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On the characterisation of anti-fuzzy multigroups

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Abstract. The concept of fuzzy multigroups applies fuzzy multisets to the theory of groups. Fuzzy multigroup is the study of group structures in fuzzy multiset context. This article introduces anti-fuzzy multigroup and characterises certain of its properties. It is established that a fuzzy multiset with group as its underlying set is a fuzzy multigroup if and only if the complement of the fuzzy multiset is an anti-fuzzy multigroup. Certain results that discuss the alpha-cuts of anti-fuzzy multigroups are explored.

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1. Introduction

The idea of fuzzy subset of a set initiated by Zadeh [1] marked a new direction in mathematics and stirred the interest of myriad of researchers to apply the concept to diverse mathematics notions. Fuzzy sets provided a vehicle for the modelling of vagueness and fuzziness enmeshed in phenomena that lack sharp boundaries. In 1971, Rosenfeld [2] utilised fuzzy subset of a set to initiate the idea of fuzzy subgroup of a group, which caught algebraists' attention like wild fire, and there seems to be no end to its ramifications. Since inception, certain group's related ideas have been investigated in fuzzy group setting, as seen in [3, 4, 5]. In 1990, Biswas [6] first introduced anti-fuzzy subgroups in fuzzy group theory. Majumdar and Sardar [7] presented some properties of anti-fuzzy subgroups. Some reviewed results on anti-fuzzy subgroups were discussed in [8, 9]. Zhengwei [10] presented some results on anti-fuzzy subgroups. Gayen et al. [11] generalised the notion of anti-fuzzy subgroup using the idea of t-conorm.

The concept of anti-fuzzy subgroups has been extended to intuitionistic fuzzy set [12] as seen in [13]. The homomorphic properties of anti-intuitionistic fuzzy subgroups have been discussed [14]. Alghazzawi et al. [15] introduced the notion of anti-intuitionistic fuzzy subgroup using a certain averaging operator. Some varieties of anti-intuitionistic fuzzy subgroup have been studied [16, 17].

In 1986, Yager [18] utilised multiset [19] to introduce fuzzy multiset. Certainly, fuzzy multisets give room for the repetition of membership grades of their elements in multiset context, and by so doing generalises fuzzy sets. Fuzzy multisets have been studied and used in making decisions [20, 21, 22, 23, 24, 25, 26].

The concept of fuzzy multigroups [27] is a group-like algebraic structure in fuzzy multiset setting, which generalises fuzzy groups. Fuzzy multigroups and fuzzy groups both generalises crisp groups, but while all fuzzy groups are fuzzy multigroups the converse are not valid. Fuzzy submultigroups, normal fuzzy submultigroups, commutative fuzzy multigroups and homomorphism in fuzzy multigroups context were explored in [28, 29, 30, 31, 32] with certain number of results. The characterisation of the concepts of direct product, cosets, quotient groups, commutators, Frattini subgroups as well as characteristic subgroups have been discussed in fuzzy multigroup context [33, 34, 35, 36, 37, 38, 39]. The notion of alpha-cuts of fuzzy multigroups together with its homomorphism were explored in [40, 41]. Some results of multigroups were fuzzified to fuzzy multigroups in [42].

The notion of fuzzy subgroup has been generalised to intuitionistic fuzzy subgroup and fuzzy multigroup (while fuzzy multigroup allows repetition of membership function of fuzzy subgroup, intuitionistic fuzzy subgroup incorporates additional function called non-membership function to the idea of fuzzy subgroup). Since the idea of anti-fuzzy subgroup has been extended to intuitionistic fuzzy subgroups, it is expedient to explore the idea in fuzzy multigroup setting. The motivation of this paper is to extend the notions of anti-fuzzy subgroups and anti-intuitionistic fuzzy subgroups to fuzzy multigroup environment and to present some new results. In this paper, we propose anti-fuzzy multigroups and obtain some relevant theorems. For easy flow the article is outlined thus; Section 2 presents certain preliminary definitions and results, Section 3 introduces anti-fuzzy multigroups with certain number of results, and Section 4 concludes the article and recommends future study directions.

2. Preliminaries

Certain preliminaries which are of relevant important are presented here.

2.1. Fuzzy multisets.

For a non-empty set Y, we take the collection of all fuzzy multisets to be $FMS(Y)$.

Definition 2.1 ([18]). A *fuzzy multiset* E of Y is characterised by a count membership function

$$CM_E : Y \to M,$$

in which M stands for collection of all multisets from $I = [0, 1]$ and

$$CM_E(a) = \{\langle \mu_E^1(a), \mu_E^2(a), ..., \mu_E^n(a), ...\rangle | a \in Y\},$$

where $\mu_E^1(a), \mu_E^2(a), ..., \mu_E^n(a), ... \in [0, 1]$ and $\mu_E^1(a) \geq \mu_E^2(a) \geq ... \geq \mu_E^n(a) \geq ....$

Suppose that E is finite. Then

$$CM_E(a) = \{\langle \mu_E^1(a), \mu_E^2(a), ..., \mu_E^n(a)\rangle | a \in Y\},$$

for $\mu_E^1(a) \geq \mu_E^2(a) \geq ... \geq \mu_E^n(a)$. We can write E in the form

$$E = \{\langle \frac{CM_E(a)}{a} \rangle \mid a \in Y\}.$$

Certain properties of fuzzy multisets have been discussed in [20, 23, 24, 25, 26].

Definition 2.2 ([22])**.** The *complement* of E in $FMS(Y)$, i.e., E^c, is a fuzzy multiset of Y given as

$$E^c = \{\langle \frac{1 - CM_E(a)}{a} \rangle | a \in Y\},$$

such that $\mu_E^1(a) \leq \mu_E^2(a) \leq ... \leq \mu_E^n(a)$. Clearly, $(E^c)^c = E$.

Definition 2.3 ([22])**.** For $E, F \in FMS(Y)$, we say E *is contained in* F, symbolised by $E \subseteq F$, if $CM_E(a) \leq CM_F(a)\ \forall a \in Y$. Again, E *is properly contained in* F, symbolised by $E \subset F$, if $E \subseteq F$ and $E \neq F$.

Definition 2.4 ([25])**.** Take $\{E_j\}_{i \in J}$ as a family of fuzzy multisets of Y. Then

(i) $CM_{\bigcup_{j \in J} E_j}(a) = \bigvee_{j \in J} CM_{E_j}(a)\ \forall a \in Y$,

(ii) $CM_{\bigcap_{j \in J} E_j}(a) = \bigwedge_{j \in J} CM_{E_j}(a)\ \forall a \in Y$,

where $\bigvee$ and $\bigwedge$ stand for maximum and minimum operations.

Definition 2.5 ([22])**.** Fuzzy multisets E and F of Y are *equal*, if $CM_E(a) = CM_F(a)\ \forall a \in Y$. E and F of Y are *comparable* if $E \subseteq F$ or $F \subseteq E$.

2.2. **Fuzzy multigroups.** Take Y as a group and $FMG(Y)$ as the collection of all fuzzy multigroups by $FMG(Y)$.

Definition 2.6 ([29])**.** Assume E is a fuzzy multiset of Y. We say E is a *fuzzy multigroupoid* of Y, if $CM_E(ab) \geq CM_E(a) \wedge CM_E(b)$ for all $a, b \in Y$. Again, E *becomes a fuzzy multigroup* of Y, if $CM_E(a^{-1}) = CM_E(a)\ \forall a \in Y$ in addition of being a fuzzy multigroupoid.

Alternatively, a fuzzy multiset E of Y is a fuzzy multigroup of a group $Y \iff CM_E(ab^{-1}) \geq CM_E(a) \wedge CM_E(b)\ \forall a, b \in Y$.

Definition 2.7 ([29])**.** If $\{E_j\}_{j \in J}, J = 1, ..., m$ is a family of fuzzy multigroups of Y. We then say $\{E_j\}_{j \in J}$ *possess inf/sup assuming chain*, if $E_1 \subseteq E_2 \subseteq ... \subseteq E_m$/ $E_1 \supseteq E_2 \supseteq ... \supseteq E_m$.

In what follows, we present a result on fuzzy multigroup.

Theorem 2.8. *Every fuzzy multigroupoid of a finite group is a fuzzy multigroup.*

Proof. Take E as a fuzzy multigroupoid of a finite group Y. Assume $a \in Y$ such that $a \neq e$. Since Y is finite, a has finite order $m \in \mathbb{N}$, where $m > 1$. Then $a^m = e$ and thus $a^{m-1} = a^{-1}$. Synthesising the definition of fuzzy multigroupoid repeatedly, we get

$$\begin{aligned} CM_E(a^{-1}) = CM_E(a^{m-1}) &= CM_E(a^{m-2}x) \\ &\geq CM_E(a^{m-2}) \wedge CM_E(a) \\ &\geq CM_E(a) \wedge \ldots \wedge CM_E(a) \\ &= CM_E(a). \end{aligned}$$

Since E is a fuzzy multigroupoid and $CM_E(a^{-1}) = CM_E(a)$, E is fuzzy multigroup of Y. □

3. Anti-fuzzy multigroups

This section presents anti-fuzzy multigroup as a fuzzy multigroup in reverse order and characterises its properties. We denote a group by Y unless otherwise stated. We represent the collection of all anti-fuzzy multigroups of Y by $AFMG(Y)$.

Definition 3.1. A fuzzy multiset E of Y is an *anti-fuzzy multigroupoid* of Y, if $CM_E(ab) \leq CM_E(a) \vee CM_E(b)$ $\forall a, b \in Y$.

A fuzzy multiset E is an anti-fuzzy multigroup of Y, if it satisfies

(i) $CM_E(ab) \leq CM_E(a) \vee CM_E(b)$ $\forall a, b \in Y$,

(ii) $CM_E(a^{-1}) \leq CM_E(a)$ $\forall a \in Y$.

Certainly, an anti-fuzzy multigroup is an anti-fuzzy multigroupoid such that $CM_E(a^{-1}) \leq CM_E(a)$ $\forall a \in Y$.

Example 3.2. Let $Y = \{e, a_1, a_2, a_3\}$ be a group, where

$$a_1a_2 = a_3, a_1a_3 = a_2, a_2a_3 = a_1, a_1^2 = a_2^2 = a_3^2 = e.$$

Then the fuzzy multiset

$$E = \{\langle\frac{0.0, 0.1}{e}\rangle, \langle\frac{0.3, 0.5}{a_1}\rangle, \langle\frac{0.2, 0.4}{a_2}\rangle, \langle\frac{0.3, 0.5}{a_3}\rangle\}$$

is an anti-fuzzy multigroup of Y.

3.1. **Certain properties of anti-fuzzy multigroups.**

Here, we discuss some properties of anti fuzzy multigroups.

Proposition 3.3. *Suppose E is an anti-fuzzy multigroup of Y. Then the following hold:*

(1) $CM_E(a^{-1}) = CM_E(a)$ $\forall a \in Y$,

(2) $CM_E(e) \leq CM_E(a)$ $\forall a \in Y$, *where e is the identity of Y,*

(3) $CM_E(a^m) \leq CM_E(a)$ $\forall a \in Y$, $m \in \mathbb{N}$.

Proof. We present the verifications of (1) to (3) as below.

(1) By Definition 3.1, $CM_E(a^{-1}) \leq CM_E(a)$ $\forall a \in Y$. Also,

$$CM_E(a) = CM_E((a^{-1})^{-1}) \leq CM_E(a^{-1}).$$

This completes the proof of (1).

(2) Suppose $a \in Y$. Then clearly, $aa^{-1} = e$. Thus we have

$$\begin{aligned} CM_E(e) = CM_E(aa^{-1}) &\leq CM_E(a) \vee CM_E(a) \\ &= CM_E(a). \end{aligned}$$

So $CM_E(e) \leq CM_E(a)$ $\forall a \in Y$.

(3) For $m \in \mathbb{N}$, we have

$$\begin{aligned} CM_E(a^m) &\leq CM_E(a^{m-1}) \vee CM_E(a) \\ &\leq CM_E(a^{m-2}) \vee CM_E(a) \vee CM_E(a) \\ &\leq CM_E(a) \vee CM_E(a) \vee ... \vee CM_E(a) \\ &= CM_E(a) \ \forall a \in Y. \end{aligned}$$

□

Definition 3.4. The *inverse* of an element $a \in Y$ in an anti-fuzzy multigroup E of Y is defined by

$$CM_E(a^{-1}) = CM_{E^{-1}}(a)\ \forall a \in Y.$$

It is deducible that, $CM_{E^{-1}}(a) = CM_E(a) = CM_{(E^{-1})^{-1}}(a)$.

Proposition 3.5. *A fuzzy multiset E is an anti-fuzzy multigroup of Y if and only if $CM_E(ab^{-1}) \leq CM_E(a) \vee CM_E(b)\ \forall a, b \in Y$.*

Proof. Suppose E is an anti-fuzzy multigroup of Y. Then the following conditions hold:

$$CM_E(ab) \leq CM_E(a) \vee CM_E(b)\ \forall a, b \in Y \text{ and } CM_E(a^{-1}) \leq CM_E(a)\ \forall a \in Y.$$

By combining the conditions, we get

$$CM_E(ab^{-1}) \leq CM_E(a) \vee CM_E(b)\ \forall a, b \in Y.$$

Conversely, suppose the given condition is satisfied. Combining the following facts:

$$CM_E(e) \leq CM_E(a),\ CM_E(a^{-1}) = CM_E(a)\ \forall a \in Y$$

and

$$\begin{aligned} CM_E(ab) \leq CM_E(a(b^{-1})^{-1}) &\leq CM_E(a) \vee CM_E(b^{-1}) \\ &= CM_E(a) \vee CM_E(b)\ \forall a, b \in Y, \end{aligned}$$

we conclude that $E \in AFMG(Y)$. □

Theorem 3.6. *Every anti-fuzzy multigroupoid of a finite group is an anti-fuzzy multigroup.*

Proof. Assume $E \in AFMG(Y)$. Since $Y \neq \varnothing$ and non-trivial, there exists atleast $a \in Y$. Since Y is finite, the order of a is finite. Then $a^m = e \Rightarrow a^{-1} = a^{m-1}$. Utilising the definition of anti-fuzzy multigroupoid repeatedly, we get

$$\begin{aligned} CM_E(a^{-1}) &= CM_E(a^{m-1}) = CM_E(a^{m-2}a) \\ &\leq CM_E(a^{m-2}) \vee CM_E(a) \\ &\leq CM_E(a) \vee ... \vee CM_E(a) \\ &= CM_E(a). \end{aligned}$$

Thus the result holds. □

Theorem 3.7. *Suppose E is a fuzzy multiset of Y. Then $E \in FMG(Y)$ if and only if $E^c \in AFMG(Y)$.*

Proof. Suppose $E \in FMG(Y)$ and let $a,\ b \in Y$. Then we have:

$$\begin{aligned} &CM_E(ab^{-1}) \geq CM_E(a) \wedge CM_E(b) \\ &\Rightarrow CM_{(E^c)^c}(ab^{-1}) \geq CM_{(E^c)^c}(a) \wedge CM_{(E^c)^c}(b) \\ &\Rightarrow 1 - CM_{E^c}(ab^{-1}) \geq 1 - CM_{E^c}(a) \wedge 1 - CM_{E^c}(b) \\ &\Rightarrow -CM_{E^c}(ab^{-1}) \geq -1 + [1 - CM_{E^c}(a) \wedge 1 - CM_{E^c}(b)] \\ &\Rightarrow CM_{E^c}(ab^{-1}) \leq 1 - [1 - CM_{E^c}(a) \wedge 1 - CM_{E^c}(b)] \end{aligned}$$

$\Rightarrow CM_{E^c}(ab^{-1}) \leq CM_{E^c}(a) \vee CM_{E^c}(b)$.
Thus $E^c \in AFMG(Y)$.

Conversely, suppose $E^c \in AFMG(Y)$ and let $a,\ b \in Y$. Then we have:
$CM_{E^c}(ab^{-1}) \leq CM_{E^c}(a) \vee CM_{E^c}(b)$
$\Rightarrow 1 - CM_E(ab^{-1}) \leq 1 - CM_E(a) \vee 1 - CM_E(b)$
$\Rightarrow -CM_E(ab^{-1}) \leq -1 + [1 - CM_E(a) \vee 1 - CM_E(b)]$
$\Rightarrow CM_E(ab^{-1}) \geq 1 - [1 - CM_E(a) \vee 1 - CM_E(b)]$
$\Rightarrow CM_E(ab^{-1}) \geq CM_E(a) \wedge CM_E(b)$. Thus $E \in FMG(Y)$. □

Proposition 3.8. *Let $E \in AFMG(Y)$. If $CM_E(a) > CM_E(b)$ for some $a, b \in Y$, then $CM_E(ab) = CM_E(a) = CM_E(ba)$.*

Proof. Suppose $CM_E(a) > CM_E(b)$ for some $a, b \in Y$. Now,

$$CM_E(ab) \leq CM_E(a) \vee CM_E(b) = CM_E(a).$$

Similarly,

$$CM_E(a) = CM_E(abb^{-1}) \leq CM_E(ab) \vee CM_E(b) = CM_E(ab).$$

Then $CM_E(ab) = CM_E(a)$. In the same vein, $CM_E(ba) = CM_E(a)$. Thus the result follows. □

Proposition 3.9. *Suppose $E \in AFMG(Y)$. Then $CM_E(ab^{-1}) = CM_E(e)$ if and only if $CM_E(a) = CM_E(b)$.*

Proof. Suppose $CM_E(ab^{-1}) = CM_E(e)\ \forall a, b \in Y$. Then

$$\begin{aligned} CM_E(a) = CM_E(a(b^{-1}b)) &= CM_E((ab^{-1})b) \\ &\leq CM_E(ab^{-1}) \vee CM_E(b) \\ &= CM_E(b). \end{aligned}$$

Similarly,

$$\begin{aligned} CM_E(b) = CM_E((a^{-1}a)b^{-1}) &= CM_E(a^{-1}(ab^{-1}) \\ &\leq CM_E(a) \vee CM_E(ab^{-1}) \\ &\leq CM_E(a). \end{aligned}$$

Thus $CM_E(a) = CM_E(b)$.

Conversely, suppose $CM_E(a) = CM_E(y)\ \forall a, b \in Y$. Then we have

$$CM_E(ab^{-1}) = CM_E(bb^{-1}) \Rightarrow CM_E(ab^{-1}) = CM_E(e).$$

□

Proposition 3.10. *Let $E \in AFMG(Y)$. Then $CM_E(ab) = CM_E(b)\ \forall a, b \in Y$ if and only if $CM_E(a) = CM_E(e)$.*

Proof. Take $CM_E(ab) = CM_E(b)\ \ \forall a, b \in Y$. If we let $b = e$, then we have $CM_E(a) = CM_E(e)\ \ \forall a \in Y$.

Conversely, suppose $CM_E(a) = CM_E(e)$. Then $CM_E(b) \geq CM_E(a)$ and thus

$$CM_E(ab) \leq CM_E(a) \vee CM_E(b) = CM_E(b).$$

Also,

$$\begin{aligned} CM_E(b) = CM_E(a^{-1}ab) &\leq CM_E(a) \vee CM_E(ab) \\ &= CM_E(ab). \end{aligned}$$

So $CM_E(ab) = CM_E(b)\ \forall b \in Y$. ☐

Corollary 3.11. *Suppose $E \in AFMG(Y)$. Then $CM_E(ab) = CM_E(a) \vee CM_E(b)$ $\forall a, b \in Y$ with $CM_E(a) \neq CM_E(b)$.*

Proof. Given that $a, b \in Y$, and assume $CM_E(a) < CM_E(b)$. Then

$$CM_E(ab) \leq CM_E(a) \vee CM_E(b) = CM_E(b)\ \forall a, b \in Y$$

and

$$\begin{aligned} CM_E(a) \vee CM_E(b) = CM_E(a^{-1}ab) &\leq CM_E(a^{-1}) \vee CM_E(ab) \\ &= CM_E(a) \vee CM_E(ab) \\ &= CM_E(ab). \end{aligned}$$

Thus $CM_E(ab) = CM_E(a) \vee CM_E(b)$. ☐

Theorem 3.12. *Let $E \in AFMG(Y)$ and if $a, b \in Y$ with $CM_E(a) \neq CM_E(b)$, then $CM_E(ab) = CM_E(ba) = CM_E(a) \vee CM_E(b)$.*

Proof. Assume $a, b \in Y$. Since $CM_E(a) \neq CM_E(b)$, it implies that $CM_E(a) < CM_E(b)$ or $CM_E(b) < CM_E(a)$. Suppose $CM_E(a) < CM_E(b)$. Then

$$CM_E(ab) \leq CM_E(b)$$

and

$$\begin{aligned} CM_E(b) = CM_E(a^{-1}ab) &\leq CM_E(a^{-1}) \vee CM_E(ab) \\ &= CM_E(a) \vee CM_E(ab) \\ &= CM_E(ab). \end{aligned}$$

Thus

$$\begin{aligned} CM_E(b) \leq CM_E(ab) &\leq CM_E(a) \vee CM_E(b) \\ &= CM_E(b). \end{aligned}$$

From here, we see that

$$CM_E(ab) \leq CM_E(a) \vee CM_E(b)$$

and

$$CM_E(a) \vee CM_E(b) \leq CM_E(ab)$$

implying that $CM_E(ab) = CM_E(a) \vee CM_E(b)$.

Similarly, suppose $CM_E(b) < CM_E(a)$. Then we have

$$CM_E(ba) \leq CM_E(a)$$

and

$$\begin{aligned} CM_E(a) = CM_E(b^{-1}ba) &\leq CM_E(b^{-1}) \vee CM_E(ba) \\ &= CM_E(b) \vee CM_E(ba) \\ &= CM_E(ba). \end{aligned}$$

Thus we get

$$\begin{aligned} CM_E(a) \leq CM_E(ba) &\leq CM_E(b) \vee CM_E(a) \\ &= CM_E(a). \end{aligned}$$

Clearly, $CM_E(ba) = CM_E(b) \vee CM_E(a)$. So the result follows. □

Now, we deduce some results connecting union, intersection and sum to anti fuzzy multigroups.

Proposition 3.13. *If $E, F \in AFMG(Y)$, then $E \cap F \in AFMG(Y)$.*

Proof. Since $Y \neq \varnothing$, assume $a, b \in Y$. Then we have

$$\begin{aligned} CM_{E\cap F}(ab^{-1}) &= CM_E(ab^{-1}) \wedge CM_F(ab^{-1}) \\ &\leq [CM_E(a) \vee CM_E(b)] \wedge [CM_F(a) \vee CM_F(b)] \\ &= [CM_E(a) \wedge CM_F(a)] \vee [CM_E(b) \wedge CM_F(b)] \\ &= CM_{E\cap F}(a) \vee CM_{E\cap F}(b). \end{aligned}$$

Thus the result holds. □

Remark 3.14. From the ongoing, the following observations hold:
(1) If $E, F \in AFMG(Y)$, then $E \cup F \in AFMG(Y) \iff$ either $E \subseteq F$ or $F \subseteq E$.
(2) If $\{E_i\}_{j\in J} \in AFMG(Y)$, then $\bigcap_{j\in J} E_j \in AFMG(Y)$.

Lemma 3.15. *The family of anti-fuzzy multigroups $\{E_j\}_{j\in J}$ of Y have inf/sup assuming chain if $E_1 \subseteq E_2 \subseteq ... \subseteq E_m / E_1 \supseteq E_2 \supseteq ... \supseteq E_m$.*

Theorem 3.16. *Suppose $\{E_j\}_{j\in J} \in AFMG(Y)$. If $\{E_j\}_{j\in J}$ possess sup/inf assuming chain, then $\bigcup_{j\in J} E_j \in AFMG(Y)$.*

Proof. Let $E = \bigcup_{j\in J} E_j$. Then $CM_E(a) = \bigvee_{j\in J} CM_{E_j}(a)$. We show that

$$CM_E(ab^{-1}) \leq CM_E(a) \vee CM_E(b) \ \forall a, b \in Y.$$

If $CM_E(a) > 0$ and $CM_E(b) > 0$, then we have:

$$\bigvee_{j\in J} CM_{E_j}(a) > 0, \ \bigvee_{j\in J} CM_{E_j}(b) > 0.$$

From the fact $\{E_j\}_{j\in J}$ possess sup/inf assuming chain, we have $j_0 \in J$ for $CM_{E_{j_0}}(a) = \bigvee_{j\in J} CM_{E_j}(a)$, and also $k_o \in J$ for $CM_{E_{k_0}}(a) = \bigvee_{j\in J} CM_{E_j}(a)$. Thus we have:
Case I: $E_{j_0} \subseteq E_{k_0}$ or
Case II: $E_{k_0} \subseteq E_{j_0}$.
By Case I, we get $CM_{E_{j_0}}(a) \leq CM_{E_{k_0}}(a)$. And so

$$\begin{aligned} CM_E(ab^{-1}) &= CM_{E_{k_0}}(ab^{-1}) \\ &\leq CM_{E_{k_0}}(a) \vee CM_{E_{k_0}}(b) \\ &\leq CM_{E_{j_0}}(a) \vee CM_{E_{j_0}}(b) \\ &= \bigvee_{j\in J} CM_{E_j}(a) \vee \bigvee_{j\in J} CM_{E_j}(b) \\ &= CM_E(a) \vee CM_E(b). \end{aligned}$$

By Case II, it implies that $CM_{E_{k_0}}(a) \leq CM_{E_{j_0}}(a)$. Thus

$$\begin{aligned} CM_E(ab^{-1}) &= CM_{E_{j_0}}(ab^{-1}) \\ &\leq CM_{E_{j_0}}(a) \vee CM_{E_{j_0}}(b) \\ &\leq CM_{E_{k_0}}(a) \vee CM_{E_{k_0}}(b) \\ &= \bigvee_{j\in J} CM_{E_j}(a) \vee \bigvee_{j\in J} CM_{E_j}(b) \\ &= CM_E(a) \vee CM_E(b). \end{aligned}$$

The proof is completed. □

Definition 3.17. Suppose $E, F \in AFMS(Y)$. Then the *sum* of E and F represented by $E \oplus F$, is given by

$$CM_{E\oplus F}(a) = CM_E(a) \oplus CM_F(x)\ \forall a \in Y$$

such that $CM_E(a)$ and $CM_F(a)$ are merged in a descending order, i.e., $\oplus$ is a merging operation.

Example 3.18. Given that $Y = \{1, -1, j, -j\}$ is a group. If

$$E = \{\langle \frac{0, 0.2}{1}\rangle, \langle \frac{0.3, 0.4}{-1}\rangle, \langle \frac{0.4, 0.5}{j}\rangle, \langle \frac{0.4, 0.5}{-j}\rangle\}$$

and

$$F = \{\langle \frac{0.1, 0.3}{1}\rangle, \langle \frac{0.3, 0.4}{-1}\rangle, \langle \frac{0.6, 0.8}{j}\rangle, \langle \frac{0.6, 0.8}{-j}\rangle\}$$

are anti-fuzzy multigroups of Y, then

$$E \oplus F = \{\langle \frac{0, 0.1, 0.2, 0.3}{1}\rangle, \langle \frac{0.3, 0.3, 0.4, 0.4}{-1}\rangle, \langle \frac{0.4, 0.5, 0.6, 0.8}{j}\rangle, \langle \frac{0.4, 0.5, 0.6, 0.8}{-j}\rangle\}.$$

Theorem 3.19. *If $E, F \in AFMG(Y)$, then $E \oplus F \in AFMG(Y)$.*

Proof. Given that $a, b \in Y$, then we have

$$\begin{aligned} CM_{E\oplus F}(ab^{-1}) &= CM_E(ab^{-1}) \oplus CM_F(ab^{-1}) \\ &\leq [CM_E(a) \vee CM_E(b)] \oplus [CM_F(a) \vee CM_F(b)] \\ &= [CM_E(a) \oplus CM_F(a)] \vee [CM_E(b) \oplus CM_F(b)] \\ &= CM_{E\oplus F}(a) \vee CM_{E\oplus F}(b). \end{aligned}$$

Thus $E \oplus F \in AFMG(Y)$. □

Remark 3.20. Let $\{E_j\}_{j\in J} \in AFMG(Y)$. Then $\sum_{j\in J} E_j \in AFMG(Y)$.

3.2. **$\alpha-$cuts of anti-fuzzy multigroups.** Let us recall the idea of $\alpha-$cuts of a fuzzy multigroup [40] before defining the $\alpha-$cut of anti-fuzzy multigroups.

Definition 3.21. Let $\mathbf{E} \in FMG(Y)$. Then, the set $\mathbf{E}_\alpha$ for $\alpha \in [0, 1]$ defined by

$$\mathbf{E}_\alpha = \{a \in Y \mid CM_{\mathbf{E}}(a) \geq \alpha\}$$

is an *α-cut* of $\mathbf{E}$.

Definition 3.22. Let $E \in AFMG(Y)$. Then, the set E_α for $\alpha \in [0,1]$ defined by

$$E_\alpha = \{a \in Y \mid CM_E(a) \leq \alpha\}$$

is an $\alpha-$cut of E.

Clearly, $E_\alpha = Y$ if $\alpha = 1$. Also, $\mathbf{E}_\alpha \cup E_\alpha = Y$ for $\alpha \in [0,1]$.

Proposition 3.23. *Assume $E \in AFMG(Y)$. Then for $\alpha \in [0,1]$ such that $\alpha \geq CM_E(e)$, $E_\alpha \subseteq Y$.*

Proof. For all $a, b \in E_\alpha$, it follows that

$$CM_E(ab^{-1}) \leq [CM_E(a) \vee CM_E(b)] \leq \alpha,$$

which concludes the proof. □

Proposition 3.24. *Suppose $E \in FMS(Y)$ such that $E_\alpha \subseteq Y \ \forall \ \alpha \in [0,1]$ with $\alpha \geq CM_E(e)$. Then $E \in AFMG(Y)$.*

Proof. Given that $a, b \in Y$ and $CM_E(a) = \alpha_1$, $CM_E(b) = \alpha_2$. Suppose $\alpha_2 \geq \alpha_1$. Then $a, b \in E_\alpha$ and thus $ab^{-1} \in E_\alpha$. So

$$CM_E(ab^{-1}) \leq \alpha_2 = \alpha_1 \vee \alpha_2 = CM_E(a) \vee CM_E(b).$$

□

4. Conclusion

We have proposed anti-fuzzy multigroups and their $\alpha-$cuts. Some properties of anti fuzzy multigroups were characterised with certain number of results. Among the relevant results, we established that $FMS(X)$ are $FMG(X)$ if and only if the complements of $FMS(X)$ are $AFMG(X)$. The idea of anti-fuzzy multigroups is quite interesting when explore side by side with fuzzy multigroups. Some analogous results in fuzzy multigroups could be investigated in anti-fuzzy multigroup context for future research.

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