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Soft topological modules

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Abstract. In this paper, we have introduced the concept of the soft topological module over soft topological rings as a hybrid of algebraic and soft topological structures. Also, we have studied the properties of soft topological module and their subsystem.

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1. Introduction

In 1999, Molodtsov [1] proposed the soft set theory as a new approach to managing uncertainties as he introduced the concept of the soft set to be a set associated with a set of parameters applied in several directions, that makes it a comprehensive extension for the theories of fuzzy sets [2] and rough sets [3].

Recently, many concepts have developed in soft theory. For instance, [4] studied the soft algebraic structures of rings by introducing the idea of soft rings, which studied later by [5]. Shabir and Naz [6] studied the soft topological structures by adding the notion of soft topology, which has been extensively studied and investigated by several authors like [7, 8, 9, 10, 11]. After that, some authors went to examine the connection between the soft topological structures and the soft algebraic structures such as the concept of soft topological soft groups and rings [12], the concept of soft topological rings [13] and the concept of soft topological soft modules [14].

To promote this type of study, we introduce the concept of soft topological soft modules in this paper. In Section 2, we present well-known results of the essential preliminaries related to soft set and soft topological spaces. In Section 3, we have introduced the concept of soft topological modules over soft topological rings and studied some of their properties. In Section 4, we have introduced the concept of soft topological submodules over soft topological rings and studied some of their properties.

2. Preliminaries

In this section, we recall some basic concepts and results which we will use in this paper. Throughout this paper, R, X, Y, and Z are assumed to be initially universal sets and E assumed to be the fixed set of parameters.

Definition 2.1 ([15]). A *soft set* F_A *over* X is defined to be a mapping $F_A : A \to P(X)$, where A is a subset of the fixed parameters set E. If $A = E$, we put $F_E = F$.

Let $S(X)$ denotes the class of all soft sets over X. Let $F_A \in S(X)$. We may write $F_A = \{(a, F_A(a)) \mid a \in A\}$. If F_A is defined such that $F_A(a) = \phi, \forall a \in A$, then F_A is called a *null soft set* over X and denoted by $\tilde{\phi_A}$. And if F_A is defined such that $F_A(a) = X, \forall a \in A$, then F_A is called an *absolute soft set* over X, and denoted by $\tilde{X_A}$ (See [15]).

Definition 2.2 ([15, 16]). Let $F_A, G_B \in S(X)$.

(i) F_A is called a *soft subset* of G_B, denoted by $F_A \subseteq G_B$, if $A \subseteq B$ and $F_A(a) \subseteq G_B(a), \forall a \in A$.

(ii) F_A is said to be *equal to* G_B, denoted by $F_A = G_B$, if $F_A \subseteq G_B$ and $G_B \subseteq F_A$.

(iii) The *intersection* of F_A and G_B is the soft set $(F \cap G)_{A\cap B} \in S(X)$ defined by: for all $a \in A \cap B$,

$$(F \cap G)_{A\cap B}(a) = F_A(a) \cap G_B(a).$$

(iv) The *union* of F_A and G_B is the soft set $(F \cup G)_{A\cup B} \in S(X)$ defined by: for all $a \in A \cup B$,

$$(F \cup G)_{A\cup B}(a) = \begin{cases} F_A(a) \cup G_B(a) & \text{if } a \in A \cap B \\ F_A(a) & \text{if } a \in A \setminus B \\ G_B(a) & \text{if } a \in B \setminus A. \end{cases}$$

Definition 2.3 ([17]). Let $F_A \in S(X)$ and $G_B \in S(Y)$. Then the *Cartesian product* of F_A and G_B is defined to be the soft set $(F \times G)_{A\times B} \in S(X \times Y)$ such that $(F \times G)_{A\times B}(a, b) = F_A(a) \times G_B(b)$, $\forall (a, b) \in A \times B$.

Definition 2.4 ([18]). Let $F_A \in S(X)$ and $G_B \in S(Y)$. Let $\mu : X \to Y$ and $\varphi : A \to B$ be two mappings.

(i) (μ, φ) is called a *soft mapping from* F_A *to* G_B, denoted by $(\varphi, \mu) : F_A \to G_B$, if for all $a \in A$,

$$\mu(F_A(a)) = G_B(\varphi(a)).$$

(ii) The *image of* F_A *under* μ *with respect to* φ is the soft set $(\mu(F_A))_{\varphi(A)} \in S(Y)$ defined as follows: for all $b \in \varphi(A)$,

$$(\mu(F_A))_{\varphi(A)}(b) = \bigcup_{\varphi(a)=b} (\mu(F_A(a)).$$

(iii) The *inverse image of* G_B *under* μ *with respect to* φ is the soft set $(\mu^{-1}(G_B))_A \in S(X)$ defined as follows: for all $a \in A$,

$$(\mu^{-1}(G_B))_A(a) = \mu^{-1}\Big(G_B(\varphi(a))\Big).$$

Remark 2.5 ([12]). Let $F_A \in S(X)$ and $G_B \in S(Y)$. Let $\mu : X \to Y$ and $\varphi : A \to B$ be two mappings.

(1) If $\varphi \times \mu : A \times X \to B \times Y$ such that $(\varphi \times \mu)(a, x) = ((\varphi(a), \mu(x)), \forall a \in A, x \in X$, then we have

$$(\mu(F_A))_{\varphi(A)} = (\varphi \times \mu)(F_A).$$

This means that the image of F_A under μ with respect to φ is the image of F_A under $\varphi \times \mu$.

(2) $(\mu^{-1}(G_B))_A = (\varphi \times \mu)^{-1}(G_B)$.

(3) If (φ, μ) is a soft mapping from F_A to G_B, then $(\mu(F_A))_{\varphi(A)} = G_B \mid_{\varphi(A)}$ (the restriction of G_B over $\varphi(A)$) and $(\mu^{-1}(G_B))_A = F_A$, if μ is injective.

(4) If $A = B$ and $\varphi = id_A$ (the identity on A), then $(\mu(F_A))_{\varphi(A)}(a) = \mu(F_A(a)),\ \forall a \in A$.

Note that If $A = B = E$ and $\varphi = id_E$ (the identity on E), we denote the soft mapping by $\tilde{f}$ instead of (id_E, f).

Definition 2.6 ([6])**.** Let $F_A \in S(X)$ and $x \in X$. If $x \in \bigcap_{a \in A} F_A(a)$, then we say that x is a *soft element in* F_A and denoted by $x \tilde{\in} F_A$.

Definition 2.7 ([12])**.** Suppose that X is a ring and $F, H \in S(X)$. The soft sets $FH, F+H, F-H, -F \in S(X)$ are defined as follows: for all $e \in E$,

(i) $(FH)(e) = F(e) \cdot H(e) = \{xy \mid x \in F(e), y \in H(e)\}$,

(ii) $(F + H)(e) = F(e) + H(e) = \{x + y \mid x \in F(e), y \in H(e)\}$,

(iii) $(F - H)(e) = F(e) - H(e) = \{x - y \mid x \in F(e), y \in H(e)\}$,

(iv) $-F(e) = -(F(e)) = \{-x \mid x \in F(e)\}$.

Definition 2.8 ([6])**.** Let τ be a family of soft sets in $S(X)$. Then τ is called a *soft topology* on X, if it satisfies the following axioms:

(i) $\tilde{\phi}, \tilde{X} \in \tau$,

(ii) τ is closed under finite intersections,

(iii) τ is closed under arbitrary unions.

In this case, the pair (X, τ) is called a *soft topological space* (in short, S.T.S) and the members of τ are called *soft open sets* in X.

Definition 2.9 ([19])**.** Let (X, τ) be an S.T.S and $G \subseteq X$. The soft topology $\tau_G = \{\tilde{G} \cap H \mid H \in \tau\}$ on G is called a *soft relative topology generated by* the set G and the soft topological space (G, τ_G) is called a *soft subspace* of (X, τ).

Proposition 2.10 ([13])**.** *Suppose that (X, τ) is an S.T.S, $H \subseteq G \subseteq X$ and (G, τ_G), (H, τ_H) are soft subspaces of (X, τ). Also suppose that $(H, (\tau_G)_H)$ be a soft subspace of (G, τ_G). Then $\tau_H = (\tau_G)_H$*

Definition 2.11 ([6])**.** Let (X, τ) be an S.T.S. If $\tau = \{\tilde{\phi}, \tilde{X}\}$, then τ is called the *soft indiscreet topology* on X and if τ is the collection of all soft subsets $F \in S(X)$, then τ is called the *soft discrete topology* on X.

Proposition 2.12. *Let (X, τ) be an S.T.S.*

(1) *$\tau^e = \{H(e) \mid H \in \tau\}$ is a topology on X for each $e \in E$ (See [6]).*

(1) *$\tau^* = \{F \in S(X) \mid F(e) \in \tau^e,\ \forall e \in E\}$ is a soft topology on X and $(\tau^*)^e = \tau^e\ \forall e \in E$ (See [8]).*

Definition 2.13 ([5])**.** Let G be an additive group and let R be a ring.

(i) The topological space (G, τ) is called a *topological group*, if the mapping $(x, y) \mapsto x - y$ from $(G \times G, \tau \times \tau)$ to (G, τ) is continuous.

(ii) The topological space (R, τ) is called a *topological ring* (in short, T.R), if the following conditions hold:

(a) The mapping $(x, y) \mapsto x - y$ from $(R \times R, \tau \times \tau)$ to (R, τ) is continuous,
(b) The mapping $(x, y) \mapsto xy$ from $(R \times R, \tau \times \tau)$ to (R, τ) is continuous.

Definition 2.14 ([5])**.** Let R be a ring, M a left (right) R-module and (R, ν) a topological ring. Then a topological space (M, τ) is called a *left (right) topological R-module*, if the following conditions hold:

(i) the mapping $(x, y) \mapsto x + y$ from $(M \times M, \tau \times \tau)$ to (M, τ) is continuous,

(ii) the mapping $x \mapsto -x$ from (M, τ) to (M, τ) is continuous,

(iii) the mapping $(r, x) \mapsto rx$ from $(R \times M, \nu \times \tau)$ to (M, τ) is continuous.

Definition 2.15 ([6])**.** Let (X, τ) be an S.T.S. Let $x \in X$. A soft set $U_x \in S(X)$ is called a *soft neighborhood of* x, if there exists $H \in \tau$ such that $x \tilde{\in} H \subseteq U_x$. A soft neighborhood U_x of x is called a *soft open neighborhood*, if $U_x \in \tau$.

Remark 2.16. Let (X, τ) be an S.T.S. Let $x \in X$ and $U \in S(X)$. Then directly from definition of soft neighborhood and the constructions of τ^* and τ^e. Note that

(1) U is a soft (open) neighborhood of x in $(X, \tau) \Rightarrow U(e)$ is an (open) neighborhood of x in (X, τ^e) for all $e \in E$,

(2) $U(e)$ is a (open) neighborhood of x in (X, τ^e), for all $e \in E \Rightarrow U$ is a soft (open) neighborhood of x in (X, τ^*).

Definition 2.17 ([20])**.** Suppose that (X, τ) and (Y, ν) are two S.T.S's and $\tilde{f} : (X, \tau) \to (Y, \nu)$ be a soft mapping. Then $\tilde{f}$ is called:

(i) *soft continuous*, if for every $x \in X$ and every soft open neighborhood $U_{\tilde{f}(x)}$ of $\tilde{f}(x)$, there exists a soft open neighborhood U_x of x such that $\tilde{f}(x) \tilde{\in} \tilde{f}(U_x) \subseteq U_{\tilde{f}(x)}$,

(ii) a *soft homeomorphism*, if $\tilde{f}$ and $\tilde{f}^{-1}$ are soft continuous,

(iii) *soft open*, if $\tilde{f}(H) \in \nu$ for every $H \in \tau$.

Proposition 2.18 ([21])**.** *Suppose that* (X, τ) *and* (Y, υ) *are two S.T.S's and* φ *is a mapping from* (X, τ) *to* (Y, υ)*. Then*

(1) $\varphi(\tau) = \{V \in S(Y) \mid \varphi^{-1}(V) \in \tau\}$ *is a soft topology on* Y*,*

(2) $\varphi^{-1}(\upsilon) = \{U \in S(X) \mid \varphi(U) \in \upsilon\}$ *is a soft topology on* X*.*

Definition 2.19 ([22])**.** Let τ be a soft topology on X. Then τ is called a *group soft topology* on X, if the following conditions are satisfied:

(i) the soft mapping $\tilde{f} : (X \times X, \tau \times \tau) \to (X, \tau)$ is soft continuous, where

$$\begin{aligned} f : X \times X &\to X \\ (x, y) &\mapsto x + y, \end{aligned}$$

(ii) the soft mapping $\tilde{j} : (X, \tau) \to (X, \tau)$ is soft continuous, where

$$\begin{aligned} j : X &\to X \\ x &\mapsto -x. \end{aligned}$$

The soft topological space (X, τ) is called a *soft topological group* and denoted by S.T.G, where τ is a group soft topology on X.

Theorem 2.20 ([13])**.** *Let* τ *be a soft topology on a group* X*. Then* (X,τ) *is an S.T.G if and only if the soft mapping* $\tilde{f} : (X \times X, \tau \times \tau) \to (X,\tau)$ *is soft continuous, where*

$$\begin{aligned} f : X \times X &\to X \\ (x, y) &\mapsto x - y. \end{aligned}$$

Definition 2.21 ([13])**.** Let τ be a soft topology on X. Then τ is called a *ring soft topology* on X, if the following conditions are satisfied:

(i) the soft mapping $\tilde{f} : (X \times X, \tau \times \tau) \to (X,\tau)$ is soft continuous, where

$$\begin{aligned} f : X \times X &\to X \\ (x, y) &\mapsto x - y, \end{aligned}$$

(ii) the soft mapping $\tilde{g} : (X \times X, \tau \times \tau) \to (X,\tau)$ is soft continuous, where

$$\begin{aligned} g : X \times X &\to X \\ (x, y) &\mapsto xy. \end{aligned}$$

The soft topological space (X,τ) is called a *soft topological ring* and denoted by S.T.R, where τ is a ring soft topology on X.

3. Soft topological modules

In literature [23], the concept of a topological module is a module that is defined in a topological setting under certain conditions. This section will introduce the concept of the soft topological module as a hybrid of algebraic and soft topological structures.

From now on, we consider that all soft sets are defined on the set of parameters E and all soft mappings are defined with respect to the identity on E.

Let R be a ring, Y and Z be left R-modules.

Definition 3.1. Let (R,ν) be an S.T.R over R and let Y be a left R-module and (Y,τ) an S.T.S. Then τ is called a *module soft topology* on Y, if the following conditions are satisfied:

(i) $\tilde{f} : (Y \times Y, \tau \times \tau \to (Y,\tau)$ is soft continuous, where

$$\begin{aligned} f : Y \times Y &\to Y \\ (x, y) &\mapsto x + y, \end{aligned}$$

(ii) $\tilde{j} : (Y,\tau) \to (Y,\tau)$ is soft continuous, where

$$\begin{aligned} j : Y &\to Y \\ x &\mapsto -x, \end{aligned}$$

(iii) $\tilde{g} : (R \times Y, \nu \times \tau) \to (Y,\tau)$ is soft continuous, where

$$\begin{aligned} g : R \times Y &\to Y \\ (r, y) &\mapsto ry. \end{aligned}$$

The soft topological space (Y,τ) is called a *left soft topological module over* (R,ν) (or *left soft topological* R*-module*), denoted by S.T.M, where τ is a ring soft topology on Y.

Similarly, one can define right soft topological module over (R,ν).

Example 3.2. Let $E = \{e_1, e_2\}$ and $R = \mathbb{Z}_8$. Let ν be the discrete soft topology on R. Then (R, ν) is an S.T.R over $\mathbb{Z}_8$. Let $Y = \{\bar{0}, \bar{4}\}$ Y is R-modules. Now, let $\tau = \{\tilde{\phi}, \tilde{Y}, \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}\})\}, \{(e_1, \bar{4}), (e_2, \{\bar{4}\})\}\}$. Then by checking conditions of Definition 3.1, it is easy to show that (Y, τ) is an S.T.M.

Remark 3.3. Note that conditions (i) and (ii) in the Definition 3.1 guarantee that each S.T.M is an S.T.G.

Proposition 3.4. *Let (R, ν) be an S.T.R, Y a R-module and (Y, τ) an S.T.S. Then (Y, τ) is a S.T.M over (R, ν) if and only if the following conditions are satisfied:*

(1) *for every $x, y \in Y$ and every soft open neighborhood U_{x+y} of $x + y$, there exists a soft open neighborhood U_x of x and a soft open neighborhood U_y of y such that $U_x + U_y \subseteq U_{x+y}$,*

(2) *for every $x \in Y$ and every soft open neighborhood U_{-x} of $-x$, there exists a soft open neighborhood U_x of x such that $-U_x \subseteq U_{-x}$,*

(3) *for every $r \in R$, $x \in Y$ and every soft open neighborhood U_{rx} of rx, there exists a soft open neighborhood U_x of x and a soft open neighborhood U_r of r such that $U_r U_x \subseteq U_{rx}$.*

Proof. Let (R, ν) be an S.T.R, Y be a R-module and (R, τ) an S.T.S.

($\Rightarrow$) Let (R, τ) be a S.T.M over (R, ν). Let $x, y \in Y$ and U_{x+y} be a soft open neighborhood of $x + y$. Since (Y, τ) is a S.T.M, the soft mapping $\tilde{f} : (Y \times Y, \tau \times \tau \to (Y, \tau)$ is soft continuous, where

$$f : Y \times Y \to Y$$
$$(x, y) \mapsto x + y.$$

Then by Definition 2.17, there exists a soft open neighborhood $U_{(x,y)}$ of (x, y) such that

$$f(x, y) \in f(U_{(x,y)}) \subseteq U_{f(x.y)} = U_{x+y}.$$

Since $U_{(x,y)} \in \tau \times \tau$, there exists $V, W \in \tau$ such that $U_{(x,y)} = V \times W$. Note that $x \in V$, $y \in W$. Now let we put $V = U_x$ and $W = U_y$. Then $U_{(x,y)} = U_x \times U_y$. Note that $f(U_{(x,y)}) = f(U_x \times U_y) = U_x + U_y$. Thus we have $U_x + U_y \subseteq U_{x+y}$. So the condition (1) holds.

Also, since (Y, τ) is a S.T.M, then the soft mapping $\tilde{j} : (Y, \tau) \to (Y, \tau)$ is soft continuous, where

$$j : Y \to Y$$
$$x \mapsto -x.$$

Then from Definition 2.17, it follows that for every $x \in Y$ and for every soft open neighborhood $U_{\tilde{j}(x)}$ of $\tilde{j}(x)$, there must be a soft open neighborhood U_x of x such that $x \in \tilde{j}(U_{(x)}) \subseteq U_{\tilde{j}(x)}$. Thus $-U_x \subseteq U_{-x}$. So the condition (2) holds.

Moreover, since (Y, τ) is a S.T.M, then the soft mapping $\tilde{g} : (R \times Y, \nu \times \tau) \to (Y, \tau)$ is soft continuous, where

$$g : R \times Y \to Y$$
$$(r, y) \mapsto ry.$$

Let $r \in R$ and U_{rx} be a soft open neighborhood of $rx = \tilde{g}(\tilde{r}, \tilde{x})$. Then by Definition 2.17, there exists a soft open neighborhood $U_{(r,x)}$ of (r, x) such that

$$\tilde{g}(\tilde{r}, \tilde{x}) \in \tilde{g}(U_{(r,x)}) \subseteq U_{\tilde{g}(\tilde{r},\tilde{x})} = U_{rx}$$

Since $U_{(r,x)} \in \nu \times \tau$, there exists $V \in \nu$ and $W \in \tau$ such that $U_{(r,x)} \in V \times W$. Since $r \in V$, $x \in W$, we put $V = U_r$ and $W = U_x$. Then $\tilde{g}(U_{(r,x)}) = U_r U_x$. Thus $U_r U_x \subseteq U_{rx}$. So the condition (3) holds.

($\Leftarrow$) Follows directly by Definition 2.17.

□

Theorem 3.5. *Let (R, ν) be an S.T.R, Y a R-module and (Y, τ) an S.T.S. Then (Y, τ) is a S.T.M over (R, ν) if and only if the following conditions are satisfied:*

(1) $\tilde{f} : (Y \times Y, \tau \times \tau \to (Y, \tau)$ *is soft continuous, where*

$$\begin{aligned} f : Y \times Y &\to Y \\ (x, y) &\mapsto x - y, \end{aligned}$$

(2) $\tilde{g} : (R \times Y, \nu \times \tau) \to (Y, \tau)$ *is soft continuous, where*

$$\begin{aligned} g : R \times Y &\to Y \\ (r, y) &\mapsto ry. \end{aligned}$$

Proof. ($\Rightarrow$) Let (X, τ) be an S.T.M. Then from Definition 3.1, it follows that $\tilde{g} : (R \times Y, \nu \times \tau) \to (Y, \tau)$ is soft continuous, where

$$\begin{aligned} g : R \times Y &\to Y \\ (r, y) &\mapsto ry. \end{aligned}$$

From Remark 3.3, (Y, τ) is an S.T.G. Thus from Theorem 2.20, it follows that the soft mapping $\tilde{f}$: $(X \times X, \tau \times \tau) \to (X, \tau)$ is soft continuous, where

$$\begin{aligned} f : X \times X &\to X \\ (x, y) &\mapsto x - y. \end{aligned}$$

So the conditions (1) and (2) hold.

($\Leftarrow$) Suppose (1) and (2) hold. Then from (2) and Theorem 2.20, it follows that (Y, τ) is an S.T.G. Thus by (1), (Y, τ) is an S.T.M. □

Theorem 3.6. *Let (R, ν) be an S.T.R and let Y a R-module and (Y, τ) an S.T.S. Let (S, τ) and (T, υ) be two S.T.S such that each is an S.T.M, where S and T are submodules of Y. Then $(S, \tau) \cap (T, \upsilon) = (S \cap T, \tau \cap \upsilon)$ is an S.T.M.*

Proof. Let (S, τ) and (T, υ) be two S.T.S such that each is an S.T.M, where S and T are submodules of Y. Let x, $y \in S \cap T$. then by Theorem 3.5 (1), the soft mappings $\tilde{f_1} : (S \times S, \tau \times \tau \to (S, \tau)$ and $\tilde{g_1} : (T \times T, \upsilon \times \upsilon \to (T, \upsilon)$ are soft continuous, where

$$\begin{aligned} f_1 : S \times S &\to S \\ (x, y) &\mapsto x - y \end{aligned} \quad , \quad \begin{aligned} g_1 : T \times T &\to T \\ (x, y) &\mapsto x - y. \end{aligned}$$

By Definition 2.17, for any arbitrary soft open neighborhoods $U_{x-y} \in \tau$ and $V_{x-y} \in \upsilon$ of $x - y$, there exist soft open neighborhoods $U_x \in \tau$ and $V_x \in \upsilon$ of x, and there exist soft open neighborhoods $U_y \in \tau$ and $V_y \in \upsilon$ of y such that

$$x - y \tilde{\in} U_x - U_y \subseteq U_{x-y} \quad \text{and} \quad x - y \tilde{\in} V_x - V_y \subseteq V_{x-y}.$$

Thus we have

$$x-y\tilde{\in}(U_x-U_y)\cap(V_x-V_y)\subseteq U_{x-y}\cap V_{x-y}.$$

So we get

$$x-y\tilde{\in}(U\cap V)_x-(U\cap V)_y\subseteq(U\cap V)_{x-y}.$$

Hence the soft mapping $\tilde{k_1}:(S\cap T\times S\cap T,\tau\cap\upsilon\times\tau\cap\upsilon\to(S\cap T,\tau\cap\upsilon)$ is soft continuous, where

$$\begin{aligned} k: S\cap T\times S\cap T &\to S\cap T\\ (x,y)&\mapsto x-y.\end{aligned}$$

Again, let $r\in R$ and $y\in S\cap T$. Then by Theorem 3.5 (2), the soft mappings $\tilde{f_2}:(R\times S,\nu\times\tau\to(S,\tau)$ and $\tilde{g_2}:(R\times T,\nu\times\upsilon\to(T,\upsilon)$ are soft continuous, where

$$\begin{aligned} f_2: R\times S&\to S\\ (r,y)&\mapsto ry\end{aligned},\qquad \begin{aligned} g_2: R\times T&\to T\\ (r,y)&\mapsto ry.\end{aligned}$$

By Definition 2.17, for any arbitrary soft open neighborhoods $U_{ry}\in\tau$ and $V_{ry}\in\upsilon$ of ry, there exists soft open neighborhood $U_r\in\nu$ of r, and there exist soft open neighborhoods $U_y\in\tau$ and $V_y\in\upsilon$ of y such that

$$ry\tilde{\in}U_rU_y\subseteq U_{ry}\quad\text{and}\quad ry\tilde{\in}U_rV_y\subseteq V_{ry}.$$

Thus we have

$$ry\tilde{\in}(U_rU_y)\cap(U_rV_y)\subseteq U_{ry}\cap V_{ry}.$$

So we get

$$ry\tilde{\in}U_r(U\cap V)_y\subseteq(U\cap V)_{ry}.$$

Hence the soft mapping $\tilde{k_2}:(S\cap T\times S\cap T,\tau\cap\upsilon\times\tau\cap\upsilon\to(S\cap T,\tau\cap\upsilon)$ is soft continuous, where

$$\begin{aligned} k_2: S\cap T\times S\cap T &\to S\cap T\\ (r,y)&\mapsto ry.\end{aligned}$$

Therefore by Theorem 3.5, $(S\cap T,\tau\cap\upsilon)$ is an S.T.M. □

Proposition 3.7. *Let $F\in S(X)$ and $G\in S(Y)$. Let (F,τ) and (G,ν) be soft subspace of (X,ω) and (Y,Ω),respectively.*

(1) *If the soft mapping $\tilde{f}:(X,\tau)\to(Y,\nu)$ is soft continuous, then $f_e:(X,\tau^e)\to(Y,\nu^e)$ is continuous for every $e\in E$.*

(2) *If the mapping $f_e:(X,\tau^e)\to(Y,\nu^e)$ is continuous for every $e\in E$, then $\tilde{f}:(X,\tau^*)\to(Y,\nu^*)$ is soft continuous.*

Proof. (1) Suppose $\tilde{f}:(X,\tau)\to(Y,\nu)$ is a soft continuous and let $x\in X$. Then every soft open neighborhood $U_{\tilde{f}(x)}$ of $\tilde{f}(x)$, there exists a soft open neighborhood U_x of x such that $\tilde{f}(x)\tilde{\in}\tilde{f}(U_x)\subseteq U_{\tilde{f}(x)}$. Thus $f(x_e)\in f(U_x(e))\subseteq U_{f(x_e)}\ \forall e\in E$. But by Remark 2.16 (1), $U_x(e)$ is an open neighborhood of x in (X,τ^e) and $U_{f(x)}$ is an open neighborhood of $f(x)$ in (Y,ν) for all $e\in E$. Thus f_e is continuous for all $e\in E$.

Suppose $f_e:(X,\tau^e)\to(Y,\nu^e)$ is continuous for all $e\in E$. Let $x\in X$ and $U_{\tilde{f}(x)}$ be a soft open neighborhood of $\tilde{f}(x)$ in (Y,ν^*). Then from Remark 2.16 (1), it follows that $U_{\tilde{f}(x)}(e)$ is an open neighborhood of $f(x)$ in (Y,ν^e) for all $e\in E$. Put $U_{\tilde{f}(x)}(e)=U_{f(x)}$ for all $e\in E$. Then there exists an open neighborhood U_x of x in (X,τ^e) such that

$$f_e(x)\in f_e(U_x)\subseteq U_{f_e(x)}\ \forall e\in E.$$

Let $U \in S(X)$ such that $U(e) = U_x \ \forall e \in E$. Then U is a soft open neighborhood of x in (X, τ^*) and $\tilde{f}(x) \tilde{\in} \tilde{f}(U) \subseteq U_{\tilde{f}(x)}$. Thus $\tilde{f}$ is soft continuous. □

Theorem 3.8. *Let (R, ν) be an S.T.R and Y be a R-module. If (Y, τ) is an S.T.M, then for every $e \in E$, (Y, τ^e) is a T.M over the ring R.*

Proof. Let (R, ν) be an S.T.R and suppose (Y, τ) is an S.T.M. Then by Theorem 3.5, the soft mappings $\tilde{f} : (Y \times Y, \tau \times \tau) \to (Y, \tau)$ and $\tilde{g} : (R \times Y, \nu \times \tau) \to (Y, \tau)$ are soft continuous, where

$$\begin{aligned} f : Y \times Y &\to Y \\ (x, y) &\mapsto x + y \end{aligned} \quad \text{and} \quad \begin{aligned} g : R \times Y &\to Y \\ (r, y) &\mapsto ry. \end{aligned}$$

Thus by Proposition (3.7 (1)), the mappings $f_e : (Y \times Y, \tau^e \times \tau^e) \to (Y, \tau^e)$ and $g_e : (R \times Y, \nu^e \times \tau^e) \to (Y, \tau^e)$ are continuous for each $e \in E$, where

$$\begin{aligned} f_e : Y \times Y &\to Y \\ (x, y) &\mapsto x + y \end{aligned} \quad \text{and} \quad \begin{aligned} g_e : R \times Y &\to Y \\ (r, y) &\mapsto ry. \end{aligned}$$

So (Y, τ^e) is a T.M over the ring R. □

Proposition 3.9. *Let (R, ν) be an S.T.R and let τ be a soft topology and Y be a R-module. If (Y, τ^e) is a T.M over the ring R for every $e \in E$, then (Y, τ^*) is soft topological R-module.*

Proof. Suppose (Y, τ^e) is a T.M over the ring R for each $e \in E$. Then the mappings the mappings $f_e : (Y \times Y, \tau^e \times \tau^e) \to (Y, \tau^e)$ and $g_e : (R \times Y, \nu^e \times \tau^e) \to (Y, \tau^e)$ are continuous, where

$$\begin{aligned} f_e : Y \times Y &\to Y \\ (x, y) &\mapsto x + y \end{aligned} \quad \text{and} \quad \begin{aligned} g_e : R \times Y &\to Y \\ (r, y) &\mapsto ry. \end{aligned}$$

Thus by Proposition (3.7 (2), the soft mappings $\tilde{f} : (Y \times Y, \tau \times \tau \to (Y, \tau)$ and $\tilde{g} : (R \times Y, \nu \times \tau) \to (Y, \tau)$ are soft continuous, where

$$\begin{aligned} f : Y \times Y &\to Y \\ (x, y) &\mapsto x + y \end{aligned} \quad \text{and} \quad \begin{aligned} g : R \times Y &\to Y \\ (r, y) &\mapsto ry. \end{aligned}$$

So (Y, τ^*) is soft topological R-module. □

4. Soft topological submodule

This section will introduce the concept of the soft topological submodule as a hybrid of algebraic and soft topological structures.

Definition 4.1. Let (X, τ) and (Y, υ) be two S.T.M. Then (X, τ) is called a *soft topological* of (Y, υ), denoted by $(X, \tau) \tilde{\leq} (Y, \upsilon)$, if the following conditions are satisfied:

(i) X is submodule of Y,

(ii) $\tau = \upsilon_X$.

Remark 4.2. A module soft topology on a module X clearly induces a module soft topology on any submodule of X, and unless the contrary is indicated, we shall assume that a submodule of a soft topological module is furnished with its induced soft topology.

Example 4.3. Let $E = \{e_1, e_2\}$ and $R = \mathbb{Z}_8$. Let ν be the discrete soft topology on R. Then (R, ν) is an S.T.R over $\mathbb{Z}_8$. Let $X = \{\bar{0}, \bar{4}\}$ and $Y = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. Then it is clear that X and Y is R-modules, also X is submodule of Y. Now, let

$\tau = \{\tilde{\phi}, \tilde{X}, \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}\})\}, \{(e_1, \bar{4}), (e_2, \{\bar{4})\}\}$, and

$\upsilon = \{\tilde{\phi}, \tilde{Y}, \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}\})\}, \{(e_1, \{\bar{4}\}), (e_2, \{\bar{4}\})\}, \{(e_1, \{\bar{0}, \bar{4}\}), (e_2, \{\bar{0}, \bar{4}\})\}, \{(e_1, \{\bar{0}, \bar{4}, \bar{6}\}), (e_2, \{\bar{0}, \bar{4}\})\}\}$.
Thus by checking conditions of Definition 3.1, it is easy to show that (X, τ) is an S.T.M. Similarly, one can show that (Y, κ) is an S.T.M. Moreover,

$$\tau = \upsilon_X = \{\tilde{\phi}, \tilde{X}, \{(e_1, \{\bar{0}\}), (e_2, \{\bar{0}\})\}, \{(e_1, \bar{4}), (e_2, \{\bar{4})\}\}.$$

So $(X, \tau)\tilde{\leq}(Y, \upsilon)$.

Theorem 4.4. *Let (X, τ) be an S.T.M. If H is a submodule of X, then (H, τ) is an S.T.M and $(H, \tau)\tilde{\leq}(X, \tau)$.*

Proof. Follows directly from Proposition 2.10 and Theorem 3.5. □

5. Conclusion

We have produced the concept of soft topological modules over soft topological rings and studied some of their properties. Also, we have produced the concept of soft topological submodules over soft topological rings and studied some of their properties. The reader can study properties of soft topological modules such that the separation axioms and linearly compactness.

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