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Preserving maps in complete co-residuated lattices

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Abstract. In this paper, we introduce the notions of $\ominus$-join and $\oplus$-meet preserving maps in complete co-residuated lattices. Moreover, we investigate the relations between $\ominus$-join and $\oplus$-meet preserving maps and residuated connections. We give their examples.

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1. Introduction

The complete residuated lattice introduced Ward and Dilworth [1] is an important mathematical tool as algebraic structures for many valued logics ([2, 3, 3, 5, 6, 7]). Bělohlávek [2] investigated information systems and decision rules over complete residuated lattices. Pawlak [8] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. For an extension of Pawlak's rough sets, many researchers([9, 10, 11]) developed fuzzy rough sets, L-lower and L-upper approximation operators in complete residuated lattices.

Zheng and Wang [12] introduced a complete co-residuated lattice as the generalization of t-conorm. Junsheng and Qing [13] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \vee, \wedge, \odot, \&, 0, 1)$ where $(L, \vee, \wedge, \&, 0, 1)$ is a complete residuated lattice and $(L, \vee, \wedge, \odot, 0, 1)$ is complete co-residuated lattice in a sense [12]. Kim and Ko [14] studied preserving maps and approximation operators in complete co-residuated lattices.

In this paper, we introduce the concepts of distance spaces instead of fuzzy partially ordered spaces in complete co-residuated lattices. We study the notions of $\ominus$-join and $\oplus$-meet preserving maps in complete co-residuated lattices. Moreover, we investigate the relations between $\ominus$-join and $\oplus$-meet preserving maps and residuated connections. We give their examples.

2. Preliminaries

Definition 2.1 ([12, 13, 14]). An algebra $(L, \wedge, \vee, \oplus, \perp, \top)$ is called a *complete co-residuated lattice,* if it satisfies the following conditions:

(C1) $L = (L, \vee, \wedge, \perp, \top)$ is a complete lattice, where $\perp$ is the bottom element and $\top$ is the top element,

(C2) $a = a \oplus \perp$, $a \oplus b = b \oplus a$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in L$,

(C3) $(\bigwedge_{i \in \Gamma} a_i) \oplus b = \bigwedge_{i \in \Gamma}(a_i \oplus b)$.

Let $(L, \leq, \oplus)$ be a complete co-residuated lattice. For each $x, y \in L$, we define

$$x \ominus y = \bigwedge\{z \in L \mid y \oplus z \geq x\}.$$

Then $(x \oplus y) \geq z$ iff $x \geq (z \ominus y)$.

Put $n(x) = \top \ominus x$. The condition $n(n(x)) = x$ for each $x \in L$ is called a *double negative law*. We denote

$$\top_x(y) = \begin{cases} \top & \text{if } y = x \\ \perp & \text{otherwise} \end{cases}, \ \perp_x(y) = \begin{cases} \perp & \text{if } y = x \\ \top & \text{otherwise,} \end{cases}$$

for $\alpha \in L, A \in L^X$, $(\alpha \ominus A), (\alpha \oplus A), \alpha_X \in L^X$ as $(A \ominus \alpha)(x) = A(x) \ominus \alpha,\ (\alpha \oplus A)(x) = \alpha \oplus A(x),\ \ \alpha_X(x) = \alpha$.

Remark 2.2 ([14]). (1) An infinitely distributive lattice $(L, \leq, \vee, \wedge, \oplus = \vee, \perp, \top)$ is a complete co-residuated lattice. In particular, the unit interval $([0,1], \leq, \vee, \wedge, \oplus = \vee, 0, 1)$ is a complete co-residuated lattice where

$$x \ominus y = \bigwedge\{z \in L \mid y \vee z \geq x\} = \begin{cases} 0 & \text{if } y \geq x \\ x & \text{if } y \not\geq x. \end{cases}$$

Put $n(x) = 1 \ominus x = 1$ for $x \neq 1$ and $n(1) = 0$. Then $n(n(x)) = 0$ for $x \neq 1$ and $n(n(1)) = 1$. Hence n does not satisfy a double negative law.

(2) The unit interval with a right-continuous t-conorm $\oplus$, $([0,1], \leq, \oplus)$, is a complete co-residuated lattice [7].

(3) $([1, \infty], \leq, \vee, \oplus = \cdot, \wedge, 1, \infty)$ is a complete co-residuated lattice, where

$$x \ominus y = \bigwedge\{z \in [1, \infty] \mid yz \geq x\} = \begin{cases} 1 & \text{if } y \geq x \\ \frac{x}{y} & \text{if } y \not\geq x. \end{cases}$$

$$\infty \cdot a = a \cdot \infty = \infty, \forall a \in [1, \infty], \infty \ominus \infty = 1.$$

Put $n(x) = \infty \ominus x = \infty$ for $x \neq \infty$ and $n(\infty) = 1$. Then $n(n(x)) = 1$ for $x \neq \infty$ and $n(n(\infty)) = \infty$. Hence n does not satisfy a double negative law.

(4) $([0, \infty], \leq, \vee, \oplus = +, \wedge, 0, \infty)$ is a complete co-residuated lattice, where

$$\begin{aligned} y \ominus x &= \bigwedge\{z \in [0, \infty] \mid x + z \geq y\} \\ &= \bigwedge\{z \in [0, \infty] \mid z \geq -x + y\} = (y - x) \vee 0, \\ \infty + a &= a + \infty = \infty, \forall a \in [0, \infty], \infty \ominus \infty = 0. \end{aligned}$$

Put $n(x) = \infty \ominus x = \infty$ for $x \neq \infty$ and $n(\infty) = 0$. Then $n(n(x)) = 0$ for $x \neq \infty$ and $n(n(\infty)) = \infty$. Hence n does not satisfy a double negative law.

(5) $([0,1], \leq, \vee, \oplus, \wedge, 0, 1)$ is a complete co-residuated lattice where

$$\begin{aligned} x \oplus y &= (x^p + y^p)^{\frac{1}{p}} \wedge 1, \ \ 1 \leq p < \infty, \\ x \ominus y &= \bigwedge\{z \in [0,1] \mid (z^p + y^p)^{\frac{1}{p}} \geq x\} \\ &= \bigwedge\{z \in [0,1] \mid z \geq (x^p - y^p)^{\frac{1}{p}}\} = (x^p - y^p)^{\frac{1}{p}} \vee 0, \end{aligned}$$

Put $n(x) = 1 \ominus x = (1 - x^p)^{\frac{1}{p}}$ for $1 \leq p < \infty$. Then $n(n(x)) = x$ for $x \in [0,1]$. Hence n satisfies a double negative law.

(6) Let $P(X)$ be the collection of all subsets of X. Then $(P(X), \subset, \cup, \cap, \oplus = \cup, \varnothing, X)$ is a complete co-residuated lattice where

$$\begin{aligned} A \ominus B &= \bigwedge\{C \in P(X) \mid B \cup C \supset A\} \\ &= A \cap B^c = A - B. \end{aligned}$$

Put $n(A) = X \ominus A = A^c$ for each $A \subset X$. Then $n(n(A)) = A$. Hence n satisfies a double negative law.

Lemma 2.3 ([14])**.** *Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $x \oplus y \leq x \oplus z$, $y \ominus x \leq z \ominus x$ and $x \ominus z \leq x \ominus y$.*

(2) *$(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$ and $x \ominus (\bigwedge_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$.*

(3) *$(\bigwedge_{i\in\Gamma} x_i) \ominus y \leq \bigwedge_{i\in\Gamma}(x_i \ominus y)$*

(4) *$x \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \ominus y_i)$.*

(5) *$x \ominus x = \bot$, $x \ominus \bot = x$ and $\bot \ominus x = \bot$. Moreover, $x \ominus y = \bot$ iff $x \leq y$.*

(6) *$y \oplus (x \ominus y) \geq x$, $y \geq x \ominus (x \ominus y)$ and $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.*

(7) *$x \ominus (y \oplus z) = (x \ominus y) \ominus z = (x \ominus z) \ominus y$.*

(8) *$x \ominus y \geq (x \oplus z) \ominus (y \oplus z)$, $x \ominus y \geq (x \ominus z) \ominus (y \ominus z)$, $y \ominus x \geq (z \ominus x) \ominus (z \ominus y)$ and $(x \oplus y) \ominus (z \oplus w) \leq (x \ominus z) \oplus (y \ominus w)$.*

(9) *$x \oplus y = \bot$ iff $x = \bot$ and $y = \bot$.*

(10) *$(x \oplus y) \ominus z \leq x \oplus (y \ominus z)$ and $(x \ominus y) \oplus z \geq x \ominus (y \ominus z)$.*

(11) *If L satisfies a double negative law and $n(x) = \top \ominus x$, then $n(x \oplus y) = n(x) \ominus y = n(y) \ominus x$ and $x \ominus y = n(y) \ominus n(x)$.*

Definition 2.4 ([14])**.** Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. Let X be a set. A function $d_X : X \times X \to L$ is called a *distance function* if it satisfies the following conditions:

(M1) $d_X(x, x) = \bot$ for all $x \in X$,

(M2) $d_X(x, y) \oplus d_X(y, z) \geq d_X(x, z)$ for all $x, y, z \in X$,

(M3) If $d_X(x, y) = d_X(y, x) = \bot$, then $x = y$.

The pair (X, d_X) is called a *distance space.*

Remark 2.5 ([14])**.** (1) We define a distance function $d_X : X \times X \to [0, \infty]$. Then (X, d_X) is called a pseudo-quasi-metric space.

(2) Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \to L$ as $d_L(x, y) = x \ominus y$. By Lemma 2.3 (5) and (6), (L, d_L) is a distance space. Define a function $d_{L^X} : L^X \times L^X \to L$ as $d_{L^X}(A, B) = \bigvee_{x\in X}(A(x) \ominus B(x))$. Then (L^X, d_{L^X}) is a distance space.

3. Preserving maps in complete co-residuated lattices

In this section, we assume $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ is a complete co-residuated lattice.

Definition 3.1. (i) A map $\mathcal{F} : L^X \to L^Y$ is called an $\ominus$-*join preserving map*, i it satisfies the following conditions:

(J1) $\mathcal{F}(A \ominus \alpha) = \mathcal{F}(A) \ominus \alpha$,

(J2) $\mathcal{F}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{F}(A_i)$.

(ii) A map $\mathcal{G} : L^X \to L^Y$ is an $\oplus$- *meet preserving map*, i it satisfies the following conditions:

(M1) $\mathcal{G}(\alpha \oplus A) = \alpha \oplus \mathcal{G}(A)$,

(M2) $\mathcal{G}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{G}(A_i)$.

(iii) Let $\mathcal{F} : L^X \to L^Y$ and $\mathcal{G} : L^X \to L^Y$ be maps. The pair $(\mathcal{F}, \mathcal{G})$ is called a *residuated connection*, if $d_{L^Y}(B, \mathcal{F}(A)) = d_{L^X}(\mathcal{G}(B), A)$ for each $A \in L^X, B \in L^Y$.

Theorem 3.2. *If $\mathcal{F} : L^X \to L^Y$ and $\mathcal{G} : L^Y \to L^X$ such that $d_{L^Y}(\mathcal{F}(A), B) = d_{L^X}(A, \mathcal{G}(B))$ for all $A \in L^X, B \in L^Y$, then $\mathcal{F}$ is an $\ominus$-join preserving map and $\mathcal{G}$ is an $\oplus$- meet preserving map.*

Proof. Since
$$\begin{aligned} d_{L^Y}(\mathcal{F}(\textstyle\bigvee_{i\in\Gamma} A_i), B) &= d_{L^X}(\textstyle\bigvee_{i\in\Gamma} A_i, \mathcal{G}(B)) \\ &= \textstyle\bigvee_{i\in\Gamma} d_{L^X}(A_i, \mathcal{G}(B)) \text{ [By Lemma 2.3 (2)]} \\ &= \textstyle\bigvee_{i\in\Gamma} d_{L^Y}(\mathcal{F}(A_i), B) \\ &= d_{L^Y}(\textstyle\bigvee_{i\in\Gamma} \mathcal{F}(A_i), B), \end{aligned}$$
for $B = \bot_X$, $\mathcal{F}(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \mathcal{F}(A_i)$ by Lemma 2.3 (5). Since
$$\begin{aligned} d_{L^Y}(\mathcal{F}(A \ominus \alpha), B) &= d_{L^X}(A \ominus \alpha, \mathcal{G}(B)) \\ &= \textstyle\bigvee_{x\in X}((A(x) \ominus \alpha) \ominus \mathcal{G}(B)(x)) \text{ [By Lemma 2.3 (7)]} \\ &= \textstyle\bigvee_{x\in X}(A(x) \ominus \mathcal{G}(B)(x)) \ominus \alpha \\ &= d_{L^X}(\mathcal{F}(A), B) \ominus \alpha \\ &= d_{L^X}(\mathcal{F}(A) \ominus \alpha, B), \end{aligned}$$
we have $\mathcal{F}(A \ominus \alpha) = \mathcal{F}(A) \ominus \alpha$ for all $\alpha \in L$.

Since
$$\begin{aligned} d_{L^X}(A, \mathcal{G}(\textstyle\bigwedge_{i\in\Gamma} B_i)) &= d_{L^Y}(\mathcal{F}(A), \textstyle\bigwedge_{i\in\Gamma} B_i) \\ &= \textstyle\bigvee_{i\in\Gamma} d_{L^X}(A, \mathcal{G}(B_i)) = d_{L^X}(A, \textstyle\bigwedge_{i\in\Gamma} \mathcal{G}(B_i)), \end{aligned}$$
we get $\mathcal{G}(\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{i\in\Gamma} \mathcal{G}(B_i)$. On the other hand,
$$\begin{aligned} d_{L^X}(A, \mathcal{G}(\alpha \oplus B)) &= d_{L^Y}(\mathcal{F}(A), \alpha \oplus B) \\ &= \textstyle\bigvee_{y\in Y}((\mathcal{F}(A)(y) \ominus B(y)) \ominus \alpha) \\ &= \textstyle\bigvee_{x\in X}(A(x) \ominus \mathcal{G}(B)(x)) \ominus \alpha = d_{L^X}(A, \mathcal{G}(B)) \ominus \alpha \\ &= d_{L^X}(A, \mathcal{G}(B) \oplus \alpha) \text{ [By Lemma 2.3 (7)]}. \end{aligned}$$
Thus $\mathcal{G}(\alpha \oplus A) = \alpha \oplus \mathcal{G}(A)$ for all $\alpha \in L$. □

Theorem 3.3. (1) *Let $\mathcal{G} : L^Y \to L^X$ be an $\oplus$-meet preserving map. Then there exists an $\ominus$-join preserving map $\mathcal{F} : L^X \to L^Y$ such that $\mathcal{F}(A)(y) = d_{L^X}(A, \mathcal{G}(\bot_y))$. Moreover, $d_{L^Y}(\mathcal{F}(A), B) = d_{L^X}(A, \mathcal{G}(B))$ for each $A \in L^X, B \in L^Y$.*

(2) *Let $\mathcal{G} : L^Y \to L^X$ be an $\oplus$-meet preserving map. Then there exists a fuzzy relation $R \in L^{X\times Y}$ with $\mathcal{G}(B)(x) = \bigwedge_{y\in Y}(B(y) \oplus R(x, y))$ and an $\ominus$-join preserving map $\mathcal{F}(A)(y) = \bigvee_{x\in X}(A(x) \ominus R(x, y))$ such that $d_{L^Y}(\mathcal{F}(A), B) = d_{L^X}(A, \mathcal{G}(B))$ for each $A \in L^X, B \in L^Y$.*

(3) *If L satisfies a double negative law with $n(x) = \top \ominus x$ and $\mathcal{F} : L^X \to L^Y$ is an $\ominus$-join preserving map, then there exists an $\oplus$-meet preserving map $\mathcal{G} : L^Y \to L^X$*

such that $\mathcal{G}(\perp_y)(x) = n(\mathcal{F}(n(\perp_x)))(y)$. *Moreover,* $d_{L^Y}(\mathcal{F}(A), B) = d_{L^X}(A, \mathcal{G}(B))$ *for each* $A \in L^X, B \in L^Y$.

(4) *If* L *satisfies a double negative law with* $n(x) = \top \ominus x$ *and* $\mathcal{F} : L^X \to L^Y$ *is an* $\ominus$*-join-preserving map, then there exists a fuzzy relation* $R \in L^{X \times Y}$ *with* $\mathcal{F}(A)(y) = \bigvee_{x \in X}(A(x) \ominus R(x, y))$ *and an* $\oplus$*-meet preserving map* $\mathcal{G}(B)(x) = \bigwedge_{y \in Y}(B(y) \oplus R(x, y))$ *such that* $d_{L^Y}(\mathcal{F}(A), B) = d_{L^X}(A, \mathcal{G}(B))$ *for each* $A \in L^X, B \in L^Y$.

Proof. (1) Since $\mathcal{G}(\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{i\in\Gamma} \mathcal{G}(B_i)$ and $\mathcal{G}(\alpha \oplus B) = \alpha \oplus \mathcal{G}(B)$, for $B(w) = \bigwedge_{y\in Y}(B(y) \oplus \perp_y(w))$, $\mathcal{G}(B)(x) = \mathcal{G}(\bigwedge_{y\in Y}(B(y) \oplus \perp_y)(x) = \bigwedge_{y\in Y}(B(y) \oplus \mathcal{G}(\perp_y)(x))$. Then $A \in L^X$, by Lemma 2.3,

$$\begin{array}{ll} \mathcal{F}(A)(y) & = \bigwedge\{B(y) \mid \mathcal{G}(B) \geq A\} \\ & = \bigwedge\{B(y) \mid \bigwedge_{y\in Y}(B(y) \oplus \mathcal{G}(\perp_y)(x)) \geq A(x)\} \\ & = \bigwedge\{B(y) \mid B(y) \geq \bigvee_{x\in X}(A(x) \ominus \mathcal{G}(\perp_y)(x))\} \\ & = \bigvee_{x\in X}(A(x) \ominus \mathcal{G}(\perp_y)(x)). \end{array}$$

Moreover, $\mathcal{F}(\bigvee_{i\in\Gamma} A_i) = \bigvee_{x\in X}(\bigvee_{i\in\Gamma} A_i(x) \ominus \mathcal{G}(\perp_y)(x)) = \bigvee_{i\in\Gamma}(\bigvee_{x\in X}(A_i(x) \ominus \mathcal{G}(\perp_y)(x))) = \bigvee_{i\in\Gamma} \mathcal{F}(A_i)$ and $\mathcal{F}(A \ominus \alpha) = \bigvee_{x\in X}((A \ominus \alpha) \ominus \mathcal{G}(\perp_y)(x)) = \bigvee_{x\in X}(A(x) \ominus \mathcal{G}(\perp_y)(x))) \ominus \alpha = \mathcal{F}(A) \ominus \alpha$ from Lemma 2.3 (7). For $A \in L^X, B \in L^Y$, by Lemma 2.3,

$$\begin{array}{ll} d_{L^X}(A, \mathcal{G}(B)) & = \bigvee_{x\in X}(A(x) \ominus \mathcal{G}(B)(x)) \\ & = \bigvee_{x\in X}(A(x) \ominus \mathcal{G}(\bigwedge_{y\in Y}(B(y) \oplus \perp_y)(x)) \\ & = \bigvee_{x\in X} \bigvee_{y\in Y}(A(x) \ominus (B(y) \oplus \mathcal{G}(\perp_y)(x))) \\ & = \bigvee_{x\in X} \bigvee_{y\in Y}(A(x) \ominus \mathcal{G}(\perp_y)(x)) \ominus B(y)) \\ & = \bigvee_{y\in Y}((\bigvee_{x\in X}(A(x) \ominus \mathcal{G}(\perp_y)(x))) \ominus B(y)) \\ & = \bigvee_{y\in Y}(\mathcal{F}(A)(y) \ominus B(y)) \\ & = d_{L^Y}(\mathcal{F}(A), B). \end{array}$$

(2) By (1), put $R(x, y) = \mathcal{G}(\perp_y)(x)$. Then the result holds.

(3) Since $\mathcal{F}(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \mathcal{F}(A_i)$ and $\mathcal{F}(A \ominus \alpha) = \mathcal{F}(A) \ominus \alpha$ for $A = \bigvee_{x\in X}(A(x) \ominus \perp_x) = \bigvee_{x\in X}(n(\perp_x) \ominus n(A)(x))$,

$$\mathcal{F}(A)(y) = \mathcal{F}(\bigvee_{x\in X}(n(\perp_x) \ominus n(A)(x))(y) = \bigvee_{x\in X}(\mathcal{F}(n(\perp_x))(y) \ominus n(A)(x)).$$

For $B \in L^Y$,

$$\begin{array}{ll} \mathcal{G}(B)(x) & = \bigvee\{A(x) \mid \mathcal{F}(A)(y) \leq B(y)\} \\ & = \bigvee\{A(x) \mid \bigvee_{x\in X}(\mathcal{F}(n(\perp_x))(y) \ominus n(A)(x)) \leq B(y)\} \\ & = \bigvee\{A(x) \mid \bigvee_{y\in Y}(\mathcal{F}(n(\perp_x))(y) \ominus B(y)) \leq n(A)(x)\} \\ & = \bigvee\{A(x) \mid \bigwedge_{y\in Y}(n(\mathcal{F}(n(\perp_x)))(y) \oplus B(y)) \geq A(x)\} \\ & = \bigwedge_{y\in Y}(n(\mathcal{F}(n(\perp_x)))(y) \oplus B(y)). \end{array}$$

Moreover, $\mathcal{G}(\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{i\in\Gamma} \mathcal{G}(B_i)$, $\mathcal{G}(\alpha\oplus B) = \alpha\oplus\mathcal{G}(B)$ and $\mathcal{G}(\perp_y)(x) = n(\mathcal{F}(n(\perp_x)))(y)$. For each $A\in L^X, B\in L^Y$,

$$\begin{array}{ll} d_{L^X}(A,\mathcal{G}(B)) & = \bigvee_{x\in X}(A(x)\ominus\mathcal{G}(B)(x)) \\ & = \bigvee_{x\in X}(A(x)\ominus\bigwedge_{y\in Y}(n(\mathcal{F}(n(\perp_x)))(y)\oplus B(y))) \\ & = \bigvee_{x\in X}\bigvee_{y\in Y}(A(x)\ominus(n(\mathcal{F}(n(\perp_x)))(y)\oplus B(y))) \\ & = \bigvee_{x\in X}\bigvee_{y\in Y}(A(x)\ominus n(\mathcal{F}(n(\perp_x)))(y))\ominus B(y)) \\ & = \bigvee_{y\in Y}((\bigvee_{x\in X}(\mathcal{F}(n(\perp_x))(y)\ominus n(A)(x)))\ominus B(y)) \\ & = \bigvee_{y\in Y}(\mathcal{F}(\bigvee_{x\in X}(n(\perp_x)(y)\ominus n(A)(x)))\ominus B(y)) \\ & = \bigvee_{y\in Y}(\mathcal{F}(A)(y)\ominus B(y)) \\ & = d_{L^Y}(\mathcal{F}(A),B). \end{array}$$

(4) By (3), put $R(x,y) = n(\mathcal{F}(n(\perp_x)))(y) = \mathcal{G}(\perp_y)(x))$.

$$\begin{array}{ll} \mathcal{F}(A)(y) & = \bigvee_{x\in X}(\mathcal{F}(n(\perp_x))(y)\ominus n(A)(x)) \\ & = \bigvee_{x\in X}(A(x)\ominus n(\mathcal{F}(n(\perp_x)))(y)) = \bigvee_{x\in X}(A(x)\ominus R(x,y)), \\ \mathcal{G}(B)(x) & = \bigwedge_{y\in Y}(n(\mathcal{F}(n(\perp_x)))(y)\oplus B(y)) = \bigwedge_{y\in Y}(R(x,y)\oplus B(y)). \end{array}$$

Then the result holds. □

Remark 3.4. Let L be satisfied a double negative law with $n(x) = \top\ominus x$ and $f : X\to Y$ be a map. A map $f^{\to} : L^X\to L^Y$ is defined as $f^{\to}(A)(y) = \bigvee_{x\in f^{-1}(\{y\})} A(x)$. Then

$$\begin{array}{ll} f^{\to}(A\ominus\alpha)(y) & = \bigvee_{x\in f^{-1}(\{y\})}(A\ominus\alpha)(x) \\ & = (\bigvee_{x\in f^{-1}(\{y\})} A(x))\ominus\alpha = f^{\to}(A)(y)\ominus\alpha \text{ [By Lemma 2.3 (2)]} \end{array}$$

and $f^{\to}(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} f^{\to}(A_i)$. Thus $f^{\to} : L^X\to L^Y$ is an $\ominus$-join preserving map. By Theorem 3.3 (3), there exists $\mathcal{G} : L^Y\to L^X$ defined as:

$$\begin{array}{ll} \mathcal{G}(B)(x) & = \bigwedge_{y\in Y}(n(f^{\to}(n(\perp_x)))(y)\oplus B(y)) \\ & = \bigwedge_{y\in Y}(n(\bigvee_{z\in f^{-1}(\{y\})} n(\perp_x)(z)\oplus B(y)) \\ & = \bigwedge_{y\in Y}(\bigwedge_{z\in f^{-1}(\{y\})} \perp_x(z)\oplus B(y)) \\ & = \perp_x(x)\oplus B(f(x)) = f^{\leftarrow}(B)(x) \end{array}$$

such that $\mathcal{G}(\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{i\in\Gamma}\mathcal{G}(B_i)$, $\mathcal{G}(\alpha\oplus A) = \alpha\oplus\mathcal{G}(A)$ with $f^{\to}(n(\perp_x))(y) = d_{L^X}(n(\perp_x),\mathcal{G}(\perp_y)) = n(\mathcal{G}(\perp_y))(x)$.

Moreover, $d_{L^Y}(f^{\to}(A),B) = d_{L^X}(A,\mathcal{G}(B))$ for each $A\in L^X, B\in L^Y$.

Example 3.5. Let $X = \{a,b,c\}$ and $Y = \{x,y,z\}$ be sets. We define $f : X\to Y$ with $f(a) = x, f(b) = f(c) = y$.

(1) A map $f^{\to} : L^X\to L^Y$ is defined as $f^{\to}(A)(y) = \bigvee_{x\in f^{-1}(\{y\})} A(x)$. Then

$$f^{\to}(A)(x) = A(a),\ f^{\to}(A)(y) = A(b)\vee A(c),\ f^{\to}(A)(z) = 0$$

$$f^{\to}(n(0_a)) = (1,0,0),\ f^{\to}(n(0_b)) = (0,1,0),\ f^{\to}(n(0_c)) = (0,1,0)$$

and $f^{\to} : L^X\to L^Y$ is an $\ominus$-join preserving map. Then there exists $\mathcal{G} : L^Y\to L^X$ defined as $\mathcal{G}(B)(x) = \bigwedge_{y\in Y}(n(f^{\to}(n(0_x)))(y)\oplus B(y)) = f^{\leftarrow}(B)(x)$ as follows:

$$\mathcal{G}(B)(a) = f^{\leftarrow}(B)(a) = B(x),\ \mathcal{G}(B)(b) = B(y),\ \mathcal{G}(B)(c) = B(y)$$

such that $\mathcal{G}(\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{i\in\Gamma}\mathcal{G}(B_i)$, $\mathcal{G}(\alpha\oplus A) = \alpha\oplus\mathcal{G}(A)$ with $f^{\to}(n(0_x))(y) = d_{L^X}(n(0_x),\mathcal{G}(0_y)) = n(\mathcal{G}(0_y))(x)$. Moreover, $d_{L^Y}(f^{\to}(A),B) = d_{L^X}(A,\mathcal{G}(B) =$

$f^{\leftarrow}(B))$ for each $A \in L^X, B \in L^Y$. Put $R(a,y) = n(f^{\rightarrow}(n(0_a)))(y)$ as

$$R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then

$$f^{\rightarrow}(A)(y) = \bigvee_{a\in X}(A(a) \ominus R(a,y)), \mathcal{G}(B)(a) = f^{\leftarrow}(B)(a) = \bigwedge_{y\in Y}(B(y) \oplus R(a,y)).$$

(2) A map $f^{\leftarrow} : L^Y \to L^X$ is defined as $f^{\leftarrow}(B)(x) = B(f(x))$. Then

$$f^{\leftarrow}(B)(a) = B(f(a)),\ f^{\leftarrow}(B)(b) = B(f(b)),\ f^{\leftarrow}(B)(c) = B(f(c))$$

$$f^{\leftarrow}(n(0_x)) = (1,0,0),\ f^{\leftarrow}(n(0_y)) = (0,1,1),\ f^{\leftarrow}(n(0_z)) = (1,1,1).$$

Since $f^{\leftarrow}(\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{i\in\Gamma} f^{\leftarrow}(B_i)$, $f^{\leftarrow}(\alpha \oplus B) = \alpha \oplus f^{\leftarrow}(B)$, $f^{\leftarrow} : L^Y \to L^X$ is an $\oplus$-meet preserving map. By Theorem 3.3 (1), there exists an $\ominus$-join preserving map $\mathcal{F} : L^X \to L^Y$ such that $\mathcal{F}(A)(y) = d_{L^X}(A, f^{\leftarrow}(0_y)) = \bigvee_{x\in f^{-1}(\{y\})} A(x) = f^{\rightarrow}(A)(y)$. Moreover, $d_{L^Y}(\mathcal{F}(A) = f^{\rightarrow}(A), B) = d_{L^X}(A, f^{\leftarrow}(B))$ for each $A \in L^X, B \in L^Y$. Put $R(a,y) = f^{\leftarrow}(0_y)(a)$ as

$$R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then

$$\mathcal{F}(A)(y) = f^{\rightarrow}(A)(y) = \bigvee_{a\in X}(A(a) \ominus R(a,y)),$$
$$f^{\leftarrow}(B)(a) = \bigwedge_{y\in Y}(B(y) \oplus R(a,y)).$$

(3) Let $([0,1], \oplus, \ominus, n, 0, 1)$ be a complete co-residuated lattice as $n(x) = 1 - x$ and

$$x \oplus y = (x+y) \wedge 1,\ x \ominus y = (x-y) \vee 0.$$

Let $d_X \in [0,1]^{X\times X}, d_Y \in [0,1]^{Y\times Y}$ be distance functions as follows:

$$d_X = \begin{pmatrix} 0 & 0.3 & 0 \\ 0.4 & 0 & 0.2 \\ 0.5 & 0.4 & 0 \end{pmatrix},\ d_Y = \begin{pmatrix} 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.7 \\ 0.3 & 0.6 & 0 \end{pmatrix}.$$

A map $\mathcal{G} : L^X \to L^Y$ is defined as $\mathcal{G}(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_Y(f(x), y)$. Then $\mathcal{G}$ is an $\oplus$-meet preserving map. By Theorem 3.3 (1), there exists an $\ominus$-join preserving map $\mathcal{F} : L^Y \to L^X$ defined as $\mathcal{F}(B)(x) = d_{L^Y}(B, \mathcal{G}(0_x)) = \bigvee_{y\in Y}(B(y) \ominus \mathcal{G}(0_x)(y)) = \bigvee_{y\in Y}(B(y) \ominus R(x,y))$ with $R(x,y) = \mathcal{G}(0_x)(y) = d_Y(f(x), y)$ as follows:

$$R = \begin{pmatrix} 0 & 0.6 & 0.6 \\ 0.5 & 0 & 1 \\ 0.5 & 0 & 1 \end{pmatrix}.$$

(4) In (3), a map $\mathcal{F} : L^Y \to L^X$ is defined as:

$$\mathcal{F}(B)(x) = \bigvee_{z\in X}(B(f(z)) \ominus d_X(z,x)).$$

Then by Remark 3.7 (1), $\mathcal{F}$ is an $\ominus$-join preserving map. On the other hand,

$$R(x,y) = n(\mathcal{F}(n(0_y))(x) = n(\bigvee_{z\in X}(n(0_y)(f(z)) \ominus d_X(z,x)))$$

$= n(\bigvee_{z\in X}(n(d_X(z,x)) \ominus 0_y(f(z)))$ [By Lemma 2.3 (2)]
$= \bigwedge_{z\in X}(d_X(z,x) \oplus 0_y(f(z)))$ [By Lemma 2.3 (11)]

as follows:

$$R = \begin{pmatrix} 0 & 0.4 & 1 \\ 0.3 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{F} : L^Y \to L^X$ is defined as $\mathcal{F}(B)(x) = \bigvee_{y\in Y}(B(y) \ominus R(x,y))$. By Theorem 3.3 (3), there exists an $\oplus$-meet preserving map $\mathcal{G} : L^X \to L^Y$ defined as $\mathcal{G}(A)(y) = \bigwedge_{x\in X}(A(x) \ominus R(x,y))$ with $d_{L^X}(\mathcal{F}(B), A) = d_{L^X}(B, \mathcal{G}(A))$.

Example 3.6. Let $X = \{x, y, z\}$ be a set and $([0,1], \oplus, \ominus, n, 0, 1)$ be a complete co-residuated lattice as $n(x) = 1 - x$ and

$$x \oplus y = (x + y) \wedge 1, \;\; x \ominus y = (x - y) \vee 0.$$

Put $D = (0.7, 0.4, 0.2) \in [0,1]^X$.

(1) Define a map $\mathcal{F} : L^X \to L^X$ as $\mathcal{F}(A)(y) = d_{L^X}(A, D) \ominus D(y)$. Then
$\mathcal{F}(A \ominus \alpha)(y) = d_{L^X}(A \ominus \alpha, D) \ominus D(y)$
$= (d_{L^X}(A, D) \ominus \alpha) \ominus D(y)$ [By Lemma 2.3 (2)]
$= (d_{L^X}(A, D) \ominus D(y)) \ominus \alpha = (\mathcal{F}(A) \ominus \alpha)(y)$ [By Lemma 2.3 (7)],
$\mathcal{F}(\bigvee_{i\in\Gamma} A_i)(y) = d_{L^X}(\bigvee_{i\in\Gamma} A_i, D) \ominus D(y)$
$= \bigvee_{i\in\Gamma}(d_{L^X}(A_i, D) \ominus D(y))$
$= \bigvee_{i\in\Gamma} \mathcal{F}(A_i)(y)$.
Thus $\mathcal{F} : L^X \to L^X$ is an $\ominus$-join preserving map. By Theorem 3.3 (3), there exists $\mathcal{G} : L^X \to L^X$ defined as $\mathcal{G}(B)(x) = \bigwedge_{y\in X}(n(\mathcal{F}(n(0_x)))(y) \oplus B(y))$ such that

$$\mathcal{G}(\bigwedge_{i\in\Gamma} B_i) = \bigwedge_{i\in\Gamma} \mathcal{G}(B_i), \;\; \mathcal{G}(\alpha \oplus B) = \alpha \oplus \mathcal{G}(B)$$

with $\mathcal{F}(n(0_x))(y) = d_{L^X}(n(0_x), \mathcal{G}(0_y)) = n(\mathcal{G}(0_y))(x)$.

Moreover, $d_{L^X}(F(A), B) = d_{L^X}(A, \mathcal{G}(B))$ for each $A, B \in L^X$. By Theorem 3.4 (4), put $R(x,y) = n(F(n(0_x)))(y)$. Since

$$\mathcal{F}(n(0_x))(y) = (\bigvee_{z\in X} (n(0_x)(z) \ominus D(z))) \ominus D(y) = n(D(x))) \ominus D(y),$$

we have $R(x,y) = n(\mathcal{F}(n(0_x)))(y) = n(n(D(x)) \ominus D(y)) = D(x) \oplus D(y)$ as

$$R = \begin{pmatrix} 1 & 1 & 0.9 \\ 1 & 0.8 & 0.6 \\ 0.9 & 0.6 & 0.4 \end{pmatrix}.$$

So

$$\mathcal{F}(A)(y) = \bigvee_{x\in X}(A(x) \ominus R(x,y)),$$
$$\mathcal{G}(B)(x) = \bigwedge_{y\in Y}(B(y) \oplus R(x,y)).$$

(2) Define a map $\mathcal{G} : L^X \to L^X$ as $\mathcal{G}(A)(y) = \bigwedge_{x\in X}(n(D)(x) \oplus A(x)) \oplus D(y)$. Since $\mathcal{G}(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{G}(A_i)$, $\mathcal{G}(\alpha \oplus A) = \alpha \oplus \mathcal{G}(A)$, $\mathcal{G} : L^X \to L^X$ is an $\oplus$-meet preserving map. By Theorem 3.3 (1), there exists an $\ominus$-join preserving map $\mathcal{F} : L^X \to L^X$ defined as $\mathcal{F}(B)(x) = d_{L^X}(B, G(0_x)) = \bigvee_{y\in X}(B(y) \ominus (D(y) \oplus n(D(x)))$

with $\mathcal{F}(n(0_y))(x) = d_{L^X}(n(0_y), \mathcal{G}(0_x)) = n(\mathcal{G}(0_x))(y)$. Put $R(x,y) = \mathcal{G}(0_x)(y) = n(D)(x) \oplus D(y)$ as

$$R = \begin{pmatrix} 1 & 0.7 & 0.5 \\ 1 & 1 & 0.8 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

$$\begin{array}{l} \mathcal{G}(A)(y) = \bigwedge_{x\in X}(A(x) \oplus R(x,y)) \\ \mathcal{F}(B)(x) = \bigvee_{y\in X}(B(y) \ominus R(x,y)). \end{array}$$

4. Conclusion

The distance function instead of fuzzy partially ordered set is a new notion. We investigated the relations among residuated connections, $\ominus$-join preserving maps and $\oplus$-meet preserving maps on complete co-residuated lattices.

In the future, fuzzy rough sets, information systems and decision rules are investigated by using the concepts of distance spaces in complete co-residuated lattices.

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References

[1] M. Ward and R. P. Dilworth, Residuated lattices, Trans. Amer. Math. Soc. 45 (1939) 335–354.
[2] R. Bělohlávek, Fuzzy Relational Systems, Kluwer Academic Publishers, New York, 2002.
[3] P. Hájek, Metamathematices of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.
[4] U. Höhle and E. P. Klement, Non-classical logic and their applications to fuzzy subsets, Kluwer Academic Publishers, Boston, 1995.
[5] U. Höhle and S. E. Rodabaugh, Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht, 1999.
[6] S. E. Rodabaugh and E. P. Klement, Topological and Algebraic Structures In Fuzzy Sets, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.
[7] E. Turunen, Mathematics Behind Fuzzy Logic, A Springer-Verlag Co., 1999.
[8] Z. Pawlak, Rough sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving, Kluwer Academic Publishers, Dordrecht, The Netherlands 1991.
[9] Y. C. Kim, Categories of fuzzy preorders, approximation operators and Alexandrov topologies, Journal of Intelligent and Fuzzy Systems 31 (2016) 1787–1793.
[10] Y. C. Kim and J. M Ko, Fuzzy complete lattices, Alexandrov L-fuzzy topologies and fuzzy rough sets, Journal of Intelligent and Fuzzy Systems 38 (2020) 3253–3266.
[11] H. Lai and D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory, Int. J. Approx. Reasoning 50 (2009) 695–707.
[12] M. C. Zheng and G. J. Wang, Coresiduated lattice with applications, Fuzzy systems and Mathematics 19 (2005) 1–6.
[13] Q. Junsheng and Hu. Bao Qing, On $(\odot, \&)$ -fuzzy rough sets based on residuated and co-residuated lattices, Fuzzy Sets and Systems, 336 (2018) 54–86.
[14] Y. C. Kim and J. M. Ko, Preserving maps and approximation operators in complete co-residuated lattices, Journal of the Korean Insitutute of Intelligent Systems 30 (5) (2020) 389–398.

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