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Maximal and maximum antichains of ordered multisets

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Abstract. This work studies the concept of maximal and maximum antichains on a partially ordered structure for which repetition is significant. By using set-based partitioning, maximal and maximum antichains of the ordered multiset structure are constructed. Analogous result on antichains and examples are also presented in this multiset setting.

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1. Introduction

In order theory [14, 16, 17], a *maximum antichain* is an antichain which is of the greatest size possible in a partially ordered set (or poset). Whereas, *a maximal antichain is* an antichain that is not a proper subset of another antichain. Though a maximal antichain is not necessarily maximum, maximum antichains are always maximal. Dilworth [5] showed that if A is a maximum antichain in a finite poset $(P, \preccurlyeq)$, then there is a partition of P into chains $C_1 \bigcup C_2 \bigcup \ldots \bigcup C_n$ such that $n = |A|$. Furthermore, each C_i contains exactly one element of A, and there is no partition of P into fewer number of chains. The dual of Dilworth's theorem was proved in [12]. The *width (height)* of a poset is the cardinality of a maximum antichain (chain). The width and height are very important parameters in the characterization of ordered structures.

Collections with repeated elements are natural phenomena in applications. The study of multisets (also called bags, lists or heaps) was introduced in order to adequately model this concept [1, 2, 3, 18]. A *multiset* (or mset) is an unordered collection of objects that admits repetition. A classical (or Cantorian) set can be viewed as a particular case of an mset, since the multiplicity (i.e., number of times an object occurs) of each object in a set is 1. Informally, an mset is an extended notion of a set, and by extension are quite apt to represent partial orders [13]. A rich literature of orderings on msets and their applications is available [4, 8, 9, 10, 11]. The mset ordering proposed in [4] was used for proving termination of systems and term rewriting systems. This ordering is defined on $\boldsymbol{M(S)}$, i.e., the set of all finite msets defined on the set S, and serves as a basis for a number of mset orderings used in this context [8, 9]. This paper focuses on studying corresponding notions of maximal and maximum antichains on msets by using an ordering which is defined on the points of a finite mset M.

In section 2, preliminaries on msets and related mset orderings are presented. In section 3, we present the ordering $' \preccurlyeq\leq'$ and give analogous definitions on chains and antichains using this ordering. In section 4, maximal and maximum mset antichains of a partially ordered multiset (or pomset) are constructed by using set-based partitioning studied in [9]. We conclude in section 5 by outlining related areas for further studies.

2. Multisets and orderings on multisets

In this section, basic definitions and terms on msets to be used in this work are presented (See [15]). We also discuss related mset orderings.

2.1 Preliminaries on Multisets

If S is a set, then an mset M over S is a cardinal-valued function, i.e., $M{:}S \to \mathbb{N}$, such that $x \in Dom\ (M)$ implies $M(x)$ is a cardinal number and $M(x) = m_M(x) > 0$, where $m_M(x)$ denotes the number of times an object x occurs in M(*multiplicity* of x in M). The *root set* of the mset M (denoted by M^*) is the set $\{x \in S | M(x) > 0\}$. Members of M^* are usually called objects in M, while each individual occurrence of an object in M is called an element of M. In this work, an object in M together with its multiplicity will denote a point in M. All occurrences of an object in M are usually treated without preference. The sum of the multiplicities of all the objects in M is the cardinality of M. Let M, N be msets in $\boldsymbol{M(S)}$. Then M is a submset of N ($M \subseteq N$), if $M(x) \leq N(x)$ for all $x \in S$, and $M \subset N$, if $M(x) < N(x)$ for at least one x. The multiplicity of an object is mostly assumed to be a finite natural number, howbeit cases where the multiplicities assume integer and real values have also been studied [2, 6]. An mset M is called a finite mset, if its objects and their multiplicities are finite. In this paper, the msets are assumed to be finite with nonnegative integral multiplicities. We assume the axioms developed in the theory MST [1] for the purpose of this study. The mset ordering used in this work is induced by the ordering on the underlying poset. The concepts studied here can be generalized by using an mset theory with a more general class for the set of multiplicities.

2.2 Multiset Orderings

Here we present related mset orderings that have been developed for comparing msets in $\boldsymbol{M(S)}$.

Dershowitz-Manna multiset ordering [4]

For two msets M *and* N defined over a set S, the Dershowitz-Manna mset ordering is defined as follows:

$M{<<}N$ (which reads N dominates M), if there exist two msets X *and* Y in ***M(S)*** such that
(i) $\phi \neq X \subseteq N$,
(ii) $M=(N \setminus X)+Y$,
(iii) X dominates Y, that is, for all $y \in Y$, there is some $x \in X$ such that $y < x$.

Dershowitz and Manna showed that the ordering on S induces an ordering on ***M(S)***. They also proved that the ordering $>$ on any given well-founded set S can be extended to form a well-founded ordering $>>$ on ***M(S)***. An important property of the Dershowitz-Manna ordering is its monotonicity and as observed in [9], it is the strongest monotonic ordering on msets. A known limitation with the Dershowitz-Manna mset ordering is that it is difficult to use when proving that two multisets are not related by inclusion.

Huet-Oppen multiset ordering [8]

If M and N are two msets over a set S, then
$M{<<}N$ if and only if $M \neq N$ and $[M(y) > N(y) \Longrightarrow (\exists x \in \mathrm{S}) x \succ y$ and $\mathrm{M}(x) < \mathrm{N}(x)]$.

The Huet-Oppen ordering is a more tractable mset ordering. It is total and becomes a lexicographic ordering if the ordering $<$ on the ground set is total [11]. It is also well-founded if and only if (S,<) is well-founded. This mset ordering like the Dershowitz-Manna mset ordering is also used in the proof of program termination and term rewriting systems.

Jouannaud-Lescanne multiset ordering [9]

Jouannaud and Lescanne defined two mset orderings that are extensions of the Dershowitz-Manna ordering as follows:

(i) $<<_1$
Assume that the partition $\overline{M} = \{M_i | i = 1 \ldots p\}$ of the mset M satisfies the following conditions:
(a) $x \in M_i \Longrightarrow M_i(x) = M(x)$,
(b) $x \in M_i$ and $y \in M_i \Longrightarrow x$ and y are incomparable,
(c) $\forall i \in [2, \ldots, p] x \in M_i \Longrightarrow (\exists y \in M_{i-1}) y \succ x$.

If M and N are two msets with set based partition in lexicographical order, $\overline{M}$ and $\overline{N}$ respectively, then $M <<_1 N$ if and only if $\overline{M} <<_1^{lex} \overline{N}$.
A different method for building a partition from a set is to require that different occurrences of the same element belong to different multisets of the partition.

(ii) $<<_2$
In this case, assume that the partition $\overline{M} = \{S_i | i = 1 \ldots p\}$ satisfies the following conditions:
(a) S_i is a set, i.e., $S_i(x) \leq 1$,
(b) $x \in S_i$ and $y \in S_i \Longrightarrow x$ and y are incomparable,

(c) $\forall i \in [2,\ldots,p] x \in S_i \Longrightarrow (\exists y \in S_{i-1}) y \succcurlyeq x$.

If M and N are multisets, then $M <<_2 N$ if and only if $\overline{M} <<_2^{lex} \overline{N}$.
This second construction is adopted for generating mset antichains in Section 4. Like the Dershowitz-Manna and Huet-Oppen mset orderings, the Jouannaud-Lescanne mset ordering is well-founded if the ordering on the generic set is well-founded.

Remark 2.1. Motivated by these mset orderings on $\boldsymbol{M(S)}$, the mset ordering $' \preccurlyeq\leq'$ on an mset M is presented in the next section. This ordering, though defined on the points of a finite mset M, is consistent with the Dershowitz-Manna mset ordering as it is also induced by the ordering on the generic set. Girish and Sunil [7] also presented a study on an mset ordering $' <:<'$ defined via the ordering on the generic set of the mset under consideration. However, this study presents certain notions in Sections 3 and 4 in a different perspective from $' <:<'$.

3. The partially ordered multiset $(M, \preccurlyeq\leq)$

We begin by fixing some notations. An object x_i in an mset M together with its multiplicity m_i will denote a point in M, where $i \in [1, n]$. For an mset M, $m_i x_i$ represents the atomic formula $x_i \in^{m_i} M$ of MST [1]. As usual the class of all finite msets defined over a set S will be denoted by $\boldsymbol{M(S)}$. The *base or ground set* for an mset will usually be denoted by $(S, \preccurlyeq)$, where S is a set and $\preccurlyeq$ is assumed to be a partial order on S.

Definition 3.1 ([7])**.** Let M be an mset. A submset R of $M \times M$ is called an mset relation on M, if every pair $(m_i x_i, m_j x_j)$ of R has multiplicity $m_i m_j$. We write $m_i x_i R m_j x_j$, whenever $m_i x_i$ is related to $m_j x_j$. The domain *Dom* R and the range *Ran* R of R is given by, respectively:

$$Dom\ R = \{m_i x_i \in M : \exists m_j x_j \in M \text{ such that } m_i x_i R m_j x_j\},$$

$$Ran\ R = \{m_j x_j \in M : \exists m_i x_i \in M \text{ such that } m_i x_i R m_j x_j\}.$$

For details on mset relations, see [7].

Definition 3.2. For any pair of points $m_i x_i$ and $m_j x_j$ in M, $m_i x_i \preccurlyeq\leq m_j x_j$ if and only if $x_i \preccurlyeq x_j$. The two points coincide, i.e., $m_i x_i == m_j x_j$ if and only if $x_i = x_j$.

Remark 3.3. In view of Definition 3.2 by implication, we have

$$m_i x_i \preccurlyeq\leq m_j x_j \text{ if and only if } x_i \preccurlyeq x_j \wedge (m_i \leq m_j \vee m_j \leq m_i).$$

The condition $m/x \neq= n/y$ in R if and only if $m \neq n$ and $x = y$ as stated in [7] does not hold under the ordering $\preccurlyeq\leq$. Since we assume the axioms of MST [1], if $x = y$ then $m = n$ follows directly from the exact multiplicity axiom of MST. The case $m/x \neq= n/y$ in R if and only if $m \neq n$ and $x = y$ will be inconsistent in MST.

The strict order $m_i x_i \prec< m_j x_j$ implies that $m_i x_i \preccurlyeq\leq m_j x_j$ and $m_i x_i \neq\neq m_j x_j$.

Two points $m_i x_i, m_j x_j$ are comparable if and only if $m_i x_i \preccurlyeq\leq m_j x_j \vee m_j x_j \preccurlyeq\leq m_i x_i$ otherwise they are incomparable and this will be denoted by $m_i x_i || m_j x_j$.

Using Definition 3.2, it can be shown that the ordering $\preccurlyeq\leq$ on M is reflexive, antisymmetric and transitive, i.e., it is a partial mset order. The pair $(M, \preccurlyeq\leq)$ is called a partially ordered multiset (pomset) and in most cases, we will simply denote it by $\mathcal{M}$. If in addition, we have that any two points in M are comparable under $\preccurlyeq\leq$, then the ordering $\preccurlyeq\leq$ will be called a linear mset order.

Definition 3.4. Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and let m_ix_i be a point in M.

(i) m_ix_i is said to be maximal under $\preccurlyeq\leq$, if for any other point $m_jx_j \in M$ with $m_ix_i \preccurlyeq\leq m_jx_j$, we have that $m_ix_i == m_jx_j$.

(ii) m_ix_i is said to be minimal, if for any other point $m_jx_j \in M$ with $m_jx_j \preccurlyeq\leq m_ix_i$ we have that $m_ix_i == m_jx_j$.

If such points are unique, they are called maximum and minimum points, respectively. The maximum and minimum points are also referred to as greatest and least points, respectively. A pomset with no unique maximal (resp. minimal) point will have no greatest (resp. least) point. The collection of all maximal and minimal points of $\mathcal{M}$ are usually denoted by MAX($\mathcal{M}$) and MIN($\mathcal{M}$), respectively.

Definition 3.5. Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and N a submset of M. A suborder $\preccurlyeq\leq_{\mathcal{K}}$ is the restriction of $\preccurlyeq\leq$ to pairs of points in the submset N of M such that

$$n_ix_i \preccurlyeq\leq_{\mathcal{K}} n_jx_j \Leftrightarrow m_ix_i \preccurlyeq\leq m_jx_j,$$

where $n_ix_i, n_jx_j \in N$ and $n_i \leq m_i$ for each corresponding x_i. The pair $(N, \preccurlyeq\leq_{\mathcal{K}})$ is called a subpomset of $\mathcal{M}$.

Definition 3.6. Let $\mathcal{C}$ be a subpomset of a pomset $\mathcal{M} = (M, \preccurlyeq\leq)$. Then $\mathcal{C}$ is said to be:

(i) an mset chain, if the ordering on $\mathcal{C}$ is a linear mset order.

(ii) an mset antichain, if no two points in $\mathcal{C}$ are comparable under the defined order,

(iii) maximal, if it is not strictly contained in any other subpomset of $\mathcal{M}$.

A maximum mset chain (antichain) is one which is of the greatest size possible.

Note that a pomset $\mathcal{M}$ is connected (or is an mset chain), if $(m_ix_i \preccurlyeq\leq m_jx_j) \vee (m_jx_j \preccurlyeq\leq m_ix_i)$ holds for all pairs of points $m_ix_i, m_jx_j \in M$. The pomset $\mathcal{M}$ is an mset antichain, if $m_ix_i || m_jx_j$ holds for all pairs m_ix_i, m_jx_j in M.

Remark 3.7. In [7], a chain $\mathcal{C}$ is called a maximum chain if no other chain contains more points than $\mathcal{C}$. Whereas, the analogous notion in Definition 3.5 assumes that every occurrence of an object is significant for computations in an mset.

Example 3.8. Let $M = [x_1, 2x_2, 4x_3, x_4, 2x_5]$ be an mset. Suppose the ordering $\preccurlyeq$ on the base set is given by $x_i \preccurlyeq x_j$ if and only if i/j (i.e., i divides j). Then the pomset $\mathcal{M}$ induced by $\preccurlyeq$ will be as follows:

$$x_1 \preccurlyeq\leq 2x_2 \preccurlyeq\leq x_4, \; x_1 \preccurlyeq\leq 4x_3, \; x_1 \preccurlyeq\leq 2x_5.$$

We can call these subpomsets $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, respectively, since they are mset chains in $\mathcal{M}$. The cardinalities of $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, are 4, 5, and 3, respectively. By definition, $\mathcal{C}_2$

is a maximum mset chain in the pomset $\mathcal{M}$. Mset antichains of the pomset $\mathcal{M}$ are constructed using set-based partitioning, this is discussed in the next section.

4. Set-based partitioning

Jouannaud and Lescanne [9] proposed an mset ordering for comparing two msets in $\boldsymbol{M}(S)$ by requiring that each mset is a set. It is known that if x_i and x_j are any two elements in a poset $(S, \preccurlyeq)$, then x_i and x_j are incomparable, i.e., $x_i || x_j$, if $x_i \not\preccurlyeq x_j$, $x_j \not\preccurlyeq x_i$, and $x_i \neq x_j$. To achieve an analogous concept in the mset setting, we employ the concept of set-based partitioning [9] on the given mset using the ordering $\preccurlyeq\leq$. In the sequel, mset antichains of the pomset $\mathcal{M}$ are constructed. First, we will recall the definition of a partition of an mset and the notion of set-based partitioning.

Definition 4.1. Let M be an mset. Then $\{M_i | i = 1, \ldots, p\}$ is called a partition of M, if $M = \sum_{i=1}^{p} M_i$.

Definition 4.2. Let M be an mset defined over a poset $(S, \preccurlyeq)$ and $\mathcal{M} = (M, \preccurlyeq\leq)$ is the pomset induced by $\preccurlyeq$. Then $\overline{M} = \{M_i | i = 1 \ldots p\}$ is called a partition of M, if it satisfies the following conditions:
(i) $M_i(x_i) \leq 1$ for each i,
(ii) $x_i \in M_i$ and $x_j \in M_i \Longrightarrow x_i || x_j$,
(iii) $\forall i \in [2, \ldots, p] x_i \in M_i \Longrightarrow (\exists x_j \in M_{i-1}) x_j \not\succcurlyeq x_i$.

The submsets $M_1, M_2, \ldots, M_p$ obtained via this construction will be antichains of the pomset $\mathcal{M}$

Definition 4.3. The height of a pomset $\mathcal{M}$ is the cardinality of a maximum mset chain. The width of $\mathcal{M}$ is the cardinality of a maximum mset antichain constructed using Definition 4.2 above.

Example 4.4. Consider the mset $M = [3x_1, x_2, 2x_3, 5x_4, x_5, 3x_6]$, where the base set is partially ordered as follows:

$x_i \preccurlyeq x_j$ if and only if i and j are both even (odd) and $i \leq j$, where $i, j \in \mathbb{N}$.

By Definitions 3.2 and 3.5, $\mathcal{C}_1 = 3x_1 \preccurlyeq\leq 2x_3 \preccurlyeq\leq x_5$ and $\mathcal{C}_2 = x_2 \preccurlyeq\leq 5x_4 \preccurlyeq\leq 3x_6$ are possible mset chains of the pomset $\mathcal{M} = (M, \preccurlyeq\leq)$ induced by $\preccurlyeq$. They are also maximal.

Using the construction in Definition 4.2, analogous antichains in this mset setting will be the following substructures of $\mathcal{M}$:

$$\begin{gathered}
\mathcal{A}_1 = \{x_5, x_6\}, \\
\mathcal{A}_2 = \{x_3, x_6\}, \\
\mathcal{A}_3 = \{x_3, x_6\}, \\
\mathcal{A}_4 = \{x_1, x_4\}, \\
\mathcal{A}_5 = \{x_1, x_4\}, \\
\mathcal{A}_6 = \{x_1, x_4\}, \\
\mathcal{A}_7 = \{x_4\}, \\
\mathcal{A}_8 = \{x_4\}, \\
\mathcal{A}_9 = \{x_2\}.
\end{gathered}$$

Where each $\mathcal{A}_i$ belongs in the partition $\overline{M} = \{A_i | i = 1 \dots p\}$ and $\sum_{i=1}^{9} \mathcal{A}_i = M$.

Recall that an mset admits repetition, hence each occurrence of an object is significant in computing a partition of the mset.

Remark 4.5. The antichains $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$, and $\mathcal{A}_6$ are maximum in $\mathcal{M}$. By definition, the width of the pomset $\mathcal{M}$ is 2, and the minimum number of mset chains into which $\mathcal{M}$ can be partitioned is 2 viz; $\mathcal{C}_1$ and $\mathcal{C}_2$. Also $|\mathcal{C}_1| = 6$ and $|\mathcal{C}_2| = 9$ and hence the mset chain $\mathcal{C}_2$ is maximum. By definition, the height of the pomset $\mathcal{M}$ is 9, and the minimum number of antichains into which $\mathcal{M}$ can be partitioned is 9 viz; $\mathcal{A}_1, \dots, \mathcal{A}_9$.

Applying the concept of set-based partitioning on the ordered mset structure $\mathcal{M}$ gives combinatorial parameters that are consistent with the statement of Dilworth's decomposition theorem [5] and its dual [12].

We have the following result.

Proposition 4.6. *Suppose the pomset $\mathcal{M}$ is partitioned into finite mset chains $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$. If $\mathcal{A}$ is an mset antichain, then there is at most one element of $\mathcal{A}$ in each $\mathcal{C}_i$ and thus $n \geq |\mathcal{A}|$.*

Proof. To prove this result on an ordered mset structure, antichains of $\mathcal{M}$ will be obtained via set-based partitioning such that no two elements of $\mathcal{A}$ are related. We then show that $|\mathcal{A} \cap \mathcal{C}_i| \leq 1$ for any $i \in [1, n]$.

Now, if $\mathcal{C}_i$ and $\mathcal{A}$ are disjoint, the result is straight forward. Suppose the intersection of $\mathcal{A}$ and $\mathcal{C}_i$ is not empty. Then we need to proceed as follows.

By Definition 4.2, each occurrence of a generating object must belong to a different antichain, i.e., $x_i, x_j \in A \Longrightarrow x_i \neq x_j$. Also since for any pair $x_i, x_j \in A$, we have $x_i || x_j$. thus $x_i \in \mathcal{C}_i$ implies $x_j \notin \mathcal{C}_i$. Otherwise, they will be comparable. So $|\mathcal{A} \cap \mathcal{C}_i| \leq 1$ for all i. Hence $n \geq |\mathcal{A}|$. □

5. Conclusions

The notions of maximal and maximum antichains were generalized using a partially ordered mset structure. Set-based partitioning was adopted in constructing substructures of the ordered mset such that no two elements are comparable. The notion of set-based partitioning promises to be very useful for explicating more complex analogous results on antichains in the mset setting.

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