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Uniform Hausdorff fuzzy metric on fuzzy sets

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Abstract. In this paper, the uniform Hausdorff fuzzy metric is constructed in the sense of fuzzy metric spaces. Several topological properties as precompactness, completeness and compactness are investigated. Given a fuzzy metric space, we prove that a self mapping is continuous if and only if its Zadeh's extension is continuous which generated the result of Jardón (2020) [13] and establish a new fixed point theorem for fuzzy mapping.

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1. Introduction

Fuzzy metric plays a significant role in the theory of fuzzy topology. The concept of fuzzy metric space was firstly introduced by Kramosil [14] from the generalization of probabilistic metric space. In order to overcome some limitations of this concept, George and Veeramani [4, 5] modified the former definition of fuzzy metric spaces and obtained a new class of fuzzy metric spaces which we called GV-fuzzy metric spaces in general. Moreover, they proved that every GV-fuzzy metric space is Hausforff and first countable.

Later on, several authors have contributed to study of fuzzy metric spaces. Gregori and Romaguera [7, 8, 9, 10, 11] proved that the topological space induced by a fuzzy metric space is metrizable. This important result closely make contact fuzzy metric spaces with metric spaces. They not only proved that a fuzzy metric space is compact if and only if it is precompact and complete, but also introduced some definitions of continuous mapping and demonstrated the relationship between them. Especially, given several useful examples of fuzzy metric spaces. Rodríguez-López and Romaguera [22, 23] constructed the Hausdorff fuzzy metric on the family of nonempty compact subsets in a fuzzy metric space and investigated some topological properties of it. Gregori [8], Mihet [17, 18, 19] and Vijayaraju [24] presented several types of fuzzy contractive mappings and extended the classical fixed point theorems to fuzzy metric spaces. Recently, Jardón [13] proved that the Zadeh's extension of a continuous function is also continuous in a metric space.

The aim of this paper is to establish the notion of uniform Hausdorff fuzzy metric on the family of normal and semicontinuous fuzzy sets with compact support in a given fuzzy metric space and investigate relations between a general fuzzy metric space and its uniform Hausdorff fuzzy metric space. The framework is organized as follows. In Section 2, we give some basic definitions and auxiliary results. In Section 3, we define the notion of the uniform Hausdorff fuzzy metric. After that, we discuss some topological equivalence between these two classes of fuzzy metric spaces. In Section 4, we mainly prove that a mapping is continuous if and only if its Zadeh's extension is continuous in the sense of Gregori. Finally, we present a fixed point theorem of fuzzy mapping from a fuzzy metric space to its uniform Hausdorff fuzzy metric space.

2. Preliminaries

We start this section by recalling some necessary notions and fundamental results which will be needed.

Given a metric space (X, d), a fuzzy set u on the space X is a function $u : X \to [0, 1]$. Denote by u_α the α-level cuts of u and $u_\alpha = \{x \in X : u(x) \geq \alpha\}$ for each $\alpha \in (0, 1]$. Define the support of u as $u_0 = \overline{\bigcup_{\alpha\in(0,1]} u_\alpha}$. Let $\mathfrak{F}(X)$ be the family of all normal and upper semicontinuous fuzzy sets $u : X \to [0, 1]$ with compact support. It is well known that the Hausdorff metric of d on non-empty compact sets A, B of X is defined by

$$H(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} d(x, y), \sup_{y\in B} \inf_{x\in A} d(x, y)\}.$$

And the uniform Hausdorff metric on fuzzy sets u, v of $\mathfrak{F}(X)$ is

$$D(u, v) = \sup_{\alpha\in[0,1]} \{H(u_\alpha, v_\alpha)\}.$$

Definition 2.1 ([4, 15])**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions: $*$ is associative and commutative; $*$ is continuous; $a * 1 = 1$ for each $a \in [0, 1]$; $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for arbitrary $a, b, c, d \in [0, 1]$.

There are some commonly used t-norms. The minimum t-norm $a * b = \min\{a, b\}$, the usual product t-norm $a*b = ab$, the Lukasiewicz t-norm $a*b = \max\{a+b-1, 0\}$.

Definition 2.2 ([1, 4])**.** The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1 \Leftrightarrow x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, s + t) \geq M(x, z, s) * M(z, y, t)$
(v) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous,

for all $x, y, z \in X, s, t \in (0, \infty)$. If $(X, M, *)$ is a fuzzy metric space, then $(M, *)$ or simply M is called a fuzzy metric on X.

Lemma 2.3 ([4]). *$M(x, y, \cdot)$ is nondecreasing for all x, y in X.*

George and Veeramani [4, 5] proved that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\} : x \in X, t > 0, r \in (0, 1)\}$. In addition, (X, τ_M) is a Hausdorff, first countable and metrizable topological space.

In a metric space, Jardón [13] proved that a function is continuous if and only if its Zadeh's extension is continuous under the uniform Hausdorff metric. For a given function $f : X \to X$, the Zadeh's extension of f is denoted by $\hat{f} : \mathfrak{F}(X) \to \mathfrak{F}(X)$ and defined as follows (See [21]):

$$\hat{f}(u)(x) = \begin{cases} \sup\{u(y) : y \in f^{-1}(x)\}, & f^{-1}(x) \neq \varnothing, \\ 0, & f^{-1}(x) = \varnothing. \end{cases}$$

For any $x \in X$, denoted by $\chi_{\{x\}}$ the characteristic function of x. It is clear that $\chi_{\{x\}} \in \mathfrak{F}(X)$.

Lemma 2.4 ([13]). *Let X be a Hausdorff space. If $f : X \to X$ is a continuous function, then $[\hat{f}(u)]_\alpha = f(u_\alpha)$ for each $u \in \mathfrak{F}(X)$ and $\alpha \in [0, 1]$. Moreover, for each $x \in X$, then $\hat{f}(\chi_{\{x\}}) = \chi_{f(x)}$.*

Proposition 2.5. *Let $(X, M, *)$ be a fuzzy metric space, for every $u \in \mathfrak{F}(X)$ and $\alpha \in [0, 1]$, u_α is a compact set in X.*

Proof. By the upper semicontinuity of u, it is easily to conclude that u_α is a closed set. Hence the compactness of u_α is a direct consequence of the compact support of u. □

Next, we introduce some topological properties of fuzzy metric spaces.

Definition 2.6 ([7]). A fuzzy metric space $(X, M, *)$ is called precompact if for each $t > 0$ and $r \in (0, 1)$, there is a finite subset $A \subseteq X$, such that $X = \bigcup_{a \in A} B_M(a, r, t)$.

Definition 2.7 ([7]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy sequence if and only if for each $t > 0$ and $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m \geq n_0$. A Cauchy sequence $\{x_n\}$ is convergent in $(X, M, *)$ if there exist $x \in X$ such that $\lim_{n \to \infty} M(x_n, x, t) = 1$. A fuzzy metric space is said to be complete if every Cauchy sequence is convergent in it.

3. The construction of the uniform Hausdorff fuzzy metric on $\mathfrak{F}(X)$

Jesús Rodríguez-López defined the Hausdorff fuzzy metric

$$H_M(A, B, t) = \min\{\inf_{x \in A} \sup_{y \in B} M(x, y, t), \inf_{y \in B} \sup_{x \in A} M(x, y, t)\},$$

where $A, B \in K(X)$ and $K(X)$ denotes all the compact subsets of X. And several theorems of Hausdorff fuzzy metric space are proved in [16, 22].

Lemma 3.1 ([22]). *Let $(X, M, *)$ be a fuzzy metric space, then M is a continuous functions on $X \times X \times (0, \infty)$.*

Lemma 3.2 ([22])**.** *Let $(X, M, *)$ be a fuzzy metric space, for each $a \in X$, an arbitrary compact subsets B of X and $t > 0$, there exist a $b_0 \in B$ such that $\sup_{b\in B} M(a, b, t) = M(a, b_0, t)$. In addition, $\sup_{b\in B} M(a, b, \cdot)$ is a continuous function on $(0, \infty)$.*

Lemma 3.3 ([22])**.** *Let $(X, M, *)$ be a fuzzy metric space, for each compact subsets A, B of X and $t > 0$, there exist $a_0 \in A$ and $b_{0\in B}$ such that $\inf_{a\in A} \sup_{b\in B} M(a, b, t) = M(a_0, b_0, t)$. In addition, $\inf_{a\in A} \sup_{b\in B} M(a, b, \cdot)$ is a continuous function on $(0, \infty)$.*

Theorem 3.4. *Let $(X, M, *)$ be a fuzzy metric space, define a function $D_M : \mathfrak{F}(X) \times \mathfrak{F}(X) \times (0, \infty) \to [0, 1]$ by*

$$D_M(u, v, t) = \inf_{\alpha\in[0,1]} \min\{ \inf_{x\in[u]_\alpha} \sup_{y\in[v]_\alpha} M(x, y, t), \inf_{y\in[v]_\alpha} \sup_{x\in[u]_\alpha} M(x, y, t)\},$$

for all $u, v \in \mathfrak{F}(X)$ and $t > 0$, then $(\mathfrak{F}(X), D_M, *)$ is a fuzzy metric space.

Proof. (i) Let $u, v, w \in \mathfrak{F}(X)$ and $s, t > 0$. By the continuity of H_M on $X \times X \times (0, \infty)$ and Lemma 3.2, there exists $\alpha_0 \in [0, 1]$ such that

$$D_M(u, v, t) = H_M(u_{\alpha_0}, v_{\alpha_0}, t) > 0.$$

It is clear that $u = v \Leftrightarrow D_M(u, v, t) = 1$ and $D_M(u, v, t) = D_M(v, u, t)$. Then it satisfies the conditions (ii) and (iii) of Definition 2.2.

(iv) Let $\alpha \in [0, 1]$, $x \in u_\alpha$ and $y \in w_\alpha$. By Lemma 3.3, there exist a point $z_x \in v_\alpha$ such that $\sup_{z\in v_\alpha} M(x, z, t) = M(x, z_x, t)$. Then by the triangular inequality of M and the continuity of $*$, we have

$$\sup_{y\in w_\alpha} M(x, y, t + s) \geq M(x, z_x, t) * \sup_{y\in w_\alpha} M(z_x, y, s).$$

Thus

$$\inf_{x\in u_\alpha} \sup_{y\in w_\alpha} M(x, y, t + s) \geq \inf_{x\in u_\alpha} \sup_{z\in v_\alpha} M(x, z, t) * \inf_{x\in u_\alpha} \sup_{y\in w_\alpha} M(z_x, y, s).$$

Since $\{z_x : x \in u_\alpha\} \subseteq v_\alpha$, $\inf_{x\in u_\alpha} \sup_{y\in w_\alpha} M(z_x, y, s) \geq \inf_{z\in v_\alpha} \sup_{y\in w_\alpha} M(z, y, s)$. So we have

$$\inf_{x\in u_\alpha} \sup_{y\in w_\alpha} M(x, y, t + s) \geq \inf_{x\in u_\alpha} \sup_{z\in v_\alpha} M(x, z, t) * \inf_{z\in v_\alpha} \sup_{y\in w_\alpha} M(z, y, s).$$

Similarly, we obtain

$$\inf_{y\in w_\alpha} \sup_{x\in u_\alpha} M(x, y, t + s) \geq \inf_{z\in v_\alpha} \sup_{x\in u_\alpha} M(x, z, t) * \inf_{y\in w_\alpha} \sup_{z\in v_\alpha} M(z, y, s).$$

Since $*$ is associative and commutative, we get

$$D_M(u, w, s + t) \geq D_M(u, v, s) * D_M(v, w, t).$$

(v) By Lemma 3.3, it is obvious that the function $D_M(u, v, \cdot)$ is continuous. This completes the proof. □

The fuzzy metric D_M will be called the uniform Hausforff metric of M on $\mathfrak{F}(X)$.

Example 3.5. Denote by R the real number set. Define the function $M(x,y,t) = e^{-\frac{d(x,y)}{t}}$ on R, where $d(x,y) = |x-y|$ for every $x,y \in R$, take $*$ be the minimum t-norm $\wedge$, then $(R, M, \wedge)$ is a fuzzy metric space. Thus $(\mathfrak{F}(R), D_M, \wedge)$ is a fuzzy metric space.

In the main result of this section, we investigate some topological equivalence between $(X, M, *)$ and $(\mathfrak{F}(X), D_M, *)$.

Theorem 3.6. *A fuzzy metric space $(X, M, *)$ is precompact if and only if $(\mathfrak{F}(X), D_M, *)$ is precompact.*

Proof. Suppose that $(\mathfrak{F}(X), D_M, *)$ is precompact. For each $t > 0$ and $r \in (0,1)$, without loss of generality assume that the finite subset $A = \{u_i : u_i \in \mathfrak{F}(X)\}_{i=1}^{n}, n \in \mathbb{N}$ such that $\mathfrak{F}(X) = \bigcup_{i=1}^{n} B_{D_M}(u_i, r, t)$. Take $x \in X$. Then the characteristic function of x, i.e., $\chi_{\{x\}} \in \mathfrak{F}(X)$. Thus there exist $u_i \in A$ such that $D_M(\chi_{\{x\}}, u_i, t) > 1 - r$. For each $\alpha \in [0,1]$, we have $[\chi_{\{x\}}]_\alpha = \{x\}$. So $\inf_{y\in[u_i]_\alpha} M(x,y,t) > 1-r$. We can find $y_i \in [u_i]_1 \subseteq X$ such that $M(x, y_i, t) \geq \inf_{y\in[u_i]_\alpha} M(x,y,t) > 1-r$, which means $x \in B_{D_M}(y_i, r, t)$. Hence the arbitrariness of x and finiteness of u_i imply $X = \bigcup_{i=1}^{n} B_M(y_i, r, t)$. Therefore $(X, M, *)$ is precompact.

Conversely, suppose $(X, M, *)$ is precompact. For each $t > 0$ and $r \in (0,1)$, without loss of generality assume that the finite subset $A = \{a_i : a_i \in X\}_{i=1}^{n}$ such that $X = \bigcup_{a\in A} B_M(a, \frac{r}{2}, t)$. Take $u \in \mathfrak{F}(X)$. Then $u_0 \subseteq X$. Thus there exist finite set $A_{u_0} \subseteq A$, we assume without loss of generality that $A_{u_0} = \{a_1, a_2, \cdots, a_k : k \leq n\}$ such that $u_0 \subseteq \bigcup_{a\in A_{u_0}} B_M(a, \frac{r}{2}, t)$. Moreover, $u_0 \cap B_M(a_i, \frac{r}{2}, t) \neq \varnothing$ for any $1 \leq i \leq k$. Similarly, we assume that $A_{u_1} = \{a_1, a_2, \cdots, a_m : m \leq k\}$ such that $u_1 \subseteq \bigcup_{a\in A_{u_1}} B_M(a, \frac{r}{2}, t)$ and $u_1 \cap B_M(a_i, \frac{r}{2}, t) \neq \varnothing$ for any $1 \leq i \leq m$. Since A_{u_0} is a finite set of A, there exist a partition $\{\alpha_i\}_{i=1}^{t} \subseteq (0,1)$ such that $A_{u_1} \subset A_{u_{\alpha_1}} \subset A_{u_{\alpha_2}} \subset \cdots \subset A_{u_{\alpha_t}} = A_{u_0}$ and for any $\alpha \in (\alpha_{i+1}, \alpha_i], i = 1, 2, \cdots, t-1$, we have $u_\alpha \subseteq \bigcup_{a\in A_{u_{\alpha_i}}} B_M(a, \frac{r}{2}, t)$ and $u_\alpha \cap B_M(a, \frac{r}{2}, t) \neq \varnothing$ for any $a \in A_{u_{\alpha_i}}$. Next we We construct a fuzzy set defined as

$$V_u(x) = \begin{cases} 1, & x \in A_{u_1}, \\ \alpha_1, & x \in A_{u_{\alpha_1}}/A_{u_1}, \\ \alpha_2, & x \in A_{u_{\alpha_2}}/A_{u_{\alpha_1}}, \\ \cdots, & \cdots; \\ \alpha_t, & x \in A_{u_{\alpha_t}}/A_{u_{\alpha_{t-1}}}, \\ 0, & x \notin A_{u_0}. \end{cases}$$

Then it is obvious that $V_u \in \mathfrak{F}(X)$ and for each $\alpha \in [0,1]$, there exist α_i such that $[V_u]_\alpha = A_{u_{\alpha_i}}$. Then for each $x \in u_\alpha$, we can find $a_x \in [V_u]_\alpha$ such that $x \in B_M(a_x, \frac{r}{2}, t)$. Thus we have

$$\inf_{x\in u_\alpha} \sup_{a\in[V_u]_\alpha} M(x,a,t) \geq 1 - \frac{r}{2}.$$

Similarly, for each $a \in [V_u]_\alpha$, we can find $x_a \in u_\alpha$ such that $x_a \in B_M(a, \frac{r}{2}, t)$. So

$$\inf_{a\in[V_u]_\alpha} \sup_{x\in u_\alpha} M(x, a, t) \geq 1 - \frac{r}{2}.$$

Hence we get

$$D_M(u, V_u, t) \geq 1 - \frac{r}{2} > 1 - r.$$

SincetThe finiteness of A implies the family of subsets A_u is finite, $\{V_u : u \in \mathfrak{F}(X)\}$ is a finite set. Therefore $\mathfrak{F}(X) = \bigcup_{V\in\{V_u\}} B_{D_M}(V, r, t)$, i.e., $(\mathfrak{F}(X), D_M, *)$ is precompact. □

Theorem 3.7. *A fuzzy metric space $(X, M, *)$ is complete if and only if $(\mathfrak{F}(X), D_M, *)$ is complete.*

Proof. Suppose that $(\mathfrak{F}(X), D_M, *)$ is complete. For every Cauchy sequence $\{x_n\} \subseteq X$, then $\chi_{\{x_n\}} \subseteq \mathfrak{F}(X)$ for each $n \in \mathbb{N}$. It is obvious that for each $t > 0$ and $r \in (0, 1)$, there exist $n_0 \in \mathbb{N}$ such that $D_M(\chi_{\{x_n\}}, \chi_{\{x_m\}}, t) = M(x_n, x_m, t) > 1 - r$ for all $n, m \geq n_0$. Then $\{\chi_{\{x_n\}}\}$ is a Cauchy sequence in the complete fuzzy metric space. Thus there exist $u_x \in \mathfrak{F}(X)$ such that $\lim_{n\to\infty} D_M(\chi_{\{x_n\}}, u_x, t) = 1$. Take $y_x \in [u_x]_1 \in X$. Then $\lim_{n\to\infty} M(x_n, y_x, t) = 1$. Thus $(X, M, *)$ is complete.

Conversely, suppose $(X, M, *)$ is complete. Then $(K(X), H_M, *)$ is complete. For every Cauchy sequence $\{u_n\} \subseteq \mathfrak{F}(X)$, each $t > 0$ and $r \in (0, 1)$, there exist $n_0 \in \mathbb{N}$ such that $D_M(u_n, u_m, t) > 1 - r$ for all $n, m \geq n_0$. By the definition of D_M, we have $H_M([u_n]_\alpha, [u_m]_\alpha, t) > 1 - r$ for each $\alpha \in I$. Thus $\{[u_n]_\alpha\}$ is a Cauchy sequence in the complete fuzzy metric space $(K(X), H_M, *)$. So there exist $A_\alpha \in K(X)$ such that

$$\lim_{n\to\infty} H_M([u_n]_\alpha, A_\alpha, t) = 1.$$

Next, for each $\alpha_1, \alpha_2 \in I$ and $\alpha_1 \leq \alpha_2$, we prove $A_{\alpha_1} \supseteq A_{\alpha_2}$. Suppose there exist $x_0 \in A_{\alpha_2}$ and $x_0 \notin A_{\alpha_1}$. Since $\lim_{n\to\infty} H_M([u_n]_{\alpha_2}, A_{\alpha_2}, t) = 1$, for $x_0 \in A_{\alpha_2}$, exist $n_1 \in \mathbb{N}$, for each $n \geq n_1$, there exist $y_0 \in [u_n]_{\alpha_2}$ such that $M(x_0, y_0, t) > 1 - r$.

Moreover, $\lim_{n\to\infty} H_M([u_n]_{\alpha_1}, A_{\alpha_1}, t) = 1$, exist $n_2 \in \mathbb{N}$, for each $n \geq \max\{n_1, n_2\}$ and $y_0 \in [u_n]_{\alpha_2} \subseteq [u_n]_{\alpha_1}$, there exist $x_1 \in A_{\alpha_1}$ such that $M(x_1, y_0, t) > 1 - r$. Since fuzzy matric M is continuous on $X \times X \times (0, +\infty)$ and the compactness of A_{α_1}, $x_0 \in A_{\alpha_1}$ which is contradictory with the hypothesis. Then $A_{\alpha_1} \supseteq A_{\alpha_2}$.

By the representation theorem of fuzzy numbers, there exists only $V_u \in \mathfrak{F}(X)$ such that $[V_u]_\alpha = A_\alpha$ for each $\alpha \in I$. Thus $\lim_{n\to\infty} H_M([u_n]_\alpha, [V_u]_\alpha, t) = 1$ for each $\alpha \in I$. So

$$\lim_{n\to\infty} D_M(u_n, V_u, t) = \lim_{n\to\infty} \inf_{\alpha\in I} H_M([u_n]_\alpha, [V_u]_\alpha, t) = 1.$$

i.e., $(\mathfrak{F}(X), D_M, *)$ is complete. □

Lemma 3.8. *A fuzzy metric space $(X, M, *)$ is compact if and only if it is precompact and complete.*

Theorem 3.9. *A fuzzy metric space $(X, M, *)$ is compact if and only if $(\mathfrak{F}(X), D_M, *)$ is compact.*

Proof. It is a direct consequence of Theorem 3.6 and Lemma 3.8. □

4. Zadeh's extensions on $\mathfrak{F}(X)$

Zadeh's extension principle plays an important role in fuzzy set theory. In this section, we introduce the concept of fuzzy continuous functions in fuzzy metric spaces and investigate some properties of its Zadeh's extension.

Definition 4.1. Let $(X, M_1, *)$ and $(Y, M_2, *)$ be fuzzy metric spaces. A mapping $f : (X, M_1, *) \to (Y, M_2, *)$ is called continuous if for each $x \in X$, $t > 0$ and $r \in (0, 1)$, there exist $t_x > 0$ and $r_x \in (0, 1)$ such that $M_2(f(x), f(y), t) > 1 - r$ whenever $M_1(x, y, t_x) > 1 - r_x$.

Lemma 4.2. *Let $(X, M, \wedge)$ be a fuzzy metric space, K is a compact subset of X and $f : (X, M, \wedge) \to (X, M, \wedge)$ is continuous. Then for any $x \in K$, $t > 0, r \in (0, 1)$, there exist $t_0 > 0$, $r_0 \in (0, 1)$ such that $M(f(x), f(y), t) > 1 - r$ whenever $y \in X$ and $M(x, y, t_0) > 1 - r_0$.*

Proof. Suppose that K is a compact subset of X. Let $t > 0, r \in (0, 1)$ and $x \in K$. Then from the continuity of f, there exist $t_x > 0$ and $r_x \in (0, 1)$, such that $M(f(x), f(y), \frac{t}{2}) > 1 - r$ whenever $y \in X$ and $M(x, y, t_x) > 1 - r_x$.

Clearly, $K \subseteq \bigcup_{x \in X_1} B_M(x, r_x, \frac{t_x}{2})$. Since K is a compact set, we have $K \subseteq \bigcup_{i=1}^{n} B_M(x_i, r_{x_i}, \frac{t_{x_i}}{2})$ for some $n \in \mathbb{N}$. Moreover, $K \cap B_M(x_i, r_{x_i}, \frac{t_{x_i}}{2}) \neq \varnothing$ for each $1 \le i \le n$. Let $t_0 = \min\{\frac{t_{x_i}}{2}\}_{i=1}^{n}$ and $r_0 = \min\{r_{x_i}\}_{i=1}^{n}$. Then for each $x \in X_1, y \in X$ with $M(x, y, t_0) > 1 - r_0$, there exist x_i such that $x \in B_M(x_i, r_{x_i}, \frac{t_{x_i}}{2})$. Thus

$$M(x, x_i, t_{x_i}) \ge M(x, x_i, \frac{t_{x_i}}{2}) > 1 - r_{x_i}$$

and

$$\begin{aligned} M(y, x_i, t_{x_i}) &\ge M(y, x, \frac{t_{x_i}}{2}) \wedge M(x, x_i, \frac{t_{x_i}}{2}) \ge M(y, x, t_0) \wedge (1 - r_{x_i}) \\ &\ge (1 - r_0) \wedge (1 - r_{x_i}) \\ &= (1 - r_{x_i}). \end{aligned}$$

So $M(f(y), f(x_i), \frac{t}{2}) > 1 - r$ and $M(f(x_i), f(x), \frac{t}{2}) > 1 - r$. It deduces that

$$M(f(x), f(y), t) \ge M(f(x), f(x_i), \frac{t}{2}) \wedge M(f(x_i), f(y), \frac{t}{2}) > (1 - r) \wedge (1 - r) = 1 - r.$$

□

Theorem 4.3. *Let $(X, M, \wedge)$ be a fuzzy metric space. A mapping $f : (X, M, \wedge) \to (X, M, \wedge)$ is continuous if and only if $\hat{f} : (\mathfrak{F}(X), D_M, \wedge) \to (\mathfrak{F}(X), D_M, \wedge)$ is continuous.*

Proof. Suppose that f is continuous on X and take $u \in \mathfrak{F}(X)$. Since u_0 is a compact set, by Lemma 4.2, for each $x \in u_0$, $t > 0$ and $r \in (0, 1)$, there exist $t_u > 0$ and $r_u \in (0, 1)$ such that $M(f(x), f(y), t) > 1 - \frac{r}{2}$ whenever $y \in X$ and $M(x, y, t_u) > 1 - r_u$.

Choose $v \in \mathfrak{F}(X)$ with $D_M(u, v, t_u) > 1 - r_u$ and we prove that $D_M(\hat{f}(u), \hat{f}(v), t) > 1 - r$. Pick $\alpha \in [0, 1]$. Then for each $x \in u_\alpha \subseteq u_0$, there exist $y_x \in v_\alpha \subseteq X$ such that $M(x, y_x, t_u) > 1 - r_u$. By Lemma 4.2, we have $M(f(x), f(y_x), t) > 1 - \frac{r}{2}$. From Lemma 2.4, it implies that $[\hat{f}(u)]_\alpha = f(u_\alpha)$. Thus we get

$$\inf_{x \in [\hat{f}(u)]_\alpha} \sup_{y \in [\hat{f}(v)]_\alpha} M(x, y, t) \geq 1 - \frac{r}{2}.$$

Similarly, for each $y \in v_\alpha$, there exist $x_y \in u_\alpha \subseteq u_0$ such that $M(x_y, y, t_u) > 1 - r_u$. So we obtain

$$\inf_{y \in [\hat{f}(v)]_\alpha} \sup_{x \in [\hat{f}(u)]_\alpha} M(x, y, t) \geq 1 - \frac{r}{2}.$$

Hence $D_M(\hat{f}(u), \hat{f}(v), t) \geq 1 - \frac{r}{2} > 1 - r$, i.e., $\hat{f}$ is continuous on $\mathfrak{F}(X)$.

Conversely, assume that $\hat{f}$ is continuous on $\mathfrak{F}(X)$. Take $x \in X$. Then $\chi_{\{x\}} \in \mathfrak{F}(X)$. For each $t > 0$ and $r \in (0, 1)$, there exist $u \in \mathfrak{F}(X)$, $t_x > 0$ and $r_x \in (0, 1)$, such that $D_M(\hat{f}(\chi_{\{x\}}), \hat{f}(u), t) > 1 - r$ whenever $D_M(\chi_{\{x\}}, u, t_x) > 1 - r_x$. Let $y \in X$ and $M(x, y, t_x) > 1 - r_x$. Then it is clear that $M(x, y, t_x) = D_M(\chi_{\{x\}}, \chi_{\{y\}}, t_x) > 1 - r_x$. Thus by Lemma 2.4, we can obtain

$$M(f(x), f(y), t) = D_M(\chi_{\{f(x)\}}, \chi_{\{f(y)\}}, t) = D_M(\hat{f}(\chi_{\{x\}}), \hat{f}(\chi_{\{y\}}), t) > 1 - r.$$

So f is continuous on X. □

5. Fixed point theorem of fuzzy mapping in fuzzy metric space

Heilpern [12] introduced the concept of fuzzy mappings in linear metric space. In this section, we generate it to fuzzy metric space and obtain a fixed point theorem.

Definition 5.1. Let $(X, M, *)$ be a fuzzy metric space, then T is called a fuzzy mapping, if T is a mapping from X to $\mathfrak{F}(X)$, i.e., $T(x) \in \mathfrak{F}(X)$ for each $x \in X$.

Definition 5.2. ([6, 20]) Let $u, v \in \mathfrak{F}(X)$. Then u is more accurate than v, denoted by $u \subset v$, if $u(x) \leq v(x)$ for each $x \in X$.

Definition 5.3. ([2, 3]) For a fuzzy mapping $F : (X, M, *) \to (\mathfrak{F}(X), D_M, *)$ and $\alpha \in [0, 1]$, $x \in X$ is called a fixed point of T, if $\chi_{\{x\}} \subset F(x)$, i.e., $x \in [F(x)]_1$. Equivalently, $x \in [F(x)]_\alpha$ for each $\alpha \in [0, 1]$.

Lemma 5.4. *For* $\forall x \in X, \alpha \in [0, 1], t > 0$, *if* $\chi_{\{x\}} \subset u \in \mathfrak{F}(X)$, *then* $\sup_{y \in u_\alpha} M(x, y, t) = 1$.

Lemma 5.5. *For* $\forall x, z \in X, u \in \mathfrak{F}(X), s, t > 0$, *then* $\sup_{y \in u_\alpha} M(x, y, t+s) \geq M(x, z, t) * \sup_{y \in u_\alpha} M(z, y, s)$.

Lemma 5.6. *For* $\forall x_0 \in X, \alpha \in [0, 1], u, v \in \mathfrak{F}(X), t > 0$, *if* $\chi_{\{x_0\}} \subset u$, *then* $\sup_{y \in v_\alpha} M(x_0, y, t) \geq D_M(u, v, t)$.

Theorem 5.7. *Let* $(X, M, *)$ *be a complete fuzzy metric space. Suppose* $F : (X, M, *) \to (\mathfrak{F}(X), D_M, *)$ *is a fuzzy mapping satisfying*

$$\frac{1}{D_M(F(x), F(y), t)} - 1 \leq k(\frac{1}{M(x, y, t)} - 1)$$

for each $x, y \in X, t > 0$ *and* k *is a fixed constant in* $(0,1)$. *Then there exist a* $x_* \in X$ *such that* $\chi_{\{x_*\}} \subset F(x_*)$, *i.e.,* x_* *is a fixed point of* F.

Proof. Let $x_1 \in X, \alpha \in [0,1]$ and $t > 0$. Since $F(x_1) \in \mathfrak{F}(X)$ is a normal fuzzy set, there exist $x_2 \in [F(x_1)]_1$, i.e., $\chi_{\{x_2\}} \subset F(x_1)$. Then by Lemma 5.6, we have

$$\begin{aligned}
&\sup_{y\in[F(x_2)]_\alpha} M(x_2,y,t)\\
&\geq \inf_{x\in[F(x_1)]_\alpha} \sup_{y\in[F(x_2)]_\alpha} M(x,y,t)\\
&\geq \min\{\inf_{x\in[F(x_1)]_\alpha} \sup_{y\in[F(x_2)]_\alpha} M(x,y,t), \inf_{y\in[F(x_2)]_\alpha} \sup_{x\in[F(x_1)]_\alpha} M(x,y,t)\}\\
&\geq D_M(F(x_1),F(x_2),t).
\end{aligned}$$

Since M is continuous on $X \times X \times \infty$ and $[F(x_2)]_\alpha$ is compact, there exist $x_3 \in [F(x_2)]_\alpha$ such that $M(x_2,x_3,t) = \sup_{y\in[F(x_2)]_\alpha} M(x_2,y,t) \geq D_M(F(x_1),F(x_2),t)$. Continuing in this way, we obtain a sequence $\{x_n\} \subset X$ such that $\chi_{\{x_n\}} \subset F(x_{n-1})$ and $M(x_{n+1},x_n,t) \geq D_M(F(x_n),F(x_{n-1}),t)$ for $n = 2,3,4,\cdots$. Since k is a fixed constant in $(0,1)$, we have

$$\frac{1}{M(x_{n+1},x_n,t)} - 1 \leq \frac{1}{D_M(F(x_n),F(x_{n-1}),t)} - 1 \leq k(\frac{1}{M(x_n,x_{n-1},t)} - 1).$$

We can show by induction that

$$\frac{1}{M(x_{n+1},x_n,t)} - 1 \leq k^{n-1}(\frac{1}{M(x_2,x_1,t)} - 1).$$

For each $r \in (0,1)$, there exist $n_0 > 0$ such that $k^{n-1}(\frac{1}{M(x_2,x_1,t)} - 1) < \frac{r}{1-r}$ for each $n \geq n_0$. Thus $\frac{1}{M(x_{n+1},x_n,t)} - 1 < \frac{r}{1-r}$, i.e. $M(x_{n+1},x_n,t) > 1-r$. Moreover, for each $s \in (0,1)$, there exist $r \in (0,1)$ such that $(1-r)*(1-r) > 1-s$. So by the the triangular inequality of M, we have

$$M(x_{n+2},x_n,t) \geq M(x_{n+2},x_{n+1},\frac{t}{2}) * M(x_{n+1},x_n,\frac{t}{2}) > (1-r)*(1-r) > 1-s.$$

This is obvious that $M(x_n,x_m,t) > 1-s$ for each $n,m \geq n_0$ by induction. Hence $\{x_n\}$ is a Cauchy sequence. Since $(X,M,*)$ is complete, there exist x_* such that $\lim_{n\to\infty} M(x_n,x_*,t) = 1$.

Next, we will show that $x_* \subset F(x_*)$. By the hypothesis, Lemma 5.5 and 5.6, we can prove that

$$\begin{aligned}
\sup_{y\in[F(x_*)]_\alpha} M(x_*,y,t) &\geq M(x_*,x_n,\frac{t}{2}) * \sup_{y\in[F(x_*)]_\alpha} M(x_n,y,t)\\
&\geq M(x_*,x_n,\frac{t}{2}) * D_M(F(x_{n-1}),F(x_*),\frac{t}{2})\\
&\geq M(x_*,x_n,\frac{t}{2}) * M(x_{n-1},x_*,\frac{t}{2}).
\end{aligned}$$

Hence $\sup_{y\in[F(x_*)]_\alpha} M(x_*,y,t) = 1$, i.e. $x_* \in [F(x_*)]_\alpha$ for each $\alpha \in [0,1]$. This shows that x_* is a fixed point of F. □

6. Conclusion

In the present paper, we have introduced the notion of uniform Hausdorff fuzzy metric on fuzzy sets in the sense of fuzzy metric spaces and discuss some interesting properties for this fuzzy metric. Also, we generalized Jardón′s result from the classical metric spaces to fuzzy metric spaces. The question of extending this uniform Hausdorff fuzzy metric to weak continuous and contractive mappings may aries in a natural way and deserves attention in a further work.

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