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Separation axioms in (L, M)-fuzzy topology (L, M)-fuzzy convexity spaces

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Abstract. In this paper, we define r-L-fuzzy closed convex sets and r-L-fuzzy closed neighbourhoods in an (L, M)-fuzzy topology (L, M)-fuzzy convexity spaces. Also, r-L-fuzzy neighbourhood separation properties r-L-FNS_i were studied where $i = \{0, 1, 2, 3, 4\}$. In addition, we also study the invariance or otherwise of these separation properties under subspace and product.

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1. Introduction

Mathematicians worked to find a mathematical expression of uncertainty in order to solve various real-life problems. Zadeh [41], Molodtsov [11] and Lee et al [7] presented the fuzzy sets, soft sets and octahedron sets as different mathematical models for this mathematical expression, which were applied in various fields of mathematics, engineering and medicine, etc. Later, some researchers (See, for example, [1, 2, 18, 22, 23, 24, 25]) introduced and studied these models.

One of the important branches of mathematics that has been accepted which it has been studied by researchers recent years is the abstract convexity theory [31], [29] which plays an important role in various branches of mathematics. It deals with set-theoretic structures which satisfies axioms similar to that usual convex sets fulfill. Here, by " usual convex sets ", we mean convex sets in real linear spaces. Also, many different mathematical research fields applied abstract convexity theory, such as topological spaces, lattices, metric spaces and graphs (See, for example, [6, 10, 28, 32, 33, 35, 40]). The concept of convex structures as a topology-like structure, it can be also treated as a special kind of spatial structures and some topology-like properties.

As we all know fuzzy mathematics has been applied in many different fields of mathematics as well and through the theory of fuzzy sets was applied fuzzy mathematics in convex structures, Rosa has worked to generalize the convex structure, where he introduced the idea of a fuzzy convex structure in [19, 20] which is called an I-convex structure. Also, Rosa studied a fuzzy topology together with a fuzzy convexity on the same underlying set X, and introduced fuzzy topology fuzzy convexity spaces and the notion of fuzzy local convexity. By framework, which proposed in [27], Li [8] presented a categorical approach to enrich (L, M)-fuzzy convex structures, Xiu et al [37] presented a degree approach to study the relationship between (L, M)-fuzzy convex structures and (L, M)-fuzzy closure systems and Wu and Li [36] introduced (L, M)-fuzzy domain finiteness, (L, M)-fuzzy restricted hull spaces and several characterizations of the category (L, M)-CS of (L, M)-fuzzy convex spaces. Recently, there has been significant research on fuzzy convex structures (See [9, 12, 13, 14, 15, 16, 26, 34, 38, 39]).

The main contributions of the present paper are to give investigations on an (L, M)-fuzzy topology (L, M)-fuzzy convexity spaces where we define an r-L-fuzzy neighbourhood separation properties $r\text{-}L\text{-}FNS_i$ with respect to (L, M)-fuzzy topology (L, M)-fuzzy convexity space where $i = \{0, 1, 2, 3, 4\}$. Also, We study their properties and discuss the relationships between these concepts.

2. Preliminaries

Throughout this paper, let X be a non-empty set, both L and M be completely distributive lattices with order reversing involution $'$ where $\bot_M$ $(\bot_L)$ and $\top_M(\top_L)$ denote the least and the greatest elements in $M(L)$ respectively, and $M_{\bot_M} = M - \{\bot_M\}(L_{\bot_L} = L - \{\bot_L\})$. An L-fuzzy subset of X is a mapping $\mu : X \longrightarrow L$ and the family L^X denoted the set of all fuzzy subsets of a given X [2]. The least and the greatest elements in L^X are denoted by $\chi_\varnothing$ and χ_X, respectively. For each $\alpha \in L$, let $\underline{\alpha}$ denote the constant L-fuzzy subset of X with the value α. The complementation of a fuzzy subset are defined as $\mu'(x) = (\mu(x))'$ for all $x \in X$, (e.g. $\mu'(x) = 1 - \mu(x)$ in the case of $L = [0, 1]$). Let $X = \prod_{i\in\Gamma} X_i$ and $\mu_i \in L^{X_i}$, then $\mu \in L^X$ denote the product of all $\mu_i \in L^{X_i}$ is defined as follows: $\mu(x) = \wedge_{i\in\Gamma}\mu_i(x_i)$ for all $x \in X$ [30].

Definition 2.1 ([4])**.** Let $\varnothing \neq Y \subseteq X$ and $\mu \in L^X$. Then the restriction of μ on Y, is denoted by $\mu|Y$. The extension of $\mu \in L^Y$ on X, denoted by μ_X, is defined by:

$$\mu_X(x) = \begin{cases} \mu(x), & \text{if } x \in Y, \\ \bot_L, & \text{if } x \in X - Y. \end{cases}$$

Definition 2.2 ([3, 17])**.** A fuzzy point x_t for $t \in L_{\bot_L}$ is an element of L^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ \bot_L, & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $P_t(X)$. Two fuzzy points x_t and y_s are distinct if $x \neq y$.

Definition 2.3 ([41]). Let $f : X \longrightarrow Y$. Then the image $f^{\rightarrow}(\mu)$ of $\mu \in L^X$ and the preimage $f^{\leftarrow}(\nu)$ of $\nu \in L^Y$ are defined by:

$$f^{\rightarrow}(\mu)(y) = \bigvee\{\mu(x) : x \in X, f(x) = y\} \text{ and } f^{\leftarrow}(\nu) = \nu \circ f, \text{ respectively.}$$

Definition 2.4 ([27, 42]). The pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex structure, where $\mathcal{C} : L^X \longrightarrow M$ satisfies the following axioms:
(LMC1) $\mathcal{C}(\chi_{\varnothing}) = \mathcal{C}(\chi_X) = \top_M$,
(LMC2) if $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty, then $\mathcal{C}(\bigwedge_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{C}(\mu_i)$,
(LMC3) if $\{\mu_i : i \in \Gamma\} \subseteq L^X$ is nonempty and totally ordered by inclusion, then $\mathcal{C}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{C}(\mu_i)$.

The mapping $\mathcal{C}$ is called an (L, M)-fuzzy convexity on X and $\mathcal{C}(\mu)$ can be regarded as the degree to which μ is an L-convex fuzzy set.

Theorem 2.5 ([27]). *Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex structure, $\varnothing \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y)$ is an (L, M)-fuzzy convex structure on Y, where*

$$(\mathcal{C}|Y)(\mu) = \bigvee\{\mathcal{C}(\nu) : \nu \in L^X, \nu|Y = \mu\},$$

for each $\mu \in L^Y$. The pair $(Y, \mathcal{C}|Y)$ is called an (L, M)-fuzzy convex sub-structure of $(X, \mathcal{C})$.

Definition 2.6 ([27]). Let $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$ be a set of (L, M)-fuzzy convex structures, X be the product of the sets X_i for $i \in \Gamma$ and $\pi_i : X \longrightarrow X_i$ be the projection for each $i \in \Gamma$. Define a mapping $\varphi : L^X \longrightarrow M$ by

$$\varphi(\mu) = \bigvee_{i\in\Gamma} \bigvee_{\pi_i^{\leftarrow}(\nu)=\mu} \mathcal{C}_i(\nu), \qquad \text{for each } \mu, \nu \in L^X.$$

Then the product convexity $\mathcal{C}$ of X is the one generated by subbase φ. The resulting (L, M)-fuzzy convex structure $(X, \mathcal{C})$ is called the product of $\{(X_i, \mathcal{C}_i) : i \in \Gamma\}$ and is denoted by $\prod_{i\in\Gamma}(X_i, \mathcal{C}_i)$.

Definition 2.7 ([5, 30]). An (L, M)-fuzzy topology on X is a map $\mathcal{T} : L^X \longrightarrow M$ with the following conditions:
(i) $\mathcal{T}(\chi_{\varnothing}) = \mathcal{T}(\chi_X) = \top_M$,
(ii) $\mathcal{T}(\mu \wedge \nu) \geq \mathcal{T}(\mu) \wedge \mathcal{T}(\nu), \quad \forall \mu, \nu \in L^X$,
(iii) $\mathcal{T}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(\mu_i), \quad \forall \mu_i \in L^X, i \in \Gamma$.
The pair $(X, \mathcal{T})$ is called an (L, M)-fuzzy topological space.

Definition 2.8 ([21]). A triple $(X, \mathcal{C}, \mathcal{T})$ consisting of a set X, an (L, M)-fuzzy convexity, and an (L, M)-fuzzy topology is called an (L, M)-fuzzy topology (L, M)-fuzzy convexity space (briefly, (L, M)-ftfcs).

Proposition 2.9 ([1, 18]). *Let $(X, \mathcal{T})$ be an (L, M)-fuzzy topological space and $A \subseteq X$. Define a mapping $\mathcal{T}_A : L^X \longrightarrow M$ by*

$$\mathcal{T}_A(\mu) = \bigvee\{\mathcal{T}(\nu) : \nu \in L^X, \nu|A = \mu\}.$$

($\bigvee$ being the supremum operation on M). Then $\mathcal{T}_A$ is an (L, M)-fuzzy topology A.

3. r-L-FNS_0, r-L-FNS_1 AND r-L-FNS_2 SPACES

Definition 3.1. Let $(X, \mathcal{C}, \mathcal{T})$ be an (L, M)-ftfcs and $\mu \in L^X$. Then μ is called:
(i) r-L-fuzzy closed convex set, if $\mathcal{T}(\mu') \geq r$ and $\mathcal{C}(\mu) \geq r$,
(ii) r-L-fuzzy closed convex neighbourhood of $x_t \in P_t(X)$, if it is an r-L-fuzzy closed convex set and an r-L-fuzzy neighbourhood of x_t.

Definition 3.2. Let $(X, \mathcal{C}, \mathcal{T})$ be an (L, M)-ftfcs. Then $(X, \mathcal{C}, \mathcal{T})$ is said to be:
(i) r-L-FNS_0 space, if for any two distinct fuzzy points there exists r-L-fuzzy closed convex neighbourhood containing one and not containing the other,
(ii) r-L-FNS_1 space if for any two distinct fuzzy points there exists r-L-fuzzy closed convex neighbourhood of each of them not containing the other,
(iii) r-L-FNS_2 space if for any two distinct fuzzy points there exist disjoint r-L-fuzzy closed convex neighbourhoods of each of them.

Theorem 3.3. *Let $(X, \mathcal{C}, \mathcal{T})$ be an r-L-FNS_i space for $i \in \{0, 1, 2\}$ and $\varnothing \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ is an r-LFS_i space.*

Proof. Let $(X, \mathcal{C}, \mathcal{T})$ be an r-L-FNS_2 space and $x_t, y_s \in P_t(Y)$ such that $x \neq y$. Then $x_t, y_s \in P_t(X)$ such that $x \neq y$. Thus there exist disjoint r-L-fuzzy closed convex neighbourhoods μ and ν for x_t and y_s in X, respectively. So $\mu|Y$ and $\nu|Y$ are disjoint r-L-fuzzy closed convex neighbourhoods of x_t and y_s in Y, respectively. Hence $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ is an r-LFS_2 space.
Similarly, we can prove the result for $i \in \{0, 1\}$. □

Theorem 3.4. *Let $(X, \mathcal{C}, \mathcal{T})$ be the product of $\{(X_i, \mathcal{C}_i, \mathcal{T}_i) : i \in \Gamma\}$. Then, $(X, \mathcal{C}, \mathcal{T})$ is an r-L-FNS_α space for $\alpha \in \{0, 1, 2\}$ if $(X_i, \mathcal{C}_i, \mathcal{T}_i)$ is an r-L-FNS_α space for each $i \in \Gamma$.*

Proof. Consider the case when $\alpha = 2$.
Let $\{(X_i, \mathcal{C}_i, \mathcal{T}_i) : i \in \Gamma\}$ be an r-L-FNS_2 space and $x_t, y_s \in P_t(X)$ such that $x \neq y$ with $X = \prod_{i \in \Gamma} X_i$ and $\pi_i : X \longrightarrow X_i$ be the projection map for each $i \in \Gamma$. Then for some $i \in \Gamma$, $(x_i)_t$ and $(y_i)_s$ are distinct fuzzy points in X_i and there exist disjoint r-L-fuzzy closed convex neighbourhoods μ_i and ν_i in X_i for $(x_i)_t$ and $(y_i)_s$, respectively. Since π_i is the projection map, $\mu = \pi_i^{\leftarrow}(\mu_i)$ and $\nu = \pi_i^{\leftarrow}(\nu_i)$ are disjoint r-L-fuzzy closed convex neighbourhoods in X of x_t and y_s respectively. Thus $(X, \mathcal{C}, \mathcal{T})$ is an r-L-FNS_2 space. Similarly, we can prove the result when $i \in \{0, 1\}$. □

Proposition 3.5. *For $r \in M_\perp$, we have*
(1) *an r-L-FNS_2 space is always r-L-FNS_1 space,*
(2) *an r-L-FNS_1 space is always r-L-FNS_0 space.*

Proof. By Definition 3.2, the proofs are is trivial. □

The next examples shows that the converse of Proposition 3.5 is not true.

Example 3.6. Let $L = M = [0, 1]$ and μ_i be fuzzy subsets of $X = \{a, b, c\}$, where $i = \{1, 2, 3, 4, 5\}$ is defined as follows:

$$\begin{array}{lll} \mu_1(a) = 1.0, & \mu_1(b) = 0.0, & \mu_1(c) = 0.0, \\ \mu_2(a) = 0.5, & \mu_2(b) = 1.0, & \mu_2(c) = 0.0, \\ \mu_3(a) = 0.5, & \mu_3(b) = 0.0, & \mu_3(c) = 0.0, \\ \mu_4(a) = 0.0, & \mu_4(b) = 0.0, & \mu_4(c) = 1.0, \\ \mu_5(a) = 1.0, & \mu_5(b) = 0.0, & \mu_5(c) = 1.0. \end{array}$$

Define an (L, M)-fuzzy topology in [5, 30] $\mathcal{T} : [0, 1]^X \longrightarrow [0, 1]$ and an (L, M)-fuzzy convexity $\mathcal{C} : [0, 1]^X \longrightarrow [0, 1]$ on X as follows:

$$\mathcal{T}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_1, \\ \frac{1}{4}, & \text{if } \nu = \underline{1} - \mu_1, \\ \frac{1}{4}, & \text{if } \nu = \mu_2, \\ \frac{1}{4}, & \text{if } \nu = \underline{1} - \mu_2, \\ \frac{1}{4}, & \text{if } \nu = \mu_3, \\ \frac{1}{4}, & \text{if } \nu = \underline{1} - \mu_3, \\ \frac{1}{2}, & \text{if } \nu = \mu_4, \\ \frac{1}{2}, & \text{if } \nu = \mu_5, \\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{C}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_1, \\ \frac{1}{4}, & \text{if } \nu = \mu_2, \\ \frac{1}{4}, & \text{if } \nu = \mu_3, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X, \mathcal{C}, \mathcal{T})$ is an r-L-FNS_0 space but it is not r-L-FNS_1 space because μ_2, μ_3 are $\frac{1}{4}$-fuzzy closed convex neighbourhood of $b_{1.0}, a_{0.5}$ and $a_{0.5} \in \mu_2$.

Example 3.7. Let $L = M = [0, 1]$ and μ_i be fuzzy subsets of $X = \{a, b, c\}$, where $i = \{1, 2, 3, 4, 5\}$ is defined as follows:

$$\begin{array}{lll} \mu_1(a) = 1.0, & \mu_1(b) = 0.0, & \mu_1(c) = 0.0, \\ \mu_2(a) = 0.0, & \mu_2(b) = 1.0, & \mu_2(c) = 0.0, \\ \mu_3(a) = 0.0, & \mu_3(b) = 0.0, & \mu_3(c) = 1.0, \\ \mu_4(a) = 0.5, & \mu_4(b) = 0.0, & \mu_4(c) = 1.0, \\ \mu_5(a) = 0.5, & \mu_5(b) = 0.0, & \mu_5(c) = 0.0. \end{array}$$

Define an (L,M)-fuzzy topology in [5, 30] $\mathcal{T}:[0,1]^X \longrightarrow [0,1]$ and an (L,M)-fuzzy convexity $\mathcal{C}:[0,1]^X \longrightarrow [0,1]$ on X as follows:

$$\mathcal{T}(\nu)=\begin{cases} 1, & \text{if } \nu\in\{\underline{0},\underline{1}\},\\ \frac{1}{7}, & \text{if } \nu=\mu_1,\\ \frac{1}{7}, & \text{if } \nu=\underline{1}-\mu_1,\\ \frac{1}{6}, & \text{if } \nu=\mu_2,\\ \frac{1}{6}, & \text{if } \nu=\underline{1}-\mu_2,\\ \frac{1}{5}, & \text{if } \nu=\mu_3,\\ \frac{1}{5}, & \text{if } \nu=\underline{1}-\mu_3,\\ \frac{1}{4}, & \text{if } \nu=\mu_4,\\ \frac{1}{4}, & \text{if } \nu=\underline{1}-\mu_4,\\ \frac{1}{3}, & \text{if } \nu=\mu_5,\\ \frac{1}{3}, & \text{if } \nu=\underline{1}-\mu_5,\\ 0, & \text{otherwise.} \end{cases} \qquad \mathcal{C}(\nu)=\begin{cases} 1, & \text{if } \nu\in\{\underline{0},\underline{1}\},\\ \frac{1}{7}, & \text{if } \nu=\mu_1,\\ \frac{1}{7}, & \text{if } \nu=\underline{1}-\mu_1,\\ \frac{1}{4}, & \text{if } \nu=\mu_4,\\ \frac{1}{4}, & \text{if } \nu=\underline{1}-\mu_4,\\ \frac{1}{3}, & \text{if } \nu=\mu_5,\\ \frac{1}{6}, & \text{if } \nu=\mu_2,\\ \frac{1}{5}, & \text{if } \nu=\mu_3,\\ 0, & \text{otherwise.} \end{cases}$$

Then $(X,\mathcal{C},\mathcal{T})$ is an r-L-FNS_1 space but it is not r-L-FNS_2 space because $\underline{1}-\mu_1$ is $\frac{1}{7}$-fuzzy closed convex neighbourhood of $b_{1.0}, c_{1.0}$.

4. Pseudo r-L-FNS_3 and r-L-FNS_3 spaces

Definition 4.1. Let $(X,\mathcal{C},\mathcal{T})$ be an (L,M)-ftfcs, μ be an r-L-fuzzy closed convex set in L^X and $x_t\in P_t(X)$ such that supports of x_t and μ are disjoint. Then $(X,\mathcal{C},\mathcal{T})$ is said to be:

(i) pseudo r-L-FNS_3 space, if there exists r-L-fuzzy closed convex neighbourhood ν of μ such that $x_t\notin\nu$,

(ii) r-L-FNS_3 space, if there exists r-L-fuzzy closed convex neighbourhoods ν of μ and λ of x_t.

Theorem 4.2. *Let $(X,\mathcal{C},\mathcal{T})$ be an r-L-FNS_3 space (resp. Pseudo r-L-FNS_3 space) and $\varnothing\neq Y\subseteq X$. Then $(Y,\mathcal{C}|Y,\mathcal{T}_Y)$ is r-LFS_3 space (resp. Pseudo r-L-FNS_3 space).*

Proof. Let $(X,\mathcal{C},\mathcal{T})$ be an r-L-FNS_3 space, $x_t\in P_t(Y)$ and μ be an r-L-fuzzy closed convex set in L^Y such that supports of x_t and μ are disjoint and $\varnothing\neq Y\subseteq X$. Then $\mu=\nu|Y$ is an r-L-fuzzy closed convex set in L^X, where ν is an r-L-fuzzy closed convex set in L^X. Since supports of x_t and μ are disjoint, we have supports of x_t and ν are disjoint in X. Thus there exists r-L-fuzzy closed convex neighbourhoods

λ_1, λ_2 of x_t and ν, respectively. So $\lambda_1|Y$ and $\lambda_2|Y$ are disjoint r-L-fuzzy closed convex neighbourhoods of x_t and μ, respectively in Y. Hence $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ is r-L-FNS_3 space.

Similarly, we can prove result Pseudo r-L-FNS_3 space. □

Theorem 4.3. *Let $(X, \mathcal{C}, \mathcal{T})$ be the product of $\{(X_i, \mathcal{C}_i, \mathcal{T}_i) : i \in \Gamma\}$. Then $(X, \mathcal{C}, \mathcal{T})$ is an r-L-FNS_3 space (resp. pseudo r-L-FNS_3 space), if $(X_i, \mathcal{C}_i, \mathcal{T}_i)$ is an r-L-FNS_3 space (resp. Pseudo r-L-FNS_3 space) for each $i \in \Gamma$.*

Proof. Let $\{(X_i, \mathcal{C}_i, \mathcal{T}_i) : i \in \Gamma\}$ be an r-L-FNS_3 space, $x_t \in P_t(X)$ and μ be an r-L-fuzzy closed convex set in L^X such that supports of x_t and μ are disjoint and $\pi_i : X \longrightarrow X_i$ is the projection map for each $i \in \Gamma$. Then

$$\mu = \pi_i^{\leftarrow}(\nu_i) \text{ where } \nu_i \text{ is } r\text{-}L\text{-fuzzy closed convex set in } L^{X_i}$$

For some $i \in \Gamma$, $(x_i)_t$ and ν_i are distinct. Since X_i is an r-L-FNS_3 space, there exists r-L-fuzzy closed convex neighbourhoods λ_i, ρ_i of $(x_i)_t$ and ν_i respectively such that $(x_i)_t \notin \rho_i$ and λ_i and ν_i are disjoint. Thus $\nu = \pi_i^{\leftarrow}(\lambda_i)$ and $\rho = \pi_i^{\leftarrow}(\rho_i)$ are disjoint r-L-fuzzy closed convex neighbourhoods of x_t and μ, respectively such that $x_t \notin \rho$, ν and μ are disjoint.

Similarly, we can prove result Pseudo r-L-FNS_3 space. □

Proposition 4.4. *For $r \in M_\perp$, an r-L-FNS_3 space is always pseudo r-L-FNS_3 space.*

Proof. By Definition 4.1, the proof is trivial. □

The next example shows that the converse of Proposition 4.4 is not true.

Example 4.5. Let $L = M = [0,1]$ and μ_i be fuzzy subsets of $X = \{a, b, c\}$, where $i = \{1, 2\}$ is defined as follows:

$$\mu_1(a) = 1.0, \quad \mu_1(b) = 0.0, \quad \mu_1(c) = 0.0,$$
$$\mu_2(a) = 0.0, \quad \mu_2(b) = 1.0, \quad \mu_2(c) = 1.0.$$

Define an (L, M)-fuzzy topology in [5, 30] $\mathcal{T} : [0,1]^X \longrightarrow [0,1]$ and an (L, M)-fuzzy convexity $\mathcal{C} : [0,1]^X \longrightarrow [0,1]$ on X as follows:

$$\mathcal{T}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_1, \\ \frac{1}{3}, & \text{if } \nu = \mu_2, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{C}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \nu = \mu_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X, \mathcal{C}, \mathcal{T})$ is Pseudo r-L-FNS_3 space but it is not r-L-FNS_3 space because the only $\frac{1}{3}$-fuzzy closed convex set is μ_1 and for $b_{1.0} \in P_t(X)$ there is not r-fuzzy closed convex neighbourhood where μ_1 is $\frac{1}{3}$-fuzzy closed convex neighbourhood of μ_1.

Note 1: An r-L-FNS_3 space and so a pseudo r-L-FNS_3 need not be an r-L-FNS_2 space.

Example 4.6. Let L, M, X and μ_i be given as Example 3.5. Define an (L, M)-fuzzy topology in [5, 30] $\mathcal{T} : [0,1]^X \longrightarrow [0,1]$ and an (L, M)-fuzzy convexity $\mathcal{C} : [0,1]^X \longrightarrow [0,1]$ on X as follows:

$$\mathcal{T}(\nu) = \mathcal{C}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_1, \\ \frac{1}{3}, & \text{if } \nu = \mu_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X, \mathcal{C}, \mathcal{T})$ is r-L-FNS_3 space but it is not r-L-FNS_2 space.

5. Semi r-L-FNS_4 and r-L-FNS_4 spaces

Definition 5.1. Let $(X, \mathcal{C}, \mathcal{T})$ be an (L, M)-ftfcs and $\mu, \nu \in L^X$ are disjoint r-L-fuzzy closed convex sets. Then $(X, \mathcal{C}, \mathcal{T})$ is said to be:

(i) semi r-L-FNS_4 space, if there exists r-L-fuzzy closed convex neighbourhood λ of μ such that λ and ν are disjoint,

(ii) r-L-FNS_4 space if there exists r-L-fuzzy closed convex neighbourhoods λ_1 of μ and λ_2 of ν such that λ_1 and ν are disjoint λ_2 and μ are disjoint.

Theorem 5.2. *Let $(X, \mathcal{C}, \mathcal{T})$ be an r-L-FNS_4 space (resp. semi r-L-FNS_4 space) and $\varnothing \neq Y \subseteq X$. Then $(Y, \mathcal{C}|Y, \mathcal{T}_Y)$ is r-LFS_4 space (resp. semi r-L-FNS_4 space).*

Proof. The proof is similar to Theorem 4.2. □

Proposition 5.3. *For $r \in M_\perp$, an r-L-FNS_4 space is always semi r-L-FNS_4 space.*

Proof. By Definition 5.1, the proof is trivial. □

The next example shows that the converse of Proposition 5.3 is not true.

Example 5.4. Let $L = M = [0,1]$ and μ_i be fuzzy subsets of $X = \{a, b, c\}$, where $i = \{1, 2, 3, 4, 5, 6, 7\}$ is defined as follows:

$$\begin{array}{lll} \mu_1(a) = 0.75, & \mu_1(b) = 1.00, & \mu_1(c) = 1.00, \\ \mu_2(a) = 1.00, & \mu_2(b) = 0.75, & \mu_2(c) = 1.00, \\ \mu_3(a) = 0.75, & \mu_3(b) = 0.75, & \mu_3(c) = 1.00, \\ \mu_4(a) = 0.50, & \mu_4(b) = 1.00, & \mu_4(c) = 1.00, \\ \mu_5(a) = 0.30, & \mu_5(b) = 0.00, & \mu_5(c) = 0.00, \\ \mu_6(a) = 0.50, & \mu_6(b) = 0.75, & \mu_6(c) = 1.00, \\ \mu_7(a) = 0.50, & \mu_7(b) = 0.50, & \mu_7(c) = 0.00. \end{array}$$

Define an (L, M)-fuzzy topology in [5, 30] $\mathcal{T} : [0,1]^X \longrightarrow [0,1]$ and an (L, M)-fuzzy convexity $\mathcal{C} : [0,1]^X \longrightarrow [0,1]$ on X as follows:

$$\mathcal{T}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if } \nu = \mu_1, \\ \frac{1}{3}, & \text{if } \nu = \mu_2, \\ \frac{1}{4}, & \text{if } \nu = \mu_3, \\ \frac{1}{5}, & \text{if } \nu = \mu_4, \\ \frac{1}{5}, & \text{if } \nu = \mu_5, \\ \frac{1}{5}, & \text{if } \nu = \mu_6, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{C}(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{5}, & \text{if } \nu \leq \mu_7, \\ 0, & \text{otherwise.} \end{cases}$$

Then

(1) μ'_1 and μ'_2 are disjoint $\frac{1}{5}$-L-fuzzy closed convex sets in X, where

$$\mu'_1(a) = 0.25, \qquad \mu'_1(b) = 0.00, \qquad \mu'_1(c) = 0.00,$$
$$\mu'_2(a) = 0.00, \qquad \mu'_2(b) = 0.25, \qquad \mu'_2(c) = 0.00.$$

Also, μ'_4 and μ'_2 are disjoint $\frac{1}{5}$-L-fuzzy closed convex sets in X, where,

$$\mu'_4(a) = 0.50, \qquad \mu'_4(b) = 0.00, \qquad \mu'_4(c) = 0.00.$$

Thus μ'_4 is $\frac{1}{5}$-L-fuzzy closed convex neighbourhood of μ'_1, because

$$\mu'_1 \leq \mu_5 \leq \mu'_4 \text{ where } \mathcal{T}(\mu_5) \geq \frac{1}{5}.$$

So $(X, \mathcal{C}, \mathcal{T})$ is semi r-L-FNS_4 but it is not r-L-FNS_4, because there is no r-L-fuzzy closed convex neighbourhood of μ'_2.

(2) $(X, \mathcal{C}, \mathcal{T})$ isn't r-L-FNS_3, because there is no r-L-fuzzy closed convex neighbourhood containing μ'_3 and disjoint with c_t, where $0 < t \leq 1$ and

$$\mu'_3(a) = 0.25, \qquad \mu'_3(b) = 0.25, \qquad \mu'_3(c) = 0.00.$$

Where μ'_3 is $\frac{1}{5}$-L-fuzzy closed convex sets in X.

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