, respectively.*

The followings are immediate results of Proposition 3.16 (13) and Definition 6.2.

Proposition 6.4. *Let X, Y, Z be INCTSs.*
(1) *The identity mapping $id : X \to X$ is continuous.*
(2) *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Remark 6.5. From Proposition 6.4, we can easily see that the class of all INCTSs and continuous mappings, denoted by **INCTop**, forms a concrete category.

Also, the followings are immediate results of Definition 6.2.

Proposition 6.6. *Let $X,\ Y$ be INCTSs.*
(1) *If X is an INCDTS, the $f : X \to Y$ is continuous.*
(2) *If Y is an INCITS, then $f : X \to Y$ is continuous.*

Theorem 6.7. *Let $X,\ Y$ be INCTSs and let $f : X \to Y$ be a mapping. Then the followings are equivalent:*
(1) *f is continuous,*
(2) *$f^{-1}(C) \in INCC(X)$ for each $C \in INCC(Y)$,*
(3) *$f^{-1}(S) \in INCO(X)$ for each member S of the subbase for the INCT on Y,*
(4) *$INcl(f^{-1}(B)) \subset f^{-1}(INcl(B))$ for each $B \in INC(Y)$,*
(5) *$f(INcl(A)) \subset INcl(f(A))$ for each $A \in INC(X)$.*

Proof. The proofs of (1)⇒(2)⇒(3)⇒(1) are obvious.

(2)⇒(4): Suppose the condition (2) holds and let $B \in INC(Y)$. By Proposition 5.5 [INCK2], $B \subset INcl(B)$. Then by Proposition 3.16 (2), $f^{-1}(B) \subset f^{-1}(INcl(B))$. Thus by Proposition 5.5 [INCK0], $INcl(f^{-1}(B)) \subset INcl(f^{-1}(INcl(B)))$. Since $INcl(B) \in INCC(Y)$, $f^{-1}(INcl(B)) \in INCC(X)$ by the condition (2). So by Theorem 5.4 (1), $INcl(f^{-1}(INcl(B))) = f^{-1}(INcl(B))$. Hence $INcl(f^{-1}(B)) \subset f^{-1}(INcl(B))$.

(4)⇒(5): Suppose the condition (4) holds and let $B = f(A)$ for each $A \in INC(X)$. Then we have $INcl(f^{-1}(f(A))) \subset f^{-1}(INcl(f(A)))$. Thus by Proposition 3.16 (3), $INcl(A) \subset f^{-1}(INcl(f(A)))$. So by Proposition 3.16 (1) and (4), $f(INcl(A)) \subset INcl(f(A))$.

(5)⇒(4): The proof is similar to (4)⇒(5). □

Theorem 6.8. *Let $X,\ Y$ be INCTSs and let $f : X \to Y$ be a mapping. Then f is continuous if and only if $f^{-1}(INint(B)) \subset INint(f^{-1}(B))$ for each $B \in INC(Y)$.*

Proof. The proof is straightforward. □

Definition 6.9. Let $(X, \tau),\ (Y, \delta)$ be two INCTSs. Then a mapping $f : X \to Y$ is said to be:
(i) intuitionistic neutrosophic crisp open, if $f(U) \in \delta$ for each $U \in \tau$,
(ii) intuitionistic neutrosophic crisp closed, if $f(C) \in INCC(Y)$ for each $C \in INCC(X)$.

Proposition 6.10. *Let $X,\ Y,\ Z$ be INCTSs, let $f : X \to Y$ and $g : Y \to Z$ be mappings. If $f,\ g$ are open [resp. closed], then $g \circ g$ is open [resp. closed].*

Proof. The proof is straightforward. □

Theorem 6.11. *Let $X,\ Y$ be INCTSs and let $f : X \to Y$ be a mapping. Then f is open if and only if $INint(f(A)) \subset f(INint(A))$ for each $A \in INC(X)$.*

Proof. The proof is straightforward. □

Proposition 6.12. *Let $X,\ Y$ be INCTSs and let $f : X \to Y$ be injective. If f is continuous, then $f(INint(A)) \subset INint(f(A))$ for each $A \in INC(X)$.*

Proof. The proof is straightforward. □

The following is the immediate result of Theorem 6.11 and Proposition 6.12.

Corollary 6.13. *Let X, Y be INCTSs and let $f : X \to Y$ be continuous, open and injective. Then $f(INint(A)) = INint(f(A))$ for each $A \in INC(X)$.*

Theorem 6.14. *Let X, Y be INCTSs and let $f : X \to Y$ be a mapping. Then f is close if and only if $INcl(f(A)) \subset f(INcl(A))$ for each $A \in INC(X)$.*

Proof. The proof is straightforward. □

The following is the immediate result of Theorems 6.7 and 6.14.

Corollary 6.15. *Let X, Y be INCTSs and let $f : X \to Y$ be a mapping. Then f is continuous and closed if and only if $f(INcl(A)) = INcl(f(A))$ for each $A \in INC(X)$.*

Definition 6.16. Let (X, τ), (Y, δ) be two INCTSs. Then a mapping $f : X \to Y$ is called an intuitionistic neutrosophic crisp homeomorphism, if f is bijective, continuous and open.

Theorem 6.17. *Let X, Y be INCDTSs and let $f : X \to Y$ be a mapping. Then f is a homeomorphism if and only if f is bijective.*

Proof. The proof is straightforward. □

Definition 6.18. Let (X, τ) be an INCTS, let Y be a set and let $f : X \to Y$ be a surjective mapping. Let δ be the family of INCSs in Y given by:

$$\delta = \{B \in INC(Y) : f^{-1}(B) \in \tau\}.$$

Then δ is called the intuitionistic neutrosophic crisp quotient topology (briefly, INCQT) on Y.

It can be easily see that $\delta \in INCT(Y)$. Also, it is obvious that for each $B \in INC(Y)$, B is closed in δ if and only if $f^{-1}(B)$ is closed in X.

Proposition 6.19. *Let (X, τ), (Y, δ) be two INCTSs, where δ is the INCQT on Y. Then a surjection $f : X \to Y$ is continuous and open. Moreover, δ is the finest topology on Y which f is continuous.*

Proof. The proof is similar to the classical case. □

The following is the immediate result of Proposition 6.19.

Corollary 6.20. *Let (X, τ), (Y, δ) be two INCTSs. If a mapping $f : X \to Y$ is continuous, open and sujective, then δ is the INCQT on Y. But the converse does not hold in general (See Example 6.21).*

Example 6.21. Let $([0, 1], \tau)$ be an INCTS and let $A = [\frac{1}{2}, 1]$. Consider the characteristic function $\chi_A : [0, 1] \to \{0, 1\}$, where $\{0, 1\}$ be the intuitionistic neutrosophic crisp Sierpin$'$ski space (See Example 4.3 (4)). Then we can easily see that the topology on $\{0, 1\}$ is the INCQT. On the other hand, $(\frac{1}{2}, 1)_{INCI} \in \tau$ but $\chi_A((\frac{1}{2}, 1)_{INCI})$ is not open in $\{0, 1\}$. Thus χ_A is not an open mapping.

Theorem 6.22. *Let (X, τ), (Y, δ), (Z, σ) be INCTSs, where δ is the INCQT on Y. Let $f : X \to Y$ and $g : Y \to Z$ be mappings. Then g is continuous if and only if $g \circ f$ is continuous.*

Proof. The proof is similar to the classical case. □

7. Conclusions

We introduced the new concept of intuitionistic neutrosophic crisp sets which are the generalization of classical sets and the specialization of intuitionistic neutrosophic sets, and obtained its various properties. Next, we introduced the notion of intuitionistic neutrosophic crisp topological spaces which are considered as an intuitionistic tri-opological space, and obtained some of its properties. Finally, we defined the notions of intuitionistic neutrosophic crisp closures and interiors, and discussed with their some properties. Also, we discussed various properties of intuitionistic neutrosophic crisp continuities and quotient topologies.

In the future, we expect that one can apply the concept of intuitionistic neutrosophic crisp sets to group and ring theory, BCK-algebra and category theory, etc. Also, we expect that one can deal with the concepts of intuitionistic soft sets and intuitionistic neutrosophic crisp soft sets.

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