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On the topological structure of spectrum of intuitionistic L-fuzzy prime submodules

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Abstract. Let R be a commutative ring with identity and let M be an R-module. We topologize $IF_LSpec(M)$, the collection of all intuitionistic L-fuzzy prime submodules of M, analogous to that for $IFSpec(R)$, the spectrum of intuitionistic fuzzy prime ideals of R, and investigate the properties of this topological space. In particular, we will study the relationship between $IF_LSpec(M)$ and $IF_LSpec(R/Ann(M))$ and obtain some results. Also, we investigate the irreducible elements in this space.

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1. Introduction

Let R be a commutative ring with identity and M be a unitary R-module. The prime spectrum $Spec(R)$ and the topological space obtained by introducing Zariski topology on the set of prime ideals of a commutative ring with identity play an important role in the field of commutative algebra, algebraic geometry and lattice theory. Also, recently the notion of prime submodules and Zariski topology on $Spec(M)$, the set of all prime submodules of a module M over a commutative ring R with identity, has been studied by many authors (for example see [1, 2, 3, 4]).

Atassanov and Stoeva [5, 6, 7] introduced the notion of Intuitionistic Fuzzy Sets (IFS) and Intuitionistic L-Fuzzy Sets (ILFS), as the generalization of both Fuzzy sets and L-fuzzy sets given by Zadeh [8] and Goguen [9] respectively. The development of Algebra in fuzzy setting was very much evident in the book of Kandasamy [10], Mordeson and Malik [11]. Davvaz, Dudek and Jun in [12], Rahman, Saikia in [13], Isaac, John in [14], Basnet in [15] and Sharma in [16, 17, 18, 19] studied some aspects of intuitionistic fuzzy submodules.

In the last few year a considerable amount of work has been done on intuitionistic fuzzy ideals in general and intuitionistic fuzzy prime ideals in particular, and some interesting topological properties of the spectrum of intuitionistic fuzzy prime ideals of a ring have been obtained (See [15, 20, 21, 22, 23, 24, 25]).

Let M be an R-module. By $N \leq M$ we mean that N is a submodule of M. For any $N \leq M$, we denote the annihilator of M/N by $M : N$, i.e., $(N : M) = \{r \in R | rM \subseteq N\}$. In particular, $M : \{\theta\}$ is called the annihilator of M and is denoted by $Ann(M)$, that is $Ann(M) = \{r \in R | rM = \theta\}$. A proper submodule N of M is called a prime submodule of M if $rm \in N$ for some $r \in R, m \in M$ implies that $m \in N$ or $r \in (N : M)$. The set of all prime submodules of M is called the prime spectrum of M or, simply the spectrum of M and is denoted by $Spec(M)$.

The authors in [17] introduced and studied the notion of intuitionistic L-fuzzy prime submodules of a module M over a commutative ring R with identity, where L is a complete lattice. The set of all intuitionistic L-fuzzy prime submodules of M has been called the intuitionistic L-fuzzy spectrum of M and is denoted by $IF_LSpec(M)$. In [25] authors defined a topology on $IFSpec(R)$, the spectrum of intuitionistic fuzzy prime ideals of R and studied some properties of this topology. In this paper analogous to that for $IFSpec(R)$, we investigate the Zariski topology on $IF_LSpec(M)$, the set of all intuitionistic L-fuzzy submodules of M and show that for L-top modules Zariski topology on $IF_LSpec(M)$ exists. The paper provides the suitable tools to define and study the properties of Zarisky topology of intuitionistic L-fuzzy prime submodules. And hence it can be considered as an introduction to intuitionistic fuzzy spectral theory.

2. Preliminaries

Throughout this paper R is a commutative ring with identity, M a unitary R-module and L stand for a complete lattice with least element 0 and greatest element 1. θ denotes the zero element of M. An element $\alpha \in L, 1 \neq \alpha$, is called a prime element in L if for all $a; b \in L$ if $a \wedge b \leq \alpha$ implies $a \leq \alpha$ or $b \leq \alpha$ (See [26]).

Definition 2.1 ([22, 27])**.** Let $(L, \leq)$ be a complete lattice with an evaluative order reversing operation $N : L \to L$. Let X is a non-empty set. An *L-fuzzy set A in X* is defined as an object of the form $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, where $\mu_A : X \to L$ and $\nu_A : X \to L$ define the degree of membership and the degree of non membership for every $x \in X$ satisfying $\mu_A(x) \leq N(\nu_A(x))$.

We also denote an intuitionistic L-fuzzy set by simply $ILFS$ and the set of all $ILFS^{'}s$ on X by $ILFS(X)$.

Remark 2.2. When $\mu_A(x) = N(\nu_A(x))$ for all $x \in X$, then A is called an *L-fuzzy set*. We use the notion $A = (\mu_A, \nu_A)$ to denote the intuitionistic L-fuzzy set $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$.

For $A,\ B \in ILFS(X)$ we say $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

If $f : X \to Y$ is a mapping $A \in ILFS(X)$ and $B \in ILFS(Y)$. Then $f(A) \in ILFS(Y)$ and $f^{-1}(B) \in ILFS(X)$ are defined as follows: for all $y \in Y$ and $x \in X$,

$$f(A)(y) = \begin{cases} (Sup\{\mu_A(x) : x \in f^{-1}(y)\}, Inf\{\nu_A(x) : x \in f^{-1}(y)\}) & \text{if } f^{-1}(y) \neq \phi \\ (0,1) & \text{otherwise} \end{cases}$$

and

$$f^{-1}(B)(x) = (\mu_B(f(x)), \nu_B(f(x))).$$

Let $f : X \to Y$ be a mapping and $A \in ILFS(X)$. Then A is said to be *f-invariant*, if $f(x_1) = f(x_2)$ implies $\mu_A(x_1) = \mu_A(x_2)$ and $\nu_A(x_1) = \nu_A(x_2)$.

Let $Y \subseteq X$ and $\alpha, \beta \in L\backslash\{0\}$ with $\alpha \leq N(\beta)$. Define $(\alpha,\beta)_Y \in ILFS(X)$ by:

$$(\alpha,\beta)_Y(x) = \begin{cases} (\alpha,\beta) & \text{if } x \in Y \\ (0,1) & \text{otherwise.} \end{cases}$$

In special case when $Y = \{x\}$, we denote $(\alpha,\beta)_{\{x\}}$ by $x_{(\alpha,\beta)}$ and it is called an *intuitionistic L-fuzzy point* (ILFP) of X.

For $A \in ILFS(X)$ and $\alpha, \beta \in L$ with $\alpha \leq N(\beta)$, define $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$. Then $A_{(\alpha,\beta)}$ is called the *(α,β)-cut set* of A. In particular, we denote $A_{(1,0)}$ by A_*. Of course, $A_* = \{x \in X : \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}$. The *support* of an $ILFS$ A is denoted by A^* and is defined as $A^* = \{x \in X : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}$. The *image* of A is denoted by $A(X)$. Also by [**?**], we can write $A = \bigcup_{(\alpha,\beta)\in A(X)} x_{(\alpha,\beta)}$.

Definition 2.3 ([17])**.** Let $Y \subseteq X$. Then the *intuitionistic L-fuzzy characteristic function* $\chi_Y = (\mu_{\chi_Y}, \nu_{\chi_Y})$ on Y is defined as:

$$\mu_{\chi_Y}(y) = \begin{cases} 1 & \text{if } y \in Y \\ 0 & \text{otherwise} \end{cases}, \quad \nu_{\chi_Y}(y) = \begin{cases} 0 & \text{if } y \in Y \\ 1 & \text{otherwise.} \end{cases}$$

The following are two very basic definitions given in [22, 24].

Definition 2.4 ([22])**.** Let $A \in ILFS(R)$. Then A is called an *intuitionistic L-fuzzy ideal* ($ILFI$) of R, if the following conditions are satisfied: for all $x, y \in R$,
(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,
(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$,
(iii) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$,
(iv)$\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Definition 2.5 ([27, 17])**.** Let $A \in ILFS(M)$. Then A is called an *intuitionistic L-fuzzy module* ($ILFM$) of M, if the following conditions are satisfied: for all $x, y \in M, r \in R$,
(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,
(ii) $\mu_A(rx) \geq \mu_A(x)$,

(iii) $\mu_A(\theta) = 1$,
(iv) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$,
(v)$\nu_A(rx) \leq \nu_A(x)$,
(vi)$\nu_A(\theta) = 0$.

Let $IF_L(M)$ denote the set of all intuitionistic L-fuzzy R-modules of M and $ILFI(R)$ denote the set of all intuitionistic L-fuzzy ideals of R. We note that when $R = M$, then $A \in IF_L(M)$ if and only if $\mu_A(\theta) = 1, \nu_A(\theta) = 0$ and $A \in ILFI(R)$.

The following lemma can be found in [20, 22]. It gives the basic operations between intuitionistic L-fuzzy ideals and intuitionistic L-fuzzy modules where L is a complete lattice satisfying the infinite distributive law.

Lemma 2.6. *Let $C \in ILFI(R), A, B \in IF_L(M)$ and let L be a complete lattice satisfying the infinite distributive law.*
(1) *$CB \subseteq A$ if and only if $C \circ B \subseteq A$.*
(2) *If $r_{(s,t)} \in ILFS(R)$ and $x_{(p,q)} \in ILFS(M)$ are $ILFPs$, then*

$$r_{(s,t)} \circ x_{(p,q)} = (rx)_{(s \wedge p, t \vee q)}.$$

(3) *If $\mu_C(0) = 1, \nu_C(0) = 0$, then $CA \in IF_L(M)$.*
(4) *If $r_{(s,t)} \in ILFS(R)$ is an $ILFP$, then for all $w \in M$,*

$$\mu_{r_{(s,t)} \circ B}(w) = \begin{cases} Sup[s \wedge \mu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 0 & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

and

$$\nu_{r_{(s,t)} \circ B}(w) = \begin{cases} Inf[t \vee \nu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 1 & \text{if } w \text{ is not expressible as } w = rx. \end{cases}$$

Definition 2.7 ([20, 25])**.** $A \in ILFI(R)$ is called an *intuitionistic L-fuzzy prime ideal* of R, if $A \neq \chi_{\{0\}}, \chi_R$ and for any $B, C \in ILFI(R)$ such that $BC \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$.

$IF_LSpec(R)$ denotes the set of all intuitionistic L-fuzzy prime ideals of R.

Theorem 2.8 ([25])**.** *Let $f : R \to R^{'}$ be a homomorphism from R onto $R^{'}$, A be any f-invariant intuitionistic fuzzy prime ideal of R and $A^{'}$ be any intuitionistic fuzzy prime ideal of $R^{'}$. Then $f(A)$ and $f^{-1}(A^{'})$ are intuitionistic fuzzy prime ideals of $R^{'}$ and R, respectively.*

Definition 2.9 ([17])**.** A non-constant intuitionistic L-fuzzy submodule A of M is said to be *prime*, if for $C \in IFLI(R)$ and $D \in IF_L(M)$ such that $C \cdot D \subseteq A$ then either $D \subseteq A$ or $C \subseteq (A : \chi_M)$.

In the sequel $IF_LSpec(M)$ denotes the set of all intuitionistic L-fuzzy prime submodules of M.

Definition 2.10 ([15])**.** Let $\{A_i : i \in J, |J| > 1\}$ be a family of intuitionistic L-fuzzy submodules of M. Then $\sum_{i \in J} A_i = (\mu_{\sum_{i \in J} A_i}, \nu_{\sum_{i \in J} A_i}$ is defined as: for all $x \in M$,

$$\mu_{\sum_{i \in J} A_i}(x) = \vee\{\wedge_{i \in J} \mu_{A_i}(x_i) : x_i \in M, i \in J, \sum_{i \in J} x_i = x\},$$

$$\nu_{\sum_{i\in J} A_i}(x) = \wedge\{\vee_{i\in J}\nu_{A_i}(x_i) : x_i \in M, i \in J, \sum_{i\in J} x_i = x\},$$

where in $\sum_{i\in J} x_i = x$, at most finitely many $x_i^{'}s$ are not zero. $\sum_{i\in J} A_i$ is called the *sum* of $A_i^{'}s$.

It is easy to see that $\sum_{i\in J} A_i \in IF_L(M)$.

Definition 2.11 ([22, 24])**.** For $A,\ B \in ILFS(M)$ and $C \in ILFS(R)$, define the *residual quotients* $(A : B)$ and $(A : C)$ as follows:

$$(A : B) = \bigcup\{D : D \in ILFS(R) \text{ such that } D \cdot B \subseteq A\},$$

$$(A : C) = \bigcup\{E : E \in ILFS(M) \text{ such that } C \cdot E \subseteq A\}.$$

In [24], it was proved that if $A \in IF_L(M)$, $B \in ILFS(M)$, $C \in ILFS(R)$, then

$$(A : B) = \bigcup\{D : D \in ILFI(R) \text{ such that } D \cdot B \subseteq A\}$$

and

$$(A : C) = \bigcup\{E : E \in IF_L(M) \text{ such that } C \cdot E \subseteq A\}.$$

Theorem 2.12 ([22, 24])**.** *If $A \in IF_L(M)$, $B \in ILFS(M)$, $C \in ILFS(R)$, then $(A : B) \in ILFI(R)$ and $(A : C) \in IF_L(M)$.*

Theorem 2.13 ([22, 24])**.** *For $A, B \in ILFS(M)$ and $C \in ILFS(R)$, we have*
(1) $(A : B) \cdot B \subseteq A$,
(2) $C \cdot (A : C) \subseteq A$,
(3) $C \cdot B \subseteq A \Leftrightarrow C \subseteq (A : B) \Leftrightarrow B \subseteq (A : C)$.

Theorem 2.14 ([17])**.** (1) *Let N be a prime submodule of M and α a prime element in L. If A is an ILFS of M defined by: for all $x \in M$,*

$$\mu_A(x) = \begin{cases} 1 & \textit{if } x \in N \\ \alpha & \textit{if otherwise} \end{cases}, \quad \nu_A(x) = \begin{cases} 0 & \textit{if } y \in N \\ \alpha^{'} & \textit{otherwise.} \end{cases}$$

where $\alpha^{'}$ is complement of α in L, then A is an intuitionistic L-fuzzy prime submodule of M.

(2) *Conversely, any intuitionistic L-fuzzy prime submodule can be obtained as in* (1).

Corollary 2.15 ([24])**.** *If $A \in IF_LSpec(M)$, then $(A : \chi_M) \in IF_LSpec(R)$.*

3. Topologies on $IF_LSpec(M)$

We denote the collection of all intuitionistic L-fuzzy prime submodules of a R-module M by $IF_LSpec(M)$. This collection is called the intuitionistic L-fuzzy prime spectrum of M. In this section, we introduce a topological structure on $IF_LSpec(M)$ and the resulting topology is called the Zariski topology on $IF_LSpec(M)$. We study many properties of this topological space.

Throughout this paper unless stated otherwise all rings are commutative ring with identity and all modules are unitary R-module. Let $X = IF_LSpec(M)$ and for any $A \in ILFS(M)$, denote the set $V(A) = \{P \in X : A \subseteq P\}$.

Theorem 3.1. *For any family* $\{A_i \in IF_L(M) : i \in J\}$, *the following statements are satisfied:*
(1) $V(\chi_{\{\theta\}}) = X$ *and* $V(\chi_M) = \varnothing$,
(2) $\bigcap_{i\in J} V(A_i) = V(\sum_{i\in J} A_i)$,
(3) $V(A_1) \cup V(A_2) \subseteq V(A_1 \cap A_2)$ *for* $A_1, A_2 \in IF_L(M)$.

Proof. (1) Obvious.
(2) Suppose $P \in \bigcap_{i\in J} V(A_i)$. Then $P \in V(A_i)\forall i \in J$. Thus $A_i \subseteq P, \forall i \in J$. On the other hand, we have

$$\begin{aligned}\mu_{\sum_{i\in J} A_i}(x) &= \vee\{\wedge_{i\in J}\mu_{A_i}(x_i) : \sum_{i\in J} x_i = x, x_i \in M, \forall i \in J\}\\ &\leq \vee\{\wedge_{i\in J}\mu_P(x_i) : \sum_{i\in J} x_i = x, x_i \in M, \forall i \in J\}\\ &= \mu_P(x).\end{aligned}$$

Thus $\mu_{\sum_{i\in J} A_i}(x) \leq \mu_P(x)$. Similarly, we can show that the inequality holds:

$$\nu_{\sum_{i\in J} A_i}(x) \geq \mu_P(x)\ \forall x \in M.$$

So $\sum_{i\in J} A_i \subseteq P$. This implies that $P \in V(\sum_{i\in J} A_i)$. Hence we get

$$\bigcap_{i\in J} V(A_i) \subseteq V(\sum_{i\in J} A_i). \tag{3.1}$$

For the converse, let $P \in V(\sum_{i\in J} A_i)$. Then clearly, $\sum_{i\in J} A_i \subseteq P$. Thus $A_i \subseteq \sum_{i\in J} A_i\ \forall i \in J$, i.e., $A_i \subseteq P\ \forall i \in J$. So $P \in V(A_i)\ \forall i \in J$. Hence $P \in \bigcap_{i\in J} V(A_i)$ implies that

$$V(\sum_{i\in J} A_i) \subseteq \bigcap_{i\in J} V(A_i). \tag{3.2}$$

(2) The proof is immediate from (3.1) and (3.2).
(3) Suppose that $A_1, A_2 \in IF_L(M)$ and $P \in V(A_1) \cup V(A_2)$. Then $A_1 \subseteq P$ or $A_2 \subseteq P$. Thus $A_1 \cap A_2 \subseteq P$. So $P \in V(A_1 \cap A_2)$. Hence $V(A_1) \cup V(A_2) \subseteq V(A_1 \cap A_2)$. This complete the proof. □

For any $B \in ILFS(M)$, $< B >$ denote the intuitionistic L-fuzzy submodule of M generated by B, it is the smallest intuitionistic L-fuzzy submodule of M containing B. Infact $< B >= \cap\{C \in IF_L(M) : B \subseteq C\}$.

For any $A \in IF_L(M)$, $X(A) = \{P \in X : (A : \chi_M) \subseteq (P : \chi_M)\}$ and if $B \in ILFS(M)$, by $X(B)$ we mean $X(< B >)$. Then we have the next result.

Proposition 3.2. *Let* $\{A_i \in IF_L(M) : i \in J\}$. *Then the followings hold:*
(1) $X(\chi_M) = \varnothing$ *and* $X(\chi_{\{\theta\}}) = X$,
(2) $X(A) = X(< A >)$, *for every* $A \in IF_L(M)$,
(3) $\bigcap_{i\in J} X(A_i) = X(\sum_{i\in J}(A_i : \chi_M)$,
(4) $X(A) \cup X(B) = X(A \cap B)$, *for* $A, B \in IF_L(M)$.

Proof. (1) Obvious.
(2) It is an immediate consequences of definition of $< A >$.

(3) Suppose that $P \in \bigcup_{i\in J} X(A_i)$. Then $(A_i : \chi_M) \subseteq (P : \chi_M), \forall i \in J$. Thus for all $i \in J$, we have

$$\begin{aligned}(A_i : \chi_M).\chi_M &\subseteq (P : \chi_M).\chi_M \subseteq P \\ &\Rightarrow \textstyle\sum_{i\in J}(A_i : \chi_M).\chi_M \subseteq P \\ &\Rightarrow (\textstyle\sum_{i\in J}(A_i : \chi_M).\chi_M) : \chi_M) \\ &\subseteq (P : \chi_M).\end{aligned}$$

So $P \in X(\sum_{i\in J}(A_i : \chi_M).\chi_M)$. Hence we have

$$\bigcup_{i\in J} X(A_i) \subseteq (\sum_{i\in J}(A_i : \chi_M).\chi_M). \tag{3.3}$$

Conversely, suppose that $P \in X(\sum_{i\in J}(A_i : \chi_M).\chi_M)$. Then we get

$$((\sum_{i\in J}(A_i : \chi_M) : \chi_M) \subseteq (P : \chi_M).$$

Clearly, we have $(((A_i : \chi_M).\chi_M) : \chi_M) = (A_i : \chi_M)\ \forall i \in J$. Also for each $i \in J$, we obtained that $(((A_i : \chi_M).\chi_M) : \chi_M) \subseteq ((\sum_{i\in J}(A_i : \chi_M).\chi_M) : \chi_M) \subseteq (P : \chi_M)$. Thus for each $i \in J$, it conclude that $(A_i : \chi_M) \subseteq (P : \chi_M)$. So for each $i \in J$, we have $P \in X(A_i) \Rightarrow P \in \bigcap_{i\in J} X(A_i)$. Hence we get

$$X(\sum_{i\in J}(A_i : \chi_M)) \subseteq \bigcap_{i\in J} X(A_i). \tag{3.4}$$

Therefore by (3.3) and (3.4), the result hold.

(4) Suppose that $A, B \in IF_L(M)$ and $P \in X(A) \cup X(B)$. Then $P \in X(A)$ or $P \in X(B)$. Without loss of generality, suppose that $P \in X(A)$. Then we have

$(A : \chi_M) \subseteq (P : \chi_M) \Rightarrow ((A \cap B) : \chi_M) \subseteq (A : \chi_M) \subseteq (P : \chi_M) \Rightarrow P \in X(A\cap B)$. Thus we get

$$X(A) \cup X(B) \subseteq X(A \cap B). \tag{3.5}$$

For converse, suppose that $P \in X(A \cap B)$. Then $((A \cap B) : \chi_M) \subseteq (P : \chi_M)$. But we have $((A \cap B) : \chi_M) = (A : \chi_M) \cap (B : \chi_M)$. This implies that

$$(A : \chi_M)(B : \chi_M) \subseteq (A : \chi_M) \cap (B : \chi_M).$$

Thus $(A : \chi_M)(B : \chi_M) \subseteq (P : \chi_M)$. Since $(P : \chi_M)$ is an intuitionistic L-fuzzy prime ideal of R, $(A : \chi_M) \subseteq (P : \chi_M)$ or $(B : \chi_M) \subseteq (P : \chi_M)$. So $P \in X(A)$ or $P \in X(B)$, i.e.,. $P \in X(A) \cup X(B)$. Hence we get

$$X(A \cap B) \subseteq X(A) \cup X(B). \tag{3.6}$$

Therefore by (3.5) and (3.6), the result hold. □

Now, we put $\mathcal{C}^*(M) = \{V(A)|A \in IF_L(M)\}$, $\mathcal{C}'(M) = \{V(C.\chi_M)|C \in ILFI(R)\}$, $\mathcal{C}(M) = \{X(A)|A \in IF_L(M)\}$.

We consider the topologies of X induced, respectively by these three sets. From Proposition 3.1, we can easily see that there exists a topology τ^* (say), on X having $\mathcal{C}^*(M)$ as the collection of all closed sets if and only if $\mathcal{C}^*(M)$ is closed under finite union.

In this case, we call the topology τ^* the *quasi-Zariski topology* on X. As in [4], a module M is called an *L-top module*, if the $\mathcal{C}^*(M)$ induces the topology τ^* on X. In contrast with $\mathcal{C}^*(M)$, $\mathcal{C}^{'}(M)$, always induces the topology $\tau^{'}$ on X, since $V(C_1.\chi_M) \cup V(C_2.\chi_M) = V((C_1.C_2).\chi_M)$. Also, $\mathcal{C}^{'}(M)$ is closed under finite union. Clearly, $\tau^{'}$ is coarser than the quasi-Zariski topology τ^*, when M is a L-top module.

For any R-module M and $A_1, A_2 \in IF_L(M)$, we have the next result.

Proposition 3.3. *If $(A_1 : \chi_M) = (A_2 : \chi_M)$, then $X(A_1) = X(A_2)$. Moreover, the converse is true if both A_1, A_2 are intuitionistic L-fuzzy prime submodules of M.*

Proof. First suppose that $(A_1 : \chi_M) = (A_2 : \chi_M)$ and $B \in X(A_1)$. Then $(A_1 : \chi_M) \subseteq (B : \chi_M)$. Thus $(A_1 : \chi_M) \subseteq (B : \chi_M)$, i.e., $B \in X(A_2)$. So $X(A_1) \subseteq X(A_2)$. Similarly, we obtain that $X(A_2) \subseteq X(A_1)$. Hence $X(A_1) = X(A_2)$.

For the converse part, suppose that A_1, A_2 are intuitionistic L-fuzzy prime submodules of M and $X(A_1) = X(A_2)$. Then $A_1 \subseteq X(A_1) = X(A_2)$. Thus we have

$$(A_2 : \chi_M) \subseteq (A_1 : \chi_M) \tag{3.7}$$

and $A_2 \subseteq X(A_2) = X(A_1)$ implies that

$$(A_1 : \chi_M) \subseteq (A_2 : \chi_M). \tag{3.8}$$

So by (3.7) and (3.8), we obtain that $(A_1 : \chi_M) = (A_2 : \chi_M)$. □

For $p \in IF_LSpec(R)$, we denote the set $X_p = \{A \in X : (A : \chi_M) = p\}$. Then we have the following results.

Proposition 3.4. (1) $X(A) = \bigcup\{X_p : p \in X(A : \chi_M)\}$ *for* $A \in IF_L(M)$.

(2) $X(C.\chi_M) = V(C.\chi_M)$ *for every* $C \in ILFI(R)$. *Moreover, if* $A \in IF_L(M)$, *then*

$$X(A) = X((A : \chi_M).\chi_M) = V((A : \chi_M).\chi_M).$$

Proof. (1) Let $B \in X(A)$. Then $(A : \chi_M) \subseteq (B : \chi_M) = p$ (say) and thus we have

$$p \in X(A : \chi_M).$$

Also, $(B : \chi_M) = p$ implies $B \in X_p$. So $B \in \bigcup\{X_p : p \in X(A : \chi_M)\}$. Hence

$$X(A) \subseteq \bigcup\{X_p : p \in X(A : \chi_M)\}. \tag{3.9}$$

Conversely, let $B \in \bigcup\{X_p : p \in X(A : \chi_M)\}$. Then there exist some $p \in IF_LSpec(R)$ such that $(A : \chi_M) \subseteq p$. Also, $B \in X_p$ implies $(B : \chi_M) = p$. Thus we have $(A : \chi_M) \subseteq (B : \chi_M)$ implies $B \in X(A)$. So we get

$$\bigcup\{X_p : p \in X(A : \chi_M)\} \subseteq X(A). \tag{3.10}$$

Hence by (3.9) and (3.10), (1) holds.

(2) Let $P \in V(C.\chi_M)$. Then we have

$C.\chi_M \subseteq P \Rightarrow (C.\chi_M : \chi_M) \subseteq (P : \chi_M) \Rightarrow P \in X(C.\chi_M)$.

Thus we get

$$V(C.\chi_M) \subseteq X(C.\chi_M). \tag{3.11}$$

Suppose that $P \in X(C.\chi_M)$. Then $(C.\chi_M : \chi_M) \subseteq (P : \chi_M)$. Obviously, $C \subseteq (C.\chi_M : \chi_M)$. Thus $C \in (P : \chi_M) \Rightarrow C.\chi_M \subseteq P$, i.e., $P \in V(C.\chi_M)$. So we have

$$X(C.\chi_M) \subseteq V(C.\chi_M). \tag{3.12}$$

Hence from (3.11) and (3.12), we have

$$X(C.\chi_M) = V(C.\chi_M). \tag{3.13}$$

Also, by the previous discussion immediately. Replacing C by $(A : \chi_M)$ in (3.13) we obtain that

$$X((A : \chi_M).\chi_M) = V((A : \chi_M).\chi_M). \tag{3.14}$$

Now for $P \in X(A) \Rightarrow (A : \chi_M) \subseteq (P : \chi_M)$. Then $(A : \chi_M).\chi_M \subseteq A$. Thus $(((A : \chi_M).\chi_M) : \chi_M) \subseteq (A : \chi_M) \subseteq (P : \chi_M) \Rightarrow P \in X((A : \chi_M).\chi_M)$. So

$$X(A) \subseteq X((A : \chi_M).\chi_M). \tag{3.15}$$

Further suppose that $P \in V((A : \chi_M).\chi_M) \Rightarrow (A : \chi_M).\chi_M \subseteq P$. Then we have

$$(A : \chi_M) \subseteq (P : \chi_M) \ \Rightarrow P \in X(A).$$

So we get

$$V((A : \chi_M).\chi_M) \subseteq X(A). \tag{3.16}$$

Hence from (3.15) and (3.16), we obtain that

$$X(A) = V((A : \chi_M).\chi_M). \tag{3.17}$$

Therefore from (3.14) and (3.17), (2) holds. □

Note that from Proposition 3.4, we obtain that $\mathcal{C}(M) = \mathcal{C}^{'}(M) \subseteq \mathcal{C}^{*}(M)$.

Example 3.5. (1) Consider the ring of integers $M = Z$ as Z-module and let L be a complemented lattice. Suppose that $p \in Z$ is prime. Then for every prime element $\alpha \in L$ with complement $\alpha^{'}$, define $P_{\alpha,\alpha'} \in ILFS(M)$ as

$$\mu_{P_{\alpha,\alpha'}}(x) = \begin{cases} 1 & \text{if } x \in <p> \\ \alpha & \text{if otherwise} \end{cases}, \quad \nu_{P_{\alpha,\alpha'}}(x) = \begin{cases} 0 & \text{if } x \in <p> \\ \alpha^{'} & \text{otherwise,} \end{cases}$$

for all $x \in M$. Then by Theorem 2.14, $P_{\alpha,\alpha'}$ is an intuitionistic L-fuzzy prime submodule of M. Thus we have

$IF_L Spec(M)$
$= \{P_{\alpha,\alpha'},$ where $\alpha \in L$ is a prime element and p is prime element of $Z\}$,

while for $L = [0, 1]$, we get

$IF - Spec(M)$
$= \{P_{\alpha,\beta},$ where $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ and p is prime element of $Z\}$.

(2) Consider $M = \mathbf{R}[x]$ as $\mathbf{R}[x]$-module, where $\mathbf{R}$ is the field of real numbers. Let L be a complemented lattice and $\alpha \in L$ be a prime element with $\alpha^{'}$ its complement in L. Then for every $p \in \mathbf{R}[x]$, define the ILFS $P_{\alpha,\alpha'} \in \mathbf{R}[x]$ by:

$$\mu_{P_{\alpha,\alpha'}}(x) = \begin{cases} 1 & \text{if } x \in <p> \\ \alpha & \text{if otherwise} \end{cases}, \quad \nu_{P_{\alpha,\alpha'}}(x) = \begin{cases} 0 & \text{if } x \in <p> \\ \alpha^{'} & \text{otherwise,} \end{cases}$$

for all $x \in M$. Then by Theorem 2.14, $P_{\alpha,\alpha'}$ is an intuitionistic L-fuzzy prime submodule of M if and only if p is an irreducible. Moreover, if $L = [0,1]$, then

$IF - Spec(M)$
$= \{P_{\alpha,\beta},$ where $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$ and p is an irreducible element of $\mathbf{R}[x]\}$.

(3) Suppose M is an arbitrary R-module and P is a prime submodule of M. Let L is a complemented lattice. For every prime element $\alpha \in L$ with complement $\alpha^{'}$. Define $P_{\alpha,\alpha'} \in ILFS(M)$ as:

$$\mu_{P_{\alpha,\alpha'}}(x) = \begin{cases} 1 & \text{if } x \in< p > \\ \alpha & \text{if otherwise} \end{cases}, \quad \nu_{P_{\alpha,\alpha'}}(x) = \begin{cases} 0 & \text{if } x \in< p > \\ \alpha^{'} & \text{otherwise,} \end{cases}$$

for all $x \in M$. Then by Theorem 2.14, $P_{\alpha,\alpha'}$ is an intuitionistic L-fuzzy prime submodule of M. Thus we have

$$IF_L Spec(M) = \{P_{\alpha,\alpha'}, \text{ where } \alpha \in L \text{ is a prime element of } L\}.$$

(4) Suppose that $L = \{0, a, b, 1\}$ is a lattice which is not a chain, that is a and b are not comparable. Then $IF_L Spec(M) = \varnothing$ for every R-module M, since L has not any prime element. This example shows that $IF_L Spec(M) = \varnothing$, but $Spec(M)$ may be non-empty.

4. A base for the Zariski topology on $IF_L Spec(M)$

Proposition 4.1 ([25]). *If f is a homomorphism from R onto $R^{'}$, then*

$$f(x_{(\alpha,\beta)}) = (f(x))_{(\alpha,\beta)}, \forall x \in R$$

and for all $\alpha, \beta \in L\backslash\{0\}$ with $\alpha \leq N(\beta)$.

Corollary 4.2. *Let $x \in R$. Then for each ideal I of R and for each $\alpha, \beta \in L\backslash\{0\}$ with $\alpha \leq N(\beta)$; $\bar{x}_{(\alpha,\beta)} = \overline{(x_{(\alpha,\beta)})}$, where $\bar{x}_{(\alpha,\beta)}$ is an intuitionistic L-fuzzy point of R/I.*

For any R-module M, we consider the set $\mathbb{B} = \{D(x_{(\alpha,\beta)}{\cdot}\chi_M) | x \in R, \alpha, \beta \in L\backslash\{0\}$ with $\alpha \leq N(\beta)\}$, such that $D(x_{(\alpha,\beta)}{\cdot}\chi_M) = X\backslash X(x_{(\alpha,\beta)}{\cdot}\chi_M)$. We claim that if the lattice L is a chain then $\mathbb{B}$ form a base for Zariski topology on X.

We consider the following cases:
(i) if $x = 0, D(0_{(\alpha,\beta)}{\cdot}\chi_M) = X\backslash X((0_{(\alpha,\beta)}{\cdot}\chi_M)) = X\backslash X(\chi_{\{0\}}) = \varnothing$,
(ii) if $x = 1, D(1_{(\alpha,\beta)}{\cdot}\chi_M) = X\backslash X((1_{(\alpha,\beta)}{\cdot}\chi_M)) = X\backslash X(\chi_M) = X$.

In the sequel, for $C \in ILFI(R)$, we set $E(C) = X\backslash X(C)$.

Proposition 4.3. *If $\psi : X \to \bar{X}$ is a natural map, then*
(1) $\psi^{-1}(E(\bar{x}_{(\alpha,\beta)})) = D(x_{(\alpha,\beta)}{\cdot}\chi_M)$,
(2) $\psi(D(x_{(\alpha,\beta)}{\cdot}\chi_M)) \subseteq E((\bar{x}_{(\alpha,\beta)}))$. *Moreover if ψ is surjective, then the equality holds.*

Proof. (1) We have

$$\begin{aligned}\psi^{-1}(E(\bar{x}_{(\alpha,\beta)})) &= \psi^{-1}(\bar{X}\backslash X(\bar{x}_{(\alpha,\beta)})) = X\backslash\psi^{-1}(X(\bar{x}_{(\alpha,\beta)})) \\ &= X\backslash X(x_{(\alpha,\beta)}{\cdot}\chi_M) = D(x_{(\alpha,\beta)}{\cdot}\chi_M).\end{aligned}$$

(2) We have $\psi(\psi^{-1}(E(\bar{x}_{(\alpha,\beta)}))) = \psi(D(x_{(\alpha,\beta)}.\chi_M))$ and $\psi(\psi^{-1}(E(\bar{x}_{(\alpha,\beta)}))) \subseteq E(\bar{x}_{(\alpha,\beta)})$. Then $\psi(D(x_{(\alpha,\beta)}.\chi_M)) \subseteq E(\bar{x}_{(\alpha,\beta)})$.

Suppose ψ is surjective. Then we obtain that $\psi(\psi^{-1}(E(\bar{x}_{(\alpha,\beta)}))) = E(\bar{x}_{(\alpha,\beta)})$. Thus $\psi(D(x_{(\alpha,\beta)}.\chi_M)) = E(\bar{x}_{(\alpha,\beta)})$. □

Proposition 4.4. *If $x, y \in R$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L\backslash\{0\}$ with $\alpha_1 \le N(\beta_1)$ and $\alpha_2 \le N(\beta_2)$, then*

$$D(x_{(\alpha_1,\beta_1)}.\chi_M) \cap D(y_{(\alpha_2,\beta_2)}.\chi_M) = D((xy)_{(\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)}.\chi_M).$$

Proof. We have

$$\begin{aligned} D(x_{(\alpha_1,\beta_1)}.\chi_M) \cap D(y_{(\alpha_2,\beta_2)}.\chi_M) &= \psi^{-1}(E(\bar{x}_{(\alpha_1,\beta_1)})) \cap \psi^{-1}(E(\bar{y}_{(\alpha_2,\beta_2)})) \\ &= \psi^{-1}(E(\bar{x}_{(\alpha_1,\beta_1)}) \cap E(\bar{y}_{(\alpha_2,\beta_2)})) \\ &= \psi^{-1}(E(\overline{(xy)}_{\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2})) \\ &= D((xy)_{\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2}.\chi_M). \end{aligned}$$

□

In the sequel we assume that the lattice L is a chain.

Theorem 4.5. *For any R-module M, the set $\mathbb{B} = \{D(x_{(\alpha,\beta)}.\chi_M) | x \in R, \alpha, \beta \in L\backslash\{0\}$ with $\alpha \le N(\beta)\}$ forms a base for the Zariski topology on X.*

Proof. Suppose that U is an arbitrary open set of X. Then $U = D(A) = X\backslash X(A)$ for some $A \in IF_L(M)$. By Proposition 3.4, $X(A) = X((A : \chi_M).\chi_M)$. By considering $C = (A : \chi_M)$, $X(A) = X(C.\chi_M)$. As mentioned in the preliminaries, we can write $C = \bigcup_{(\alpha,\beta)\in C(R)} (\alpha,\beta)_{C_{(\alpha,\beta)}}$, where $(\alpha,\beta)_{C_{(\alpha,\beta)}} = \bigcup_{x\in C_{(\alpha,\beta)}} x_{(\alpha,\beta)}$. Thus we obtain that

$$\begin{aligned} X(C.\chi_M) &= X((\bigcup_{(\alpha,\beta)\in C(R)} (\bigcup_{x\in C_{(\alpha,\beta)}} x_{(\alpha,\beta)})).\chi_M) \\ &= X(\bigcup_{(\alpha,\beta)\in C(R), x\in C_{(\alpha,\beta)}} x_{(\alpha,\beta)}).\chi_M) \\ &= X(\bigcup_{(\alpha,\beta)\in C(R), x\in C_{(\alpha,\beta)}} (x_{(\alpha,\beta)}.\chi_M)) \\ &= \bigcap_{(\alpha,\beta)\in C(R), x\in C_{(\alpha,\beta)}} X(x_{(\alpha,\beta)}.\chi_M). \end{aligned}$$

So

$$\begin{aligned} D(A) &= X\backslash X(A) \\ &= X\backslash \bigcap_{(\alpha,\beta)\in C(R), x\in C_{(\alpha,\beta)}} X(x_{(\alpha,\beta)}.\chi_M) \\ &= \bigcup_{(\alpha,\beta)\in C(R), x\in C_{(\alpha,\beta)}} (X\backslash X(x_{(\alpha,\beta)}.\chi_M)) \\ &= \bigcup_{(\alpha,\beta)\in C(R), x\in C_{(\alpha,\beta)}} D(x_{(\alpha,\beta)}.\chi_M). \end{aligned}$$

This shows that $\mathbb{B}$ is a base for the Zariski topology on X. $\square$

Proposition 4.6. *Let M be an R-module. If the natural map ψ is surjective, then X is compact.*

Proof. $X = \bigcup\{D(x_{(\alpha,\beta)}.\chi_M)|x \in R, \alpha, \beta \in L\backslash\{0\} \text{ with } \alpha \leq N(\beta)\}$. Then

$$\begin{aligned}\overline{X} &= \psi(X)\\ &= \psi(\bigcup\{D(x_{(\alpha,\beta)}.\chi_M)|x \in R, \alpha, \beta \in L\backslash\{0\} \text{ with } \alpha \leq N(\beta)\})\\ &= \bigcup\{\psi(D(x_{(\alpha,\beta)}.\chi_M))|x \in R, \alpha, \beta \in L\backslash\{0\} \text{ with } \alpha \leq N(\beta)\}\\ &= \bigcup\{\bar{x}_{(\alpha,\beta)}|x \in R, \alpha, \beta \in L\backslash\{0\} \text{ with } \alpha \leq N(\beta)\}.\end{aligned}$$

Also, since $\overline{X}$ is compact, we can write $\overline{X} = \bigcup_{i=1}^{n} \overline{x_i}_{(\alpha_i,\beta_i)}$. Then we get

$$\psi^{-1}(X) = \psi^{-1}(\bigcup_{i=1}^{n}(\overline{x_i}_{(\alpha_i,\beta_i)})).$$

Thus $X = \bigcup_{i=1}^{n} \psi^{-1}(\overline{x_i}_{(\alpha_i,\beta_i)})$. So $\psi^{-1}(\overline{x_i}_{(\alpha_i,\beta_i)}) = ((x_i)_{(\alpha_i,\beta_i)}.\chi_M)$. Hence X is compact. $\square$

Example 4.7. We look at the closed and basic open sets in $IF_LSpec(M)$, where $M = Z$ as Z-module. Then by Example 3.5 (1), we have

$X = IF_LSpec(M)$
$= \{P_{\alpha,\alpha'}$, where $\alpha \in L$ is a prime element and p is prime integer of $Z\}$.

Let L be a chain. Then for any prime integer $p \in Z$, we have

$D(p_{(\alpha,\alpha')}.\chi_M)$
$= X \setminus X(p_{(\alpha,\alpha')}.\chi_M)$
$= \{Q_{\alpha,\alpha'}$, where $\alpha \in L$ is a prime element and $q \neq p$ is prime integer$\}$,

where $Q_{\alpha,\alpha'}$ is given as:

$$\mu_{Q_{\alpha,\alpha'}}(x) = \begin{cases}1 & \text{if } x \in < q >\\ \alpha & \text{if otherwise}\end{cases}, \quad \nu_{Q_{\alpha,\alpha'}}(x) = \begin{cases}0 & \text{if } x \in < q >\\ \alpha' & \text{otherwise.}\end{cases}$$

It is easy to see that for any natural number $m > 1$, $D((p^m)_{(\alpha,\alpha')}.\chi_M) = D(p_{(\alpha,\alpha')}.\chi_M)$. Thus for any integer $n = p_1^{m_1}p_2^{m_2}.......p_k^{m_k}$, we have

$D(n_{(\alpha,\alpha')}.\chi_M)$
$= D((p_1^{m_1}p_2^{m_2}.......p_k^{m_k})_{(\alpha,\alpha')}.\chi_M)$
$= D((p_1^{m_1})_{(\alpha,\alpha')}.\chi_M) \cap D((p_2^{m_2})_{(\alpha,\alpha')}.\chi_M) \cap .... \cap D((p_k^{m_k})_{(\alpha,\alpha')}.\chi_M)$
$= D((p_1)_{(\alpha,\alpha')}.\chi_M) \cap D((p_2)_{(\alpha,\alpha')}.\chi_M) \cap .... \cap D((p_k)_{(\alpha,\alpha')}.\chi_M)$
$= \{Q_{\alpha,\alpha'}$, where $\alpha \in L$ is a prime element and $q \neq p_i$ is prime integer, i = 1,2,...,k$\}$.

For a prime p, we have

$V(\chi_{<p>}) = \{P \in X : \chi_{<p>} \subseteq P\}$
$= \{P_{\alpha,\alpha'} : \alpha$ is prime element of L and p is prime integer$\}$.

It is easy to see that for any natural number $m > 1$, $V(\chi_{<p^m>}) = V(\chi_{<p>})$. So for any integer $n = p_1^{m_1}p_2^{m_2}.......p_k^{m_k}$, we have

$V(\chi_{<n>})$

$= \{P_{\alpha_i,\alpha_i'} : \alpha_i$ are prime elements in L, p_i are prime integers, $i = 1, 2, .., k\}$.

Let us denote by $\mathbb{C} = \{p = (P : \chi_M) | P \in IF_L Spec(M))\}$ and $\mathbb{C}^* = \{p_* | p \in \mathbb{C}\}$. Then we have the following.

Lemma 4.8. *$D(x_{(\alpha,\beta)}.\chi_M) = \varnothing$ if and only if $x \in \bigcap_{p \in \mathbb{C}} \{p_*\}$.*

Proof. Let $D(x_{(\alpha,\beta)}.\chi_M) = \varnothing$. Then $X(x_{(\alpha,\beta)}.\chi_M) = X$. Suppose N is a prime submodule of M and set $A = \chi_N$. Then $A \in IF_L Spec(M)$. Let $p = (A : \chi_M)$. Then $((x_{(\alpha,\beta)}.\chi_M) : \chi_M) \subseteq (A : \chi_M) = p$, but $x_{(\alpha,\beta)} \subseteq ((x_{(\alpha,\beta)}.\chi_M) : \chi_M)$. Thus $x_{(\alpha,\beta)} \subseteq p$. So $\alpha \leq \mu_p(x) = 1$, $\beta \geq \nu_p(x) = 0$ implies that $x \in p_*$. Hence $x \in \bigcap_{p \in \mathbb{C}} \{p_*\}$.

Conversely, suppose that $x \in \bigcap_{p \in \mathbb{C}} \{p_*\}$ and $P \in X$.
If $p = (P : \chi_M)$, then $x \in p_*$. Thus we have

$$\begin{aligned}
&\mu_p(x) = 1 \text{ and } \nu_p(x) = 0 \\
&\Rightarrow \mu_{(P:\chi_M)}(x) = 1 \text{ and } \nu_{(P:\chi_M)}(x) = 0 \\
&\Rightarrow x_{(\alpha,\beta)} \subseteq (P : \chi_M) \\
&\Rightarrow x_{(\alpha,\beta)}.\chi_M \subseteq P \\
&\Rightarrow (x_{(\alpha,\beta)}.\chi_M : \chi_M) \subseteq (P : \chi_M).
\end{aligned}$$

So $P \in X(x_{(\alpha,\beta)}).\chi_M)$. Hence $X(x_{(\alpha,\beta)}) = X$. Therefore $D(x_{(\alpha,\beta)}).\chi_M) = \varnothing$. □

Example 4.9. Let M be an arbitrary R-module and let N be any prime submodule of M. Consider the ILFPSMs A and B of M as follows:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in N \\ \alpha_1 & \text{if otherwise} \end{cases}, \quad \nu_A(x) = \begin{cases} 0 & \text{if } x \in N \\ \alpha_1' & \text{if otherwise} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 1 & \text{if } x \in N \\ \alpha_2 & \text{if otherwise} \end{cases}, \quad \nu_B(x) = \begin{cases} 0 & \text{if } x \in N \\ \alpha_2' & \text{if otherwise,} \end{cases}$$

where α_i' are complement of α_i in L $\forall i = 1, 2$.

Let $D(x_{(\alpha_1,\alpha_1')}.\chi_M)$ and $D(y_{(\alpha_2,\alpha_2')}.\chi_M)$ be two basic open sets such that

$$A \in D(x_{(\alpha_1,\alpha_1')}.\chi_M) \text{ and } B \in D(y_{(\alpha_2,\alpha_2')}.\chi_M).$$

Then we have

$$x_{(\alpha_1,\alpha_1')} = ((x_{(\alpha_1,\alpha_1')}.\chi_M) : \chi_M) \nsubseteq (A : \chi_M)$$

and

$$y_{(\alpha_2,\alpha_2')} = ((y_{(\alpha_2,\alpha_2')}.\chi_M) : \chi_M) \nsubseteq (B : \chi_M).$$

Now,

$$\mu_{(A:\chi_M)}(x) = \begin{cases} 1 & \text{if } x \in (N : M) \\ \alpha_1 & \text{if otherwise} \end{cases}, \quad \nu_{(A:\chi_M)}(x) = \begin{cases} 0 & \text{if } x \in (N : M) \\ \alpha_1' & \text{if otherwise.} \end{cases}$$

But $(N : M)$ is a prime ideal of R. Then $xy \notin (N : M)$. Thus $xy \notin \mathbb{C}^*$. So $D((xy)_{(\alpha_1 \wedge \alpha_2, \alpha_1' \vee \alpha_2')}.\chi_M) \neq \varnothing$ and we obtained that

$$D(x_{(\alpha_1,\alpha_1')}.\chi_M) \cap D(y_{(\alpha_2,\alpha_2')}.\chi_M) = D((xy)_{(\alpha_1 \wedge \alpha_2, \alpha_1' \vee \alpha_2')}.\chi_M).$$

This shows that X is not Hausdorff.

5. Relating $IF_LSpec(M)$ with $IF_LSpec(R/Ann(M))$

Let A be an intuitionistic L-fuzzy prime submodule of M. Then by Corollary (2.15) $(A:\chi_M)$ is an intuitionistic L-fuzzy prime ideal of R. Consider the quotient ring $R/Ann(M)$. We denote a typical element of $R/Ann(M)$ by $[x]$, where $x \in R$.

Consider the quotient map $\pi : R \to R/Ann(M)$ is defined by $\pi(x) = [x]$, we denote the image of $(A:\chi_M)$ under π by $\overline{(A:\chi_M)}$. In fact,

$$\overline{(A:\chi_M)}[x] = (\mu_{\overline{(A:\chi_M)}}([x]), \nu_{\overline{(A:\chi_M)}}([x])),$$

where

$$\mu_{\overline{(A:\chi_M)}}([x]) = \vee\{\mu_{(A:\chi_M)}(z) | z \in [x]\}$$

and

$$\nu_{\overline{(A:\chi_M)}}([x]) = \wedge\{\nu_{(A:\chi_M)}(z) | z \in [x]\}.$$

Proposition 5.1. *Let $A \in ILFS(M)$. Then $\overline{(A:\chi_M)}$ is an intuitionistic L-fuzzy prime ideal of $R/Ann(M)$.*

Proof. The quotient map π is epimorphism, it is easy to verify that $(A:\chi_M)$ is π-invariant. Then by Theorem 2.8, $\overline{(A:\chi_M)}$ is an intuitionistic L-fuzzy prime ideal of $R/Ann(M)$. □

Define the map $\varphi : IF_LSpec(M) \to IF_LSpec(R/Ann(M))$ by

$$\varphi(A) = \overline{(A:\chi_M)}, \text{ for } A \in IF_LSpec(M)$$

φ is called the *natural map*.

Lemma 5.2. *Let I be an ideal of R and $A \in ILFI(R/I)$. Then there exists $C \in ILFI(R)$ such that $A = \overline{C}$.*

Proof. Consider the quotient map $\pi : R \to R/I$. Then it is easy to verify that

$$\mu_A \circ \pi = \mu_C \text{ and } \nu_A \circ \pi = \nu_C.$$

□

Proposition 5.3. *The natural map φ is continuous for topologies on $IF_LSpec(M)$ and $IF_LSpec(R/Ann(M))$.*

Proof. Let $\overline{C} \in ILFI(R/Ann(M))$. We claim that $\varphi^{-1}(X(\overline{C})) = X(C.\chi_M)$. Suppose that $P \in X(C.\chi_M)$. Then $(C.\chi_M) \subseteq P$ and $\chi_M \nsubseteq P$. Thus $C \subseteq (P:\chi_M)$. So $\overline{C} \subseteq \overline{(P:\chi_M)}$. Hence $\overline{(P:\chi_M)} \subseteq X(\overline{C})$, but $\overline{(P:\chi_M)} = \varphi(P)$. Then we have

$$P \in \varphi^{-1}(X(\overline{C})) \;\Rightarrow\; X(C.\chi_M) \subseteq \varphi^{-1}(X(\overline{C})).$$

Similarly, we can show that $\varphi^{-1}(X(\overline{C})) \subseteq X(C.\chi_M)$. Thus $\varphi^{-1}(X(\overline{C})) = X(C.\chi_M)$. So φ is continuous. □

Proposition 5.4. *For any R-module M the following statements are equivalent:*
(1) *φ is injective,*
(2) *for $A, B \in X$, if $X(A) = X(B)$, then $A = B$,*
(3) *for every $p \in IF_LSpec(R)$, $|X_p| \leq 1$.*

Proof. (1)⇒(2): Let $A, B \in X$. If $X(A) = X(B)$, then $(A : \chi_M) = (B : \chi_M)$. By Proposition 3.3, we have $\overline{(A : \chi_M)} = \overline{(B : \chi_M)}$ which implies that $\varphi(A) = \varphi(B)$. Thus $A = B$, since φ is injective by (1).

(2)⇒ (3): Let $A, B \in X_p$. Then $(A : \chi_M) = (B : \chi_M) = p$. Thus $X(A) = X(B)$ by Proposition 3.3. So by (2)). we have $A = B$. Hence $|X_p| \leq 1$.

(3)⇒ (1): Suppose that $A, B \in X$ and $\varphi(A) = \varphi(B)$. Then

$$\overline{(A : \chi_M)} = \overline{(B : \chi_M)} \Rightarrow (A : \chi_M) = (B : \chi_M) = p \Rightarrow A, B \in X_p \Rightarrow A = B.$$

It means that φ is injective. □

In the sequel, we set $X = IF_LSpec(M)$ and $\overline{X} = IF_LSpec(R/Ann(M))$.

Theorem 5.5. *Let φ be the natural map. If φ is surjective then φ is both closed and open.*

Proof. Suppose that $\varphi : X \to \overline{X}$ is the natural map and $A \in X$. Then by the proof of Proposition 5.3,

$$\varphi^{-1}(X(\overline{(A : \chi_M)})) = X((A : \chi_M).\chi_M) = X(A)$$
$$\Rightarrow \varphi(X(A)) = \varphi \circ \varphi^{-1}(X(\overline{(A : \chi_M)})) = X(\overline{(A : \chi_M)}).$$

Thus φ is closed. Also we have

$$\begin{aligned}\varphi(X - X(A)) &= \varphi(\varphi^{-1}(\overline{X})) - \varphi^{-1}(X(\overline{(A : \chi_M)}))\\ &= \varphi(\varphi^{-1}(\overline{X} - X(\overline{(A : \chi_M)})))\\ &= \overline{X} - X(\overline{(A : \chi_M)}).\end{aligned}$$

this mean that φ is open. □

Proposition 5.6. *Let φ be the natural map from X into $\overline{X}$ and let φ be surjective. Then X is connected if and only if $\overline{X}$ is connected.*

Proof. Suppose that X IS connected. Then $\overline{X} = \varphi(X)$ is connected, since φ is continuous and surjective.

Conversely, suppose that $\overline{X}$ is connected but X is disconnected. Then X contains a non-empty proper subset Y such that it is both open as well as closed. We show that $\varphi(Y)$ is a non-empty proper subset of $\overline{X}$. Since Y is open, there exists $A \in IF_L(M)$ such that $Y = X \backslash X(A)$. Then $\varphi(Y) = \overline{X} \backslash X(\overline{(A : \chi_M)})$.

If $\varphi(Y) = \overline{X}$, then $X(\overline{(A : \chi_M)}) = \varnothing$. Thus we get

$$\overline{(A : \chi_M)} = \chi_{R/Ann(M)} \Rightarrow A = \chi_M \Rightarrow Y = X \backslash X(A) = X \backslash X(\chi_M) = X.$$

This is a contradiction.

If $\varphi(Y) = \varnothing$, then we must have $X(\overline{(A : \chi_M)}) = \overline{X}$ Thus we have

$$\overline{(A : \chi_M)} = \chi_{\{0\}} \Rightarrow A = \chi_{\{\theta\}} \Rightarrow Y = X \backslash X(\chi_{\{\theta\}}) = X \backslash X = \varnothing.$$

This is a contradiction. So $\varphi(Y)$ is a proper non-empty subset of $\overline{X}$ such that it is both open as well as closed, a contradiction. Hence X is connected. □

Proposition 5.7. *Let M and M' be R-modules . If $X = IF_LSpec(M)$ and $X' = IF_LSpec(M')$ and $f : M \to M'$ is an epimorphism, then the mapping $g : X' \to X$ is defined by $g(A) = f^{-1}(A)$, is continuous.*

Proof. Suppose that $A \in IF_L(M)$ and $X(A)$ is closed set in X. For $P^{'} \in g^{-1}(X(A))$, by Proposition 3.4 (2), we have $X(A) = V((A : \chi_M).\chi_M)$. Then we have

$$\begin{aligned} & P^{'} \in g^{-1}(V(A : \chi_M).\chi_M) \\ \Leftrightarrow\ & g(P^{'}) \in V((A : \chi_M).\chi_M) \\ \Leftrightarrow\ & (A : \chi_M).\chi_M \subseteq g(P^{'}) = f^{-1}(P^{'}) \\ \Leftrightarrow\ & f((A : \chi_M).\chi_M) \subseteq P^{'} \\ \Leftrightarrow\ & (A : \chi_M).\chi_{M'} \subseteq P^{'} \\ \Leftrightarrow\ & P^{'} \in V((A : \chi_M).\chi_{M'}) = X((A : \chi_M).\chi_{M'}). \end{aligned}$$

Thus $g^{-1}(X(A)) = X((A : \chi_M).\chi_{M'})$. So g is continuous. □

6. Irreducible subsets of $IF_LSpec(M)$

In the sequel we assume that M is an R-module and $X = IF_LSpec(M)$. For $Y \subseteq X$, we write $\Gamma(Y) = \bigcap_{P\in Y} P$ and $\overline{Y}$ = closure of Y with regard to topology on X.

Proposition 6.1. *If $Y \subseteq X$, then $X(\Gamma(Y)) = \overline{Y}$ and thus Y is closed set iff $X(\Gamma(Y)) = Y$.*

Proof. Suppose $P \in Y$. Then $\Gamma(Y) \subseteq P$. This implies $(\Gamma(Y) : \chi_M) \subseteq (P : \chi_M)$. Thus $P \in X(\Gamma(Y))$. So $Y \subseteq X(\Gamma(Y))$.

Suppose $X(A)$ is any closed set in X so that $Y \subseteq X(A)$. Then for every $P \in Y$, $P \in X(A)$, we get $(A : \chi_M) \subseteq (P : \chi_M)$ implies

$$\begin{aligned} (A : \chi_M) &\subseteq \bigcap_{P\in Y} (P : \chi_M) \\ &= (\cap_{P\in Y} P : \chi_M) \\ &= (\Gamma(Y) : \chi_M). \end{aligned}$$

Again suppose that $Q \in X(\Gamma(Y))$. Then $(\Gamma(Y) : \chi_M) \subseteq (Q : \chi_M)$, but

$$(A : \chi_M) \subseteq (\Gamma(Y) : \chi_M) \subseteq (Q : \chi_M).$$

Thus $Q \in X(A)$. So we asserts that $X(\Gamma(Y)) \subseteq X(A)$, since

$$\overline{Y} = \bigcap\{X(A) : Y \subseteq X(A)\} = X(\Gamma(Y)).$$

. From this, it is convenient to observe that Y is closed set iff $X(\Gamma(Y)) = Y$. This complete the proof. □

Lemma 6.2. *If $A \in ILFI(R)$, then A is contained in some intuitionistic L-fuzzy maximal ideal.*

Proof. Let $A \in ILFI(R)$. Take $A_* = \{r \in R : \mu_A(r) = 1, \nu_A(r) = 0\}$. Since A_* is an ideal of R, there exist a maximal ideal S of R. Then $A_* \subseteq S$. Define $B \in ILFS(R)$ such that

$$\mu_B(r) = \begin{cases} 1 & \text{if } r \in S \\ \alpha & \text{if otherwise} \end{cases}, \quad \nu_B(r) = \begin{cases} 0 & \text{if } r \in S \\ \beta & \text{if otherwise,} \end{cases}$$

where $\alpha = sup\{\mu_A(r) : r \in R\}$ and $\beta = inf\{\nu_A(r) : r \in R\}$. Clearly, B is an intuitionistic L-fuzzy maximal ideal (ILFMI) of R such that $A \subseteq B$. In other words, there exists a ILFMI B of R such that $A \subseteq B$. □

Proposition 6.3. *Let B be a ILFMI of R. Then $B.\chi_M$ is an ILFPSM of M*

Proof. Let B be an ILFMI of R. Then B_* is the maximal ideal of R.

$$\mu_B(r) = \begin{cases} 1 & \text{if } r \in B_* \\ \alpha & \text{if otherwise} \end{cases}, \quad \nu_B(r) = \begin{cases} 0 & \text{if } r \in B_* \\ \alpha^{'} & \text{if otherwise,} \end{cases}$$

where $\alpha^{'}$ is a complement of α in L. Since B_* is the maximal ideal of R, B_*M is a prime submodule of M. Hence by Theorem 2.14, $B.\chi_M$ is an ILFPSM of M. □

Definition 6.4. An ILFS A of an R-module M is termed as an *intuitionistic L-fuzzy maximal prime submodule* (ILFMPSM) of M, if $A \in IF_LSpec(M)$ and there does not exist any $B \in IF_LSpec(M)$ which contains A properly.

For example, the ILFSM A on the Z-module Z_p defined by:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in pZ_p \\ \alpha & \text{if otherwise} \end{cases}, \quad \nu_A(x) = \begin{cases} 0 & \text{if } x \in pZ_p \\ \alpha^{'} & \text{if otherwise} \end{cases}$$

is an ILFMPSM of Z_p.

Lemma 6.5. *If $A \in IF_LSpec(M)$ is maximal prime, then $(A : \chi_M)$ is a ILFMI of R.*

Proof. Let $A \in IFSpec(M)$ is maximal prime. Suppose $C \in ILFI(R)$ be such that

$$(A : \chi_M) \subseteq C. \tag{6.1}$$

Then from Lemma 6.2, there exists a ILFMI B of R such that $C \subseteq B$. Since $(A : \chi_M) \subseteq C$, $A \subseteq C.\chi_M \subseteq B.\chi_M$. Also, from Proposition 6.3, $B.\chi_M$ is an ILFPSM of M. Thus $A = B.\chi_M$. Since A is maximal prime, $A = C.\chi_M$. So we get

$$C \subseteq (A : \chi_M). \tag{6.2}$$

Now by equations (6.1) and (6.2), we have $(A : \chi_M) = C$. Hence $(A : \chi_M)$ is a ILFMI of R. □

Proposition 6.6. *For any element P of X, the subsequent affirmation are satisfied:*
(1) $\overline{\{P\}} = X(P)$,
(2) *for any $Q \in X$, $Q \in \overline{\{P\}}$ iff $(P : \chi_M) \subseteq (Q : \chi_M)$ if and only if $X(Q) \subseteq X(P)$,*
(3) *The set $\{P\}$ is closed iff*
  (a) *P is ILFMPSM of M,*
  (b) *$X_p = \{P\}$ such that $(P : \chi_M) = p$.*

Proof. (1) It is an immediate consequences of proposition 6.1.

(2) Follows from (1)

(3) Let $\{P\}$ be a closed set. Then $\{P\} = \overline{\{P\}} = X(P)$. Suppose that $A \in IF_LSpec(M)$ and $P \subseteq A$. Then $(P : \chi_M) \subseteq (A : \chi_M)$. Thus $A \in X(P) = \{P\}$. So $A = P$. This means that P is an ILFMPSM of M.

Now suppose that $A \in X_p$. Then $(A : \chi_M) = p = (P : \chi_M)$. Thus $A \in X(P) = \{P\}$. So $A = P$.

Conversely, suppose that (a) and (b) are satisfied. Let $A \in X(P)$. Then $(P : \chi_M) \subseteq (A : \chi_M)$. Since P is maximal prime, by Lemma 6.5, it is concluded that $(P : \chi_M) = p$ is a ILFMPI of R. Thus $p = (P : \chi_M) = (A : \chi_M)$. This means that $A \in X_p = \{P\}$. So $A = P$. Hence $X(P) = \{P\}$. But $\overline{\{P\}} = X(P) = \{P\}$. It means that $\{P\}$ is closed. ☐

Remark 6.7. From the last proposition, we conclude that the space X is a T_1 space iff every ILFPSM of M is maximal prime and $|X_p| \leq 1$ for every $p \in IFSpec(R)$.

Further, recall that if A_1 and A_2 be any closed subsets of a space A such that $A = A_1 \cup A_2$, then the space A is said to be irreducible if either $A = A_1$ or $A = A_2$. Also the subspace A_0 of A is irreducible if it is irreducible as a subspace of A.

In a topological space A, an irreducible component of A is a maximal irreducible subset of A.

Theorem 6.8. *For any ILFPSM P of M, the closed set $X(P)$ is an irreducible set in X.*

Proof. By Proposition 6.6 (1), $X(P) = \overline{\{P\}}$. Let $X(P) = A_1 \cup A_2$ for closed sets A_1 and A_2. Then $\overline{\{P\}} = A_1 \cup A_2$. But $P \in \overline{\{P\}}$. Thus $P \in A_1$ or $P \in A_2$. Let $P \in A_1$. Then $P \in A_1 \in \overline{\{P\}}$, which is a contradiction. Thus we must have $A_1 = \overline{\{P\}}$. This mean that $X(P)$ is irreducible. ☐

Remark 6.9. The converse of Theorem 6.8 may not be true, i.e., $X(P)$ can be irreducible even if P is not an ILFPSM of M.

For example, consider $X = IF_LSpec(M)$, where $M = Z$ is Z-module. Define the ILFS P of M as

$$\mu_P(x) = \begin{cases} 1 & \text{if } x \in <4> \\ \alpha & \text{if otherwise} \end{cases}, \quad \nu_P(x) = \begin{cases} 0 & \text{if } x \in <4> \\ \alpha' & \text{if otherwise.} \end{cases}$$

Clearly, P is not an ILFPSM of M. Also we notice that here $X(P) = X(\sqrt{P})$ is an irreducible in X.

Corollary 6.10. *Let $Y \subseteq X$. If $\Gamma(Y)$ is a ILFPSM of M, then Y is irreducible.*

Proof. Let $\Gamma(Y) = P$ be a ILFPSM of M. By Proposition 6.1, $X(P) = X(\Gamma(Y)) = \overline{Y}$ is irreducible. For closed subsets A_1 and A_2, let

$$Y = A_1 \cup A_2. \tag{6.3}$$

Then $\overline{Y} = \overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2} = A_1 \cup A_2$. Since $\overline{Y}$ is irreducible, $\overline{Y} = A_1$ or $\overline{Y} = A_2$. Without loss of generality, suppose that $\overline{Y} = A_1$. Then $Y \subseteq A_1$ and equation (6.3) implies that $A_1 \subseteq Y$. Thus $Y = A_1$. This means that Y is irreducible. ☐

Corollary 6.11. *Let $P^* = \bigcap_{P \in X} P$. Then X is irreducible if and only if P^* is a ILFPSM of M.*

Proof. If part follow from Corollary 6.8 and the only if part follow from the fact that $X(P^*) = X(P)$ is irreducible. □

Corollary 6.12. *For an R-module M, the followings hold:*
(1) *if* $Y = \{P_i : i \in J\}$ *is linearly ordered by the set inclusion, then* Y *is irreducible in* X,
(2) X_p *is irreducible for* $p \in IFSpec(R)$,
(3) *if* p *is a ILFMPI of* R, *then* X_p *is an irreducible closed subset of* X.

Proof. (1) As the members of Y are linearly ordered by the set inclusion, $\Gamma(Y)$ is a ILFPSM of M. Then by Corollary 6.10, Y is irreducible.

(2) We prove that $\Gamma(X_p)$ is a ILFPSM of M. For this, we have

$$(\bigcap_{P\in X} P : \chi_M) = \bigcap_{P\in X}(P : \chi_M) = p.$$

Then $\Gamma(X_p : \chi_M) = p$.

Now suppose that $C \in ILFI(R)$, $B \in ILFM(M)$ and $C.B \subseteq \Gamma(X_p)$ such that $B \nsubseteq \Gamma(X_p)$. Then there exists $P^{'} \in X_p$ such that $B \nsubseteq P^{'}$. Thus $C \subseteq (P^{'} : \chi_M) = p = (\Gamma(X_p) : \chi_M)$. This means that $\Gamma(X_p)$ is a ILFPSM. So X_p is irreducible by Corollary 6.10.

(3) Suppose that p is ILFMI of R. By (2), X_p is irreducible. But because p is maximal, $((p.\chi_M) : \chi_M) = p$. Also, for $Q \in X(p.\chi_M)$, we have $p = ((p.\chi_M) : \chi_M) \subseteq (Q : \chi_M)$. Since p is maximal, $(Q : \chi_M) = p \Rightarrow Q \in X_p$ implies that

$$X(p.\chi_M) \subseteq X_p. \tag{6.4}$$

But for $P \in X_p$, it is concluded that $(P : \chi_M) = p = ((p.\chi_M) : \chi_M)$. Then $P \in X(p.\chi_M)$ implies that

$$X_p \subseteq X(p.\chi_M). \tag{6.5}$$

Thus from equations (6.4) and (6.5), we obtain $X(p.\chi_M) = X_p$. Therefore X_p is closed as desired. □

Corollary 6.13. *Let* $Y \subseteq X$ *and* $(\Gamma(Y) : \chi_M) = p$ *be an ILFPI of* R. *If* $X_p \neq \varnothing$ *then* Y *is irreducible.*

Proof. Let $P \in X_p$. Then $(P : \chi_M) = (\Gamma(Y) : \chi_M) = p$. Then $X(\Gamma(Y)) = X(P)$, by Proposition 3.3 of [19]. But by Proposition 6.1, we have $X(\Gamma(Y)) = \overline{Y}$. Thus$X(P) = \overline{Y}$. So by Theorem 6.8, $X(P)$ is irreducible. Hence $\overline{Y}$. Therefore Y is irreducible. □

7. Conclusion

In this paper we have constituted a topology on $IF_LSpec(M)$, the collection of all intuitionistic L-fuzzy prime submodules of a module M over a commutative ring R with identity, which is called Zariski topology. For various R-modules M, we have constructed the set $X = IF_LSpec(M)$. By using the bases for the Zariski topology, it is shown that the space X is compact. Apart from these we have studied the relationship between $X = IF_LSpec(M)$ and $\overline{X} = IF_LSpec(R/Ann(M))$. We have shown that if the natural map $\varphi : X \to \overline{X}$ is surjective, then the space X is connected if and only if $\overline{X}$ is connected. Further we have shown that X is irreducible if and only if the intersection of all the elements of $IF_L Spec(M)$ is also an element of $IF_L Spec(M)$. Many other related results have been derived. We observe that this topological space is rich enough in the view point of topological properties. Also, in this paper, we have tried to bring the first stones of intuitionistic L-fuzzy spectral theory based on intuitionistic L-fuzzy prime submodules, and hence we hope that this paper encourage researchers in the field of intuitionistic fuzzy algebra and intuitionistic fuzzy topology to continue this way for finding further and deep results.

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