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Crossing cubic structures as an extension of bipolar fuzzy sets

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Abstract. As an extension of bipolar-valued fuzzy sets, the notion of (inner, outer) crossing cubic structures is introduced by using the notion of $\mathcal{N}$-functions and interval-valued fuzzy sets, and related properties are investigated. The same direction order and the opposite direction order in crossing cubic structures are defined, and several properties are discussed. Also, S-union, S-intersection, O-union and O-intersection of crossing cubic structures are introduced, and their related properties are considered. Crossing cubic subalgebras are studied by applying a crossing cubic structure to BCK/BCI-algebras.

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1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. In [1], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, everyone also feel a need to supply mathematical tool. To attain such object, Jun et al. [2] introduced and used a new function which is called negative-valued function. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [3] introduced the notion of cubic sets.

Fuzzy set theory is established in the paper [4]. In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0,1]$. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Only with the membership degrees ranged on the interval $[0,1]$, it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [5] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets.

In this paper, using the notion of $\mathcal{N}$-functions and interval-valued fuzzy sets, we introduce the notion of (inner, outer) crossing cubic structures which is an extension of bipolar-valued fuzzy sets, and investigate several properties. We define the same direction order and the opposite direction order in crossing cubic structures. Also, we define S-union, S-intersection, O-union and O-intersection of crossing cubic structures, and discuss their related properties. We study crossing cubic subalgebras by applying crossing cubic structures to BCK/BCI-algebras.

2. Preliminaries

A set X with a binary operation "$\rightsquigarrow$" and a special element 0) is called *BCI-algebra* if it satisfies:

$$(\forall x,y,z\in X)(((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\rightsquigarrow(z\rightsquigarrow y)=0), \tag{2.1}$$

$$(\forall x,y\in X)((x\rightsquigarrow(x\rightsquigarrow y))\rightsquigarrow y=0), \tag{2.2}$$

$$(\forall x\in X)(x\rightsquigarrow x=0), \tag{2.3}$$

$$(\forall x,y\in X)(x\rightsquigarrow y=0,\ y\rightsquigarrow x=0\ \Rightarrow\ x=y). \tag{2.4}$$

By a *BCK-algebra* we mean a BCI-algebra X satisfying the following condition:

$$(\forall x\in X)(0\rightsquigarrow x=0). \tag{2.5}$$

A subset L of a BCK/BCI-algebra X is called a *subalgebra* of X if $x\rightsquigarrow y\in L$ for all $x,y\in L$.

Denote by $\mathcal{F}(X,[-1,0])$ the collection of all functions from a set X to $[-1,0]$. We say that an element of $\mathcal{F}(X,[-1,0])$ is a *negative-valued function* from X to $[-1,0]$ (briefly, $\mathcal{N}$*-function* on X.) Define a relation $\leq$ on $\mathcal{F}(X,[-1,0])$ as follows:

$$\xi\leq\eta\ \Leftrightarrow\ (\forall x\in X)(\xi(x)\leq\eta(x)) \tag{2.6}$$

for all $\xi,\eta\in\mathcal{F}(X,[-1,0])$. The *complement* of $\xi\in\mathcal{F}(X,[-1,0])$, denoted by ξ^c, is defined as follows:

$$(\forall x\in X)(\xi^c(x)=-1-\xi(x)). \tag{2.7}$$

An *interval number* is defined to be a subinterval $\tilde{a}=[a^-,a^+]$ of $[0,1]$, where $0\leq a^-\leq a^+\leq 1$. The interval number $\tilde{a}=[a^-,a^+]$ with $a^-=a^+$ is denoted by $\mathbf{a}$. Denote by $[[0,1]]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) of two elements in $[[0,1]]$. We also define the symbols "$\succcurlyeq$", "$\preccurlyeq$", "$=$" in case of two elements in $[[0,1]]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\text{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],$$
$$\tilde{a}_1 \succcurlyeq \tilde{a}_2 \;\Leftrightarrow\; a_1^- \geq a_2^-,\; a_1^+ \geq a_2^+,$$

and similarly we may have $\tilde{a}_1 \preccurlyeq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succcurlyeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preccurlyeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [[0,1]]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

For any $\tilde{a} \in [[0,1]]$, its *complement*, denoted by $\tilde{a}^c$, is defined to be the interval number

$$\tilde{a}^c = [1 - a^+, 1 - a^-].$$

Let X be a nonempty set. A function $f : X \to [[0,1]]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[[0,1]]^X$ stand for the set of all IVF sets in X. For every $f \in [[0,1]]^X$ and $x \in X$, $f(x) = [f^-(x), f^+(x)]$ is called the *degree* of membership of an element x to f, where $f^- : X \to [0,1]$ and $f^+ : X \to [0,1]$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $f = [f^-, f^+]$. For every $f, g \in [[0,1]]^X$, we define

$$f \subseteq g \;\Leftrightarrow\; f(x) \preccurlyeq g(x) \;\text{ for all } x \in X,$$

and

$$f = g \;\Leftrightarrow\; f(x) = g(x) \;\text{ for all } x \in X.$$

The complement f^c of $f \in [[0,1]]^X$ is defined as follows: $f^c(x) = f(x)^c$ for all $x \in X$, that is,

$$f^c(x) = [1 - f^+(x), 1 - f^-(x)] \;\text{ for all } x \in X.$$

3. Crossing cubic structures

Definition 3.1. By a *crossing cubic structure* on a set X, we mean a pair $\left(X, \mathcal{C}_{(f,\xi)}\right)$ where

$$\mathcal{C}_{(f,\xi)} := \{\langle x, f(x), \xi(x)\rangle \mid x \in X\} \tag{3.1}$$

in which f is an interval-valued fuzzy set in X and ξ is an $\mathcal{N}$-function on X.

Definition 3.2. A crossing cubic structure $\left(X, \mathcal{C}_{(f,\xi)}\right)$ on a set X is said to be

- *inner*, if it satisfies:

$$(\forall x \in X)(-\xi(x) \in [f^-(x), f^+(x)]). \tag{3.2}$$

- *outer*, if it satisfies:

$$(\forall x \in X)(-\xi(x) \leq f^-(x) \text{ or } -\xi(x) \geq f^+(x)). \tag{3.3}$$

Example 3.3. 1. Let f be an interval-valued fuzzy set in X. Then $\mathcal{C}_{(f,\xi_0)} := \{\langle x, f(x), \xi_0(x)\rangle \mid x \in X\}$, $\mathcal{C}_{(f,\xi_{-1})} := \{\langle x, f(x), \xi_{-1}(x)\rangle \mid x \in X\}$ and $\mathcal{C}_{(f,\xi_c)} := \{\langle x, f(x), \xi_c(x)\rangle \mid x \in X\}$ are crossing cubic structures on X, where $\xi_0(x) = 0$, $\xi_{-1}(x) = -1$ and $\xi_c(x) = \frac{1}{2}(f^-(x) + f^+(x))$ for all $x \in X$.

2. Let $\left([0,1],\mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on $[0,1]$. If $f(x)=[0.4,0.7]$ and $\xi(x)=-0.5$ for all $x\in[0,1]$, then $\left([0,1],\mathcal{C}_{(f,\xi)}\right)$ is an inner crossing cubic structure on $[0,1]$. If $f(x)=[0.4,0.7]$ and $\xi(x)=-0.75$ for all $x\in[0,1]$, then $\left([0,1],\mathcal{C}_{(f,\xi)}\right)$ is an outer crossing cubic structure on $[0,1]$. If $f(x)=[0.4,0.7]$ and $\xi(x)=-x$ for all $x\in[0,1]$, then $\left([0,1],\mathcal{C}_{(f,\xi)}\right)$ is neither an inner crossing cubic structure nor an outer crossing cubic structure .

Example 3.4. Define an interval-valued fuzzy set f and a negative-valued function ξ on the real line $\mathbb{R}$ by

$$f:\mathbb{R}\to[[0,1]],\ x\mapsto\begin{cases}[0,0.4] & \text{if } x<0\\ [0.5,0.6] & \text{if } x=0\\ [0.7,0.9] & \text{if } x>0\end{cases}$$

and

$$\xi:\mathbb{R}\to[-1,0],x\mapsto -1+\frac{1}{1+e^{-x}}.$$

respectively, Then $\left(\mathbb{R},\mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic structure on the real line $\mathbb{R}$.

Proposition 3.5. *If a crossing cubic structure $\left(X,\mathcal{C}_{(f,\xi)}\right)$ on a set X is not outer, then $f^-(a)<-\xi(a)<f^+(a)$ for some $a\in X$.*

Proof. It is straightforward. □

Proposition 3.6. *If a crossing cubic structure $\left(X,\mathcal{C}_{(f,\xi)}\right)$ on a set X is inner and outer, then*

$$(\forall x\in X)(-\xi(x)=f^-(x)\text{ or }-\xi(x)=f^+(x)). \tag{3.4}$$

Proof. Let $\left(X,\mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X which is inner and outer. Then $-\xi(x)\in[f^-(x),f^+(x)]$ and $-\xi(x)\le f^-(x)$ or $-\xi(x)\in[f^-(x),f^+(x)]$ and $-\xi(x)\ge f^+(x)$ for all $x\in X$. It follows that $-\xi(x)=f^-(x)$ or $-\xi(x)=f^+(x)$. This completes the proof. □

Definition 3.7. Let $\left(X,\mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X. The *complement* of $\left(X,\mathcal{C}_{(f,\xi)}\right)$ is defined to be the crossing cubic structure

$$\left(X,\mathcal{C}_{(f,\xi)}\right)^c:=\left(X,\mathcal{C}_{(f^c,\xi^c)}\right),$$

where $f^c:X\to[[0,1]],x\mapsto[1-f^+(x),1-f^-(x)]$ and $\xi^c:X\to[-1,0],\ x\mapsto -1-\xi(x)$.

Example 3.8. Consider the crossing cubic structure $\left(\mathbb{R},\mathcal{C}_{(f,\xi)}\right)$ on $\mathbb{R}$ which is given in Example 3.4. Then f^c and ξ^c are calculated as follows:

$$f^c:\mathbb{R}\to[[0,1]],\ x\mapsto\begin{cases}[0.6,1] & \text{if } x<0\\ [0.4,0.5] & \text{if } x=0\\ [0.1,0.3] & \text{if } x>0\end{cases}$$

and

$$\xi^c:\mathbb{R}\to[-1,0],x\mapsto\frac{-1}{1+e^{-x}}.$$

Hence the crossing cubic structure $\big(X, \mathcal{C}_{(f,\xi)}\big)^c := \big(X, \mathcal{C}_{(f^c,\xi^c)}\big)$ is the complement of $\big(\mathbb{R}, \mathcal{C}_{(f,\xi)}\big)$.

For a crossing cubic structure $\big(X, \mathcal{C}_{(f,\xi)}\big)$ on a set X, $\tilde{a} = [a^-, a^+] \in [[0,1]]$, $t \in [-1, 0]$ and $\varepsilon \in \{\geq, >, \leq, <\}$, we define:

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge\geq,\varepsilon)} &:= f_{\tilde{a}}^{\geq\wedge\geq} \cap \xi_t^{\varepsilon} \\ &:= \{x \in X \mid f^-(x) \geq a^-,\, f^+(x) \geq a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge>,\varepsilon)} &:= f_{\tilde{a}}^{\geq\wedge>} \cap \xi_t^{\varepsilon} \\ &:= \{x \in X \mid f^-(x) \geq a^-,\, f^+(x) > a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.6}$$

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(>\wedge\geq,\varepsilon)} &:= f_{\tilde{a}}^{>\wedge\geq} \cap \xi_t^{\varepsilon} \\ &:= \{x \in X \mid f^-(x) > a^-,\, f^+(x) \geq a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.7}$$

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,\varepsilon)} &:= f_{\tilde{a}}^{>\wedge>} \cap \xi_t^{\varepsilon} \\ &:= \{x \in X \mid f^-(x) > a^-,\, f^+(x) > a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)} &:= f_{\tilde{a}}^{\leq\vee\leq} \cap \xi_t^{\varepsilon} \\ &:= \{x \in X \mid f^-(x) \leq a^- \text{ or } f^+(x) \leq a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee<,\varepsilon)} &:= f_{\tilde{a}}^{\leq\vee<} \cap \xi_t^{\varepsilon} \\ &:= \{x \in X \mid f^-(x) \leq a^- \text{ or } f^+(x) < a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(<\vee\leq,\varepsilon)} &:= f_{\tilde{a}}^{<\vee\leq} \cap \xi_t^{\varepsilon} \\ &:= \{x \in X \mid f^-(x) < a^- \text{ or } f^+(x) \leq a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(<\vee<,\varepsilon)} &:= f_{\tilde{a}}^{<\vee<} \cap \xi_t^{\varepsilon} \\ &:= \{x \in X \mid f^-(x) < a^- \text{ or } f^+(x) < a^+\} \cap \{x \in X \mid \xi(x)\,\varepsilon\, t\}.\end{aligned} \tag{3.12}$$

In a crossing cubic structure $\big(X, \mathcal{C}_{(f,\xi)}\big)$ on a set X, we define:

$$\begin{aligned}f_{a^-}^{\geq} &:= \{x \in X \mid f^-(x) \geq a^-\},\ f_{a^+}^{\geq} := \{x \in X \mid f^+(x) \geq a^+\},\\ f_{a^-}^{>} &:= \{x \in X \mid f^-(x) > a^-\},\ f_{a^+}^{>} := \{x \in X \mid f^+(x) > a^+\},\\ f_{a^-}^{\leq} &:= \{x \in X \mid f^-(x) \leq a^-\},\ f_{a^+}^{\leq} := \{x \in X \mid f^+(x) \leq a^+\},\\ f_{a^-}^{<} &:= \{x \in X \mid f^-(x) < a^-\},\ f_{a^+}^{<} := \{x \in X \mid f^+(x) < a^+\},\end{aligned}$$

for all $\tilde{a} = [a^-, a^+] \in [[0,1]]$.

Proposition 3.9. *Let* $\big(X, \mathcal{C}_{(f,\xi)}\big)$ *be a crossing cubic structure on a set* X*. For any* $\tilde{a} = [a^-, a^+] \in [[0,1]]$*,* $t \in [-1, 0]$ *and* $\varepsilon \in \{\geq, >, \leq, <\}$*, we have:*

$$\begin{aligned}\mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge\geq,\varepsilon)} &= f_{a^-}^{\geq} \cap f_{a^+}^{\geq} \cap \xi_t^{\varepsilon},\ \mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge>,\varepsilon)} = f_{a^-}^{\geq} \cap f_{a^+}^{>} \cap \xi_t^{\varepsilon},\\ \mathcal{C}_{(\tilde{a},t)}^{(>\wedge\geq,\varepsilon)} &= f_{a^-}^{>} \cap f_{a^+}^{\geq} \cap \xi_t^{\varepsilon},\ \mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,\varepsilon)} = f_{a^-}^{>} \cap f_{a^+}^{>} \cap \xi_t^{\varepsilon}.\end{aligned}$$

Proof. Straightforward. □

Proposition 3.10. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X. For any $\tilde{a} = [a^-, a^+] \in [[0,1]]$, $t \in [-1,0]$ and $\varepsilon \in \{\geq, >, \leq, <\}$, we have:*

$$\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)} = (f_{a^-}^{\leq} \cap \xi_t^{\varepsilon}) \cup (f_{a^+}^{\leq} \cap \xi_t^{\varepsilon}),$$
$$\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee<,\varepsilon)} = (f_{a^-}^{\leq} \cap \xi_t^{\varepsilon}) \cup (f_{a^+}^{<} \cap \xi_t^{\varepsilon}),$$
$$\mathcal{C}_{(\tilde{a},t)}^{(<\vee\leq,\varepsilon)} = (f_{a^-}^{<} \cap \xi_t^{\varepsilon}) \cup (f_{a^+}^{\leq} \cap \xi_t^{\varepsilon}),$$
$$\mathcal{C}_{(\tilde{a},t)}^{(<\vee<,\varepsilon)} = (f_{a^-}^{<} \cap \xi_t^{\varepsilon}) \cup (f_{a^+}^{<} \cap \xi_t^{\varepsilon}).$$

Proof. Straightforward. □

Proposition 3.11. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X. For any $\tilde{a} = [a^-, a^+] \in [[0,1]]$, $t \in [-1,0]$ and $\varepsilon \in \{\geq, >, \leq, <\}$, we have:*

$$\mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge>,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge\geq,\varepsilon)},$$
$$\mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(>\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\geq\wedge\geq,\varepsilon)},$$
$$\mathcal{C}_{(\tilde{a},t)}^{(<\vee<,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\leq\vee<,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)},$$
$$\mathcal{C}_{(\tilde{a},t)}^{(<\vee<,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(<\vee\leq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)}.$$

Proof. Straightforward. □

Proposition 3.12. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X and let $\varepsilon \in \{\geq, >, \leq, <\}$. Let $\tilde{a} = [a^-, a^+], \tilde{b} = [b^-, b^+] \in [[0,1]]$ and $t, s \in [-1,0]$ be such that $\tilde{a} \prec \tilde{b}$ and $t < s$. If $\varepsilon \in \{\geq, >\}$, then $\mathcal{C}_{(\tilde{b},s)}^{(\geq\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,>)}$ and $\mathcal{C}_{(\tilde{a},s)}^{(\leq\vee\leq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{b},t)}^{(<\vee<,>)}$. If $\varepsilon \in \{\leq, <\}$, then $\mathcal{C}_{(\tilde{b},t)}^{(\geq\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},s)}^{(>\wedge>,<)}$ and $\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{b},s)}^{(<\vee<,>)}$.*

Proof. Assume that $\tilde{a} \prec \tilde{b}$ and $t < s$. Then $a^- < b^-$ and $a^+ < b^+$. If $x \in \mathcal{C}_{(\tilde{b},s)}^{(\geq\wedge\geq,\varepsilon)}$ for $\varepsilon \in \{\geq, >\}$, then $x \in f_{b^-}^{\geq} \cap f_{b^+}^{\geq} \cap \xi_s^{\varepsilon}$. Thus $f^-(x) \geq b^- > a^-$, $f^+(x) \geq b^+ > a^+$ and $\xi(x)\,\varepsilon\, s > t$. This shows that $x \in \mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,>)}$. So $\mathcal{C}_{(\tilde{b},s)}^{(\geq\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},t)}^{(>\wedge>,>)}$. Let $x \in \mathcal{C}_{(\tilde{a},s)}^{(\leq\vee\leq,\varepsilon)}$. Then $x \in f_{a^-}^{\leq} \cap \xi_s^{\varepsilon}$ or $x \in f_{a^+}^{\leq} \cap \xi_s^{\varepsilon}$. If $x \in f_{a^-}^{\leq} \cap \xi_s^{\varepsilon}$, then $f^-(x) \leq a^- < b^-$ and $\xi(x)\,\varepsilon\, s > t$, that is, $x \in f_{b^-}^{<} \cap \xi_t^{>}$. If $x \in f_{a^+}^{\leq} \cap \xi_s^{\varepsilon}$, then $f^+(x) \leq a^+ < b^+$ and $\xi(x)\,\varepsilon\, s > t$, that is, $x \in f_{b^+}^{<} \cap \xi_t^{>}$. Hence $x \in (f_{a^+}^{\leq} \cap \xi_t^{>}) \cup (f_{b^+}^{<} \cap \xi_t^{>}) = \mathcal{C}_{(\tilde{b},t)}^{(<\vee<,>)}$. Similarly, we can verify that $\mathcal{C}_{(\tilde{b},t)}^{(\geq\wedge\geq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{a},s)}^{(>\wedge>,<)}$ and $\mathcal{C}_{(\tilde{a},t)}^{(\leq\vee\leq,\varepsilon)} \subseteq \mathcal{C}_{(\tilde{b},s)}^{(<\vee<,>)}$ for $\varepsilon \in \{\leq, <\}$. □

Applying the De Morgan's laws to Propositions 3.9 and 3.10 induces the following results.

Proposition 3.13. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on a set X, $\tilde{a} = [a^-, a^+] \in [[0,1]]$ and $t \in [-1,0]$. For $\alpha = \geq$ (resp., $>$, $\leq$ and $<$), let $\alpha^c = <$ (resp., $\leq$, $>$ and $\geq$). Then $\left(\mathcal{C}_{(\tilde{a},t)}^{(\alpha\wedge\beta,\gamma)}\right)^c = f_{a^-}^{\alpha^c} \cup f_{a^+}^{\beta^c} \cup \xi_t^{\gamma^c}$ and $\left(\mathcal{C}_{(\tilde{a},t)}^{(\alpha\vee\beta,\gamma)}\right)^c = (f_{a^-}^{\alpha^c} \cap f_{a^+}^{\beta^c}) \cup \xi_t^{\gamma^c}$ for $\alpha, \beta, \gamma \in \{\geq, >, \leq, <\}$.*

Denote by $CCS(X)$ the set of all crossing cubic structures on a set X. We define a binary relation "$\lessdot$", called the *same direction order* (briefly, S-order), on $CCS(X)$ as follows:

$$\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(g,\eta)}\right) \;\Leftrightarrow\; f \subseteq g,\; \xi \leq \eta \tag{3.13}$$

for all $\left(X, \mathcal{C}_{(f,\xi)}\right), \left(X, \mathcal{C}_{(g,\eta)}\right) \in CCS(X)$. It is clear that $(CCS(X), \lessdot)$ is a poset.

For any $\left(X, \mathcal{C}_{(f,\xi)}\right) \in CCS(X)$, $\tilde{a} = [a^-, a^+] \in [[0,1]]$ and $t \in [-1, 0]$, we define a scalar $\odot$-product and a scalar $*$-product of $\mathcal{C}_{(f,\xi)}$ by $(\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)} := \mathcal{C}_{(\tilde{a}\odot f, t\odot\xi)}$ and $(\tilde{a}, t) * \mathcal{C}_{(f,\xi)} := \mathcal{C}_{(\tilde{a}*f, t*\xi)}$ where

$$\tilde{a} \odot f : X \to [[0,1]],\; x \mapsto [\min\{a^-, f^-(x)\}, \min\{a^+, f^+(x)\}],$$

$$t \odot \xi : X \to [-1, 0],\; x \mapsto \min\{t, \xi(x)\},$$

$$\tilde{a} * f : X \to [[0,1]],\; x \mapsto [\max\{a^-, f^-(x)\}, \max\{a^+, f^+(x)\}],$$

$$t * \xi : X \to [-1, 0],\; x \mapsto \max\{t, \xi(x)\}.$$

Proposition 3.14. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on a set X, $\tilde{a} = [a^-, a^+], \tilde{b} = [b^-, b^+] \in [[0,1]]$ and $t, s \in [-1, 0]$. If $\tilde{a} \preccurlyeq \tilde{b}$ and $t \leq s$, then $(X, (\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{b}, s) \odot \mathcal{C}_{(f,\xi)})$ and $(X, (\tilde{a}, t) * \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{b}, s) * \mathcal{C}_{(f,\xi)})$. If $\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(g,\eta)}\right)$, then $(X, (\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{a}, t) \odot \mathcal{C}_{(g,\eta)})$ and $(X, (\tilde{a}, t) * \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{a}, t) * \mathcal{C}_{(g,\eta)})$.*

Proof. For any $x \in X$, we have

$$\begin{aligned}(\tilde{a} \odot f)(x) &= [\min\{a^-, f^-(x)\}, \min\{a^+, f^+(x)\}] \\ &\preccurlyeq [\min\{b^-, f^-(x)\}, \min\{b^+, f^+(x)\}] \\ &= (\tilde{b} \odot f)(x),\end{aligned}$$

$(t \odot \xi)(x) = \min\{t, \xi(x)\} \leq \min\{s, \xi(x)\} = (s \odot \xi)(x)$,

$$\begin{aligned}(\tilde{a} * f)(x) &= [\max\{a^-, f^-(x)\}, \max\{a^+, f^+(x)\}] \\ &\preccurlyeq [\max\{b^-, f^-(x)\}, \max\{b^+, f^+(x)\}] \\ &= (\tilde{b} * f)(x),\end{aligned}$$

and $(t * \xi)(x) = \max\{t, \xi(x)\} \leq \max\{s, \xi(x)\} = (s * \xi)(x)$. Then $(X, (\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{b}, s) \odot \mathcal{C}_{(f,\xi)})$ and $(X, (\tilde{a}, t) * \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{b}, s) * \mathcal{C}_{(f,\xi)})$. Assume that $\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(g,\eta)}\right)$. Then $f \subseteq g$ and $\xi \leq \eta$, that is, $[f^-(x), f^+(x)] \preccurlyeq [g^-(x), g^+(x)]$ and $\xi(x) \leq \eta(x)$ for all $x \in X$. Thus

$$\begin{aligned}(\tilde{a} \odot f)(x) &= [\min\{a^-, f^-(x)\}, \min\{a^+, f^+(x)\}] \\ &\preccurlyeq [\min\{a^-, g^-(x)\}, \min\{a^+, g^+(x)\}] \\ &= (\tilde{a} \odot g)(x)\end{aligned}$$

and $(t \odot \xi)(x) = \min\{t, \xi(x)\} \leq \min\{s, \xi(x)\} = (s \odot \xi)(x)$ for all $x \in X$, that is, $\tilde{a} \odot f \subseteq \tilde{a} \odot g$ and $t \odot \xi \leq t \odot \eta$. So $(X, (\tilde{a}, t) \odot \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{a}, t) \odot \mathcal{C}_{(g,\eta)})$. Also we have

$$\begin{aligned}(\tilde{a} * f)(x) &= [\max\{a^-, f^-(x)\}, \max\{a^+, f^+(x)\}] \\ &\preccurlyeq [\max\{a^-, g^-(x)\}, \max\{a^+, g^+(x)\}] \\ &= (\tilde{a} * g)(x),\end{aligned}$$

and $(t * \xi)(x) = \max\{t, \xi(x)\} \leq \max\{t, \eta(x)\} = (t * \eta)(x)$ for all $x \in X$. i.e., $\tilde{a} * f \subseteq \tilde{a} * g$ and $t * \xi \leq t * \eta$. Hence $(X, (\tilde{a}, t) * \mathcal{C}_{(f,\xi)}) \lessdot (X, (\tilde{a}, t) * \mathcal{C}_{(g,\eta)})$. □

Theorem 3.15. *If we define a binary operation "$\cdot$" on $CCS(X)$ as follows:*

$$\left(X, \mathcal{C}_{(f,\xi)}\right) \cdot \left(X, \mathcal{C}_{(g,\eta)}\right) = \left(X, \mathcal{C}_{(f \wedge_r g, \xi \wedge \eta)}\right), \tag{3.14}$$

where $(f \wedge_r g)(x) = \text{rmin}\{f(x), g(x)\}$ *and* $(\xi \wedge \eta)(x) = \min\{\xi(x), \eta(x)\}$ *for all* $x \in X$, *then* $(CCS(X), \cdot)$ *is a semigroup.*

Proof. Straightworwad. □

Definition 3.16. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on a set X. We define the *equality* "=" and the *opposite direction order* (briefly, O-order) "$\ll$" in $CCS(X)$ as follows:

$$\begin{aligned}\left(X, \mathcal{C}_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(g,\eta)}\right) &\Leftrightarrow f = g,\ \xi = \eta, \\ \left(X, \mathcal{C}_{(f,\xi)}\right) \ll \left(X, \mathcal{C}_{(g,\eta)}\right) &\Leftrightarrow f \subseteq g,\ \xi \geq \eta.\end{aligned}$$

Theorem 3.17. $(CCS(X), \ll)$ *is a poset.*

Proof. Straightforward. □

Definition 3.18. Let $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ be a family of crossing cubic structures on a set X, where Λ is any index set and $\mathcal{C}^i_{(f,\xi)} = \{\langle x, f_i(x), \xi_i(x)\rangle \mid x \in X\}$. Then

(i) the S-*union*, denoted by $\Cup_{S,\, i\in\Lambda} \left(X, \mathcal{C}^i_{(f,\xi)}\right)$, of $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ is defined to be the crossing cubic structure $\left(X, \Cup_{S,\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right)$ in which

$$\Cup_{S,\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)} := \left\{\left\langle x, \left(\bigcup_{i\in\Lambda} f_i\right)(x), \left(\bigvee_{i\in\Lambda} \xi_i\right)(x)\right\rangle \mid x \in X\right\},$$

(ii) the S-*intersection*, denoted by $\Cap_{S,\, i\in\Lambda} \left(X, \mathcal{C}^i_{(f,\xi)}\right)$, of $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ is defined to be the crossing cubic structure $\left(X, \Cap_{S,\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right)$ in which

$$\Cap_{S,\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)} := \left\{\left\langle x, \left(\bigcap_{i\in\Lambda} f_i\right)(x), \left(\bigwedge_{i\in\Lambda} \xi_i\right)(x)\right\rangle \mid x \in X\right\},$$

(iii) the O-*union*, denoted by $\Cup_{O,\, i\in\Lambda} \left(X, \mathcal{C}^i_{(f,\xi)}\right)$, of $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ is defined to be the crossing cubic structure $\left(X, \Cup_{O,\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right)$ in which

$$\Cup_{O,\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)} := \left\{\left\langle x, \left(\bigcup_{i\in\Lambda} f_i\right)(x), \left(\bigwedge_{i\in\Lambda} \xi_i\right)(x)\right\rangle \mid x \in X\right\},$$

(iv) the O-*intersection*, denoted by $\Cap_{O\, i\in\Lambda} \left(X, \mathcal{C}^i_{(f,\xi)}\right)$, of $\left\{\left(X, \mathcal{C}^i_{(f,\xi)}\right) \mid i \in \Lambda\right\}$ is defined to be the crossing cubic structure $\left(X, \Cap_{O\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right)$ in which

$$\Cap_{O\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)} := \left\{\left\langle x, \left(\bigcap_{i\in\Lambda} f_i\right)(x), \left(\bigvee_{i\in\Lambda} \xi_i\right)(x)\right\rangle \mid x \in X\right\},$$

where $\left(\bigcup_{i\in\Lambda} f_i\right)(x) = \operatorname{rsup}_{i\in\Lambda} f_i(x)$, $\left(\bigvee_{i\in\Lambda} \xi_i\right)(x) = \sup\{\xi_i(x) \mid i \in \Lambda\}$,
$\left(\bigcap_{i\in\Lambda} f_i\right)(x) = \operatorname{rinf}_{i\in\Lambda} f_i(x)$ and $\left(\bigwedge_{i\in\Lambda} \xi_i\right)(x) = \inf\{\xi_i(x) \mid i \in \Lambda\}$.

Note that

$$\left(X, \Cup_{S\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(\bigcup_{i\in\Lambda} f_i, \bigvee_{i\in\Lambda} \xi_i)}\right), \ \left(X, \Cap_{S\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(\bigcap_{i\in\Lambda} f_i, \bigwedge_{i\in\Lambda} \xi_i)}\right),$$
$$\left(X, \Cup_{O\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(\bigcup_{i\in\Lambda} f_i, \bigwedge_{i\in\Lambda} \xi_i)}\right), \ \left(X, \Cap_{O\, i\in\Lambda} \mathcal{C}^i_{(f,\xi)}\right) = \left(X, \mathcal{C}_{(\bigcap_{i\in\Lambda} f_i, \bigvee_{i\in\Lambda} \xi_i)}\right).$$

Proposition 3.19. *Given crossing cubic structures*

$$\left(X, \mathcal{C}_{(f,\xi)}\right), \left(X, \mathcal{C}_{(g,\eta)}\right), \left(X, \mathcal{C}_{(h,\zeta)}\right) \text{ and } \left(X, \mathcal{C}_{(k,\varrho)}\right)$$

on a set X, we have

(1) *if* $\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(g,\eta)}\right)$, *then* $\left(X, \mathcal{C}_{(g,\eta)}\right)^c \lessdot \left(X, \mathcal{C}_{(f,\xi)}\right)^c$,

(2) *if* $\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(g,\eta)}\right)$ *and* $\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(h,\zeta)}\right)$, *then*
$$\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(g,\eta)}\right) \Cap_S \left(X, \mathcal{C}_{(h,\zeta)}\right),$$

(3) *if* $\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(h,\zeta)}\right)$ *and* $\left(X, \mathcal{C}_{(g,\eta)}\right) \lessdot \left(X, \mathcal{C}_{(h,\zeta)}\right)$, *then*
$$\left(X, \mathcal{C}_{(f,\xi)}\right) \Cup_S \left(X, \mathcal{C}_{(g,\eta)}\right) \lessdot \left(X, \mathcal{C}_{(h,\zeta)}\right),$$

(4) *if* $\left(X, \mathcal{C}_{(f,\xi)}\right) \lessdot \left(X, \mathcal{C}_{(h,\zeta)}\right)$ *and* $\left(X, \mathcal{C}_{(g,\eta)}\right) \lessdot \left(X, \mathcal{C}_{(k,\varrho)}\right)$, *then*
$$\left(X, \mathcal{C}_{(f,\xi)}\right) \Cup_S \left(X, \mathcal{C}_{(g,\eta)}\right) \lessdot \left(X, \mathcal{C}_{(h,\zeta)}\right) \Cup_S \left(X, \mathcal{C}_{(k,\varrho)}\right) \text{ and}$$
$$\left(X, \mathcal{C}_{(f,\xi)}\right) \Cap_S \left(X, \mathcal{C}_{(g,\eta)}\right) \lessdot \left(X, \mathcal{C}_{(h,\zeta)}\right) \Cap_S \left(X, \mathcal{C}_{(k,\varrho)}\right),$$

(5) *if* $\left(X, \mathcal{C}_{(f,\xi)}\right) \ll \left(X, \mathcal{C}_{(g,\eta)}\right)$, *then* $\left(X, \mathcal{C}_{(g,\eta)}\right)^c \ll \left(X, \mathcal{C}_{(f,\xi)}\right)^c$,

(6) *if* $\left(X, \mathcal{C}_{(f,\xi)}\right) \ll \left(X, \mathcal{C}_{(g,\eta)}\right)$ *and* $\left(X, \mathcal{C}_{(f,\xi)}\right) \ll \left(X, \mathcal{C}_{(h,\zeta)}\right)$, *then*
$$\left(X, \mathcal{C}_{(f,\xi)}\right) \ll \left(X, \mathcal{C}_{(g,\eta)}\right) \Cap_O \left(X, \mathcal{C}_{(h,\zeta)}\right),$$

(7) *if* $\left(X, \mathcal{C}_{(f,\xi)}\right) \ll \left(X, \mathcal{C}_{(h,\zeta)}\right)$ *and* $\left(X, \mathcal{C}_{(g,\eta)}\right) \ll \left(X, \mathcal{C}_{(h,\zeta)}\right)$, *then*
$$\left(X, \mathcal{C}_{(f,\xi)}\right) \Cup_O \left(X, \mathcal{C}_{(g,\eta)}\right) \ll \left(X, \mathcal{C}_{(h,\zeta)}\right),$$

(8) *if* $\left(X, \mathcal{C}_{(f,\xi)}\right) \ll \left(X, \mathcal{C}_{(h,\zeta)}\right)$ *and* $\left(X, \mathcal{C}_{(g,\eta)}\right) \ll \left(X, \mathcal{C}_{(k,\varrho)}\right)$, *then*
$$\left(X, \mathcal{C}_{(f,\xi)}\right) \Cup_O \left(X, \mathcal{C}_{(g,\eta)}\right) \ll \left(X, \mathcal{C}_{(h,\zeta)}\right) \Cup_O \left(X, \mathcal{C}_{(k,\varrho)}\right) \text{ and}$$
$$\left(X, \mathcal{C}_{(f,\xi)}\right) \Cap_O \left(X, \mathcal{C}_{(g,\eta)}\right) \ll \left(X, \mathcal{C}_{(h,\zeta)}\right) \Cap_O \left(X, \mathcal{C}_{(k,\varrho)}\right).$$

Proof. Straightforward. □

Theorem 3.20. *If a crossing cubic structure $\left(X, \mathcal{C}_{(f,\xi)}\right)$ on a set X is inner (resp., outer), then its complement is also inner (resp., outer).*

Proof. Assume that $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is an inner crossing cubic structure on a set X. Then $-\xi(x) \in [f^-(x), f^+(x)] = f(x)$, that is, $f^-(x) \leq -\xi(x) \leq f^+(x)$ for all $x \in X$. It follows that $1 - f^+(x) \leq -\xi^c(x) \leq 1 - f^-(x)$, i.e., $-\xi^c(x) \in [1 - f^+(x), 1 - f^-(x)] = f^c(x)$ for all $x \in X$. Thus $\left(X, \mathcal{C}_{(f,\xi)}\right)^c$ is an inner crossing cubic structure on X. Now if $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is an outer crossing cubic structure on a set X, then $-\xi(x) \leq f^-(x)$ or $-\xi(x) \geq f^+(x)$ for all $x \in X$. So $-\xi^c(x) = -(-1 - \xi(x)) = 1 + \xi(x) \geq 1 - f^-(x)$ or $-\xi^c(x) = -(-1 - \xi(x)) = 1 + \xi(x) \leq 1 - f^+(x)$ for all $x \in X$. Hence $\left(X, \mathcal{C}_{(f,\xi)}\right)^c$ is an outer crossing cubic structure on X. □

Theorem 3.21. *If $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ are inner crossing cubic structures on a set X, then so is their O-union.*

Proof. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ be inner crossing cubic structures on a set X. Then $f^-(x) \leq -\xi(x) \leq f^+(x)$ and $g^-(x) \leq -\eta(x) \leq g^+(x)$ for all $x \in X$. It follows that

$$\begin{aligned}(f \cup g)^-(x) &= \max\{f^-(x), g^-(x)\} \leq \max\{-\xi(x), -\eta(x)\} \\ &= -\min\{\xi(x), \eta(x)\} = -(\xi \wedge \eta)(x)\end{aligned}$$

and

$$\begin{aligned}-(\xi \wedge \eta)(x) &= -\min\{\xi(x), \eta(x)\} = \max\{-\xi(x), -\eta(x)\} \\ &\leq \max\{f^+(x), g^+(x)\} = (f \cup g)^+(x)\end{aligned}$$

for all $x \in X$. Thus $\left(X, \mathcal{C}_{(f,\xi)}\right) \Cup_O \left(X, \mathcal{C}_{(g,\eta)}\right)$ is an inner crossing cubic structure on X. □

Theorem 3.22. *If $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ are inner crossing cubic structures on a set X, then so is their O-intersection.*

Proof. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ and $\left(X, \mathcal{C}_{(g,\eta)}\right)$ be inner crossing cubic structures on a set X. Then $f^-(x) \leq -\xi(x) \leq f^+(x)$ and $g^-(x) \leq -\eta(x) \leq g^+(x)$ for all $x \in X$. Thus

$$\begin{aligned}(f \cap g)^-(x) &= \min\{f^-(x), g^-(x)\} \leq \min\{-\xi(x), -\eta(x)\} \\ &= -\max\{\xi(x), \eta(x)\} = -(\xi \vee \eta)(x)\end{aligned}$$

and

$$\begin{aligned}-(\xi \vee \eta)(x) &= -\max\{\xi(x), \eta(x)\} = \min\{-\xi(x), -\eta(x)\} \\ &\leq \min\{f^+(x), g^+(x)\} = (f \cap g)^+(x)\end{aligned}$$

for all $x \in X$. So $\left(X, \mathcal{C}_{(f,\xi)}\right) \Cap_O \left(X, \mathcal{C}_{(g,\eta)}\right)$ is an inner crossing cubic structure on X. □

In the following example, we know that the S-union and the S-intersection of inner crossing cubic structures may not be an inner crossing cubic structure.

Example 3.23. 1. Let $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.1, 0.8]$, $\xi(x) = -0.2$, $g(x) = [0.4, 0.9]$ and $\eta(x) = -0.5$ for all $x \in [0,1]$. Then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ are inner crossing cubic structures on $[0,1]$. The S-union of $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ is

$$\left([0,1], \mathcal{C}_{(f,\xi)}\right) \Cup_S \left([0,1], \mathcal{C}_{(g,\eta)}\right) = \left([0,1], \mathcal{C}_{(f \cup g, \xi \vee \eta)}\right) = \left([0,1], \mathcal{C}_{(g,\xi)}\right)$$

We can check that $-\xi(x) = 0.2 \notin [0.4, 0.9] = g(x)$ which shows that $\big([0,1], \mathcal{C}_{(f,\xi)}\big) \Cup_S \big([0,1], \mathcal{C}_{(g,\eta)}\big)$ is not an inner crossing cubic structure on $[0,1]$.

2. Let $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.2, 0.4]$, $\xi(x) = -0.35$, $g(x) = [0.2, 0.3]$ and $\eta(x) = -0.25$ for all $x \in [0,1]$. Then $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ are inner crossing cubic structures on $[0,1]$. The S-intersection of $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ is

$$\big([0,1], \mathcal{C}_{(f,\xi)}\big) \Cap_S \big([0,1], \mathcal{C}_{(g,\eta)}\big) = \big([0,1], \mathcal{C}_{(f\cap g, \xi\wedge\eta)}\big) = \big([0,1], \mathcal{C}_{(g,\xi)}\big)$$

and it is not an inner crossing cubic structure on $[0,1]$ since $-\xi(x) = 0.35 \notin [0.2, 0.3] = g(x)$.

The following example shows that the S-union and the S-intersection of outer crossing cubic structures may not be an outer crossing cubic structure.

Example 3.24. (1) Let $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.31, 0.53]$, $\xi(x) = -0.76$, $g(x) = [0.72, 0.83]$ and $\eta(x) = -0.87$ for all $x \in [0,1]$. Then $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ are outer crossing cubic structures on $[0,1]$. The S-union of $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ is

$$\big([0,1], \mathcal{C}_{(f,\xi)}\big) \Cup_S \big([0,1], \mathcal{C}_{(g,\eta)}\big) = \big([0,1], \mathcal{C}_{(f\cup g, \xi\vee\eta)}\big) = \big([0,1], \mathcal{C}_{(g,\xi)}\big)$$

and it is not an outer crossing cubic structure on $[0,1]$ since $-\xi(x) = 0.76 \in [0.72, 0.83] = [g^-(x), g^+(x)]$.

(2) Let $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.4, 0.6]$, $\xi(x) = -0.28$, $g(x) = [0.5, 0.7]$ and $\eta(x) = -0.47$ for all $x \in [0,1]$. Then $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ are outer crossing cubic structures on $[0,1]$. The S-intersection of $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ is

$$\big([0,1], \mathcal{C}_{(f,\xi)}\big) \Cap_S \big([0,1], \mathcal{C}_{(g,\eta)}\big) = \big([0,1], \mathcal{C}_{(f\cap g, \xi\wedge\eta)}\big) = \big([0,1], \mathcal{C}_{(f,\eta)}\big)$$

and it is not an outer crossing cubic structure on $[0,1]$ since $-\eta(x) = 0.47 \in [0.4, 0.6] = [f^-(x), f^+(x)]$.

The O-union of two outer crossing cubic structures is not an outer crossing cubic structure as seen in the following example.

Example 3.25. Let $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.4, 0.7]$, $\xi(x) = -0.8$, $g(x) = [0.6, 0.9]$ and $\eta(x) = -0.5$ for all $x \in [0,1]$. Then $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ are outer crossing cubic structures on $[0,1]$. The O-union of $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ is

$$\big([0,1], \mathcal{C}_{(f,\xi)}\big) \Cup_O \big([0,1], \mathcal{C}_{(g,\eta)}\big) = \big([0,1], \mathcal{C}_{(f\cup g, \xi\wedge\eta)}\big) = \big([0,1], \mathcal{C}_{(g,\xi)}\big),$$

and it is not an outer crossing cubic structure on $[0,1]$.

The O-intersection of two outer crossing cubic structures is not an outer crossing cubic structure as seen in the following example.

Example 3.26. Let $\big([0,1], \mathcal{C}_{(f,\xi)}\big)$ and $\big([0,1], \mathcal{C}_{(g,\eta)}\big)$ be crossing cubic structures on $[0,1]$ in which $f(x) = [0.47, 0.75]$, $\xi(x) = -0.83$, $g(x) = [0.68, 0.87]$ and $\eta(x) = -0.45$

for all $x \in [0,1]$. Then $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ are outer crossing cubic structures on $[0,1]$. The O-intersection of $\left([0,1], \mathcal{C}_{(f,\xi)}\right)$ and $\left([0,1], \mathcal{C}_{(g,\eta)}\right)$ is

$$\left([0,1], \mathcal{C}_{(f,\xi)}\right) \Cap_O \left([0,1], \mathcal{C}_{(g,\eta)}\right) = \left([0,1], \mathcal{C}_{(f\cap g, \xi\vee\eta)}\right) = \left([0,1], \mathcal{C}_{(f,\eta)}\right),$$

and it is not an outer crossing cubic structure on $[0,1]$.

4. Application to BCK/BCI-algebras

In this section, let X denote a BCK/BCI-algebra unless otherwise specified.

Definition 4.1. A crossing cubic structure $\left(X, \mathcal{C}_{(f,\xi)}\right)$ on X is called a *crossing cubic subalgebra* of X, if it satisfies:

$$(\forall x, y \in X) \left(\begin{array}{l} f(x \rightsquigarrow y) \succcurlyeq \text{rmin}\{f(x), f(y)\} \\ \xi(x \rightsquigarrow y) \leq \max\{\xi(x), \xi(y)\} \end{array} \right). \tag{4.1}$$

Example 4.2. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $\rightsquigarrow$ given by Table 1.

Table 1. Cayley table for the binary operation "$\rightsquigarrow$"

$\rightsquigarrow$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on X which is given by Table 2. It is

Table 2. Tabular representation for $\left(X, \mathcal{C}_{(f,\xi)}\right)$

X	$f(x)$	$\xi(x)$
0	$[0.33, 0.83]$	-0.8
1	$[0.15, 0.56]$	-0.5
2	$[0.33, 0.83]$	-0.7
3	$[0.15, 0.56]$	-0.3

routine to verify that $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X.

Proposition 4.3. *If $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X, then $f(0) \succcurlyeq f(x)$ and $\xi(0) \leq \xi(x)$ for all $x \in X$.*

Proof. Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic subalgebra of X. Using (2.3) and (4.1), we get

$$\begin{aligned} f(0) = f(x \rightsquigarrow x) &\succcurlyeq \text{rmin}\{f(x), f(y)\} \\ &= \text{rmin}\{[f^-(x), f^-(x)], [f^+(x), f^+(x)]\} \\ &= [f^-(x), f^-(x)] = f(x) \end{aligned}$$

and $\xi(0) = \xi(x \rightsquigarrow x) \leq \max\{\xi(x), \xi(x)\} = \xi(x)$ for all $x \in X$. □

Theorem 4.4. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on X. Then it is a crossing cubic subalgebra of X if and only if f^- and f^+ are fuzzy subalgebras of X, and ξ is an $\mathcal{N}$-subalgebra of X.*

Proof. It is easy to verify that if f^- and f^+ are fuzzy subalgebras of X, and ξ is an $\mathcal{N}$-subalgebra of X, then $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X.

Conversely, assume that $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X. It is clear that ξ is an $\mathcal{N}$-subalgebra of X. For any $x, y \in X$, we have

$$\begin{aligned}
[f^-(x \rightsquigarrow y), f^+(x \rightsquigarrow y)] &= f(x \rightsquigarrow y) \succcurlyeq \mathrm{rmin}\{f(x), f(y)\} \\
&= \mathrm{rmin}\{[f^-(x), f^+(x)], [f^-(y), f^+(y)]\} \\
&= [\min\{f^-(x), f^-(y)\}, \min\{f^+(x), f^+(y)\}].
\end{aligned}$$

It follows that $f^-(x \rightsquigarrow y) \geq \min\{f^-(x), f^-(y)\}$ and $f^+(x \rightsquigarrow y) \geq \min\{f^+(x), f^+(y)\}$. Therefore f^- and f^+ are fuzzy subalgebras of X. □

Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on X. We define a level set of $\left(X, \mathcal{C}_{(f,\xi)}\right)$, written as $\ell\left(X, \mathcal{C}_{(f,\xi)}\right)$, as follows:

$$\ell\left(X, \mathcal{C}_{(f,\xi)}, [\alpha, \beta], t\right) = \ell(X, f, [\alpha, \beta]) \cap \ell(X, \xi, t) \tag{4.2}$$

where $\ell(X, f, [\alpha, \beta]) = \{x \in X \mid f(x) \succcurlyeq [\alpha, \beta]\}$ and $\ell(X, \xi, t) = \{x \in X \mid \xi(x) \leq t\}$ for $[\alpha, \beta] \in [[0, 1]]$ and $t \in [-1, 0]$. We say that $\ell(X, f, [\alpha, \beta])$ and $\ell(X, \xi, t)$ are *f-level set* and *ξ-level set* of $\left(X, \mathcal{C}_{(f,\xi)}\right)$ with level indices $[\alpha, \beta]$ and t, respectively.

Theorem 4.5. *If $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X, then its nonempty f-level set and ξ-level set are subalgebras of X for all level indices.*

Proof. Let $[\alpha, \beta] \in [[0, 1]]$ and $t \in [-1, 0]$ be level indices of $\left(X, \mathcal{C}_{(f,\xi)}\right)$ such that $\ell(X, f, [\alpha, \beta])$ and $\ell(X, \xi, t)$ are nonempty. Let $x, y \in \ell(X, f, [\alpha, \beta])$ and $a, b \in \ell(X, \xi, t)$. Then $f(x) \succcurlyeq [\alpha, \beta]$, $f(y) \succcurlyeq [\alpha, \beta]$, $\xi(a) \leq t$ and $\xi(b) \leq t$. It follows from (4.1) that $f(x \rightsquigarrow y) \succcurlyeq \mathrm{rmin}\{f(x), f(y)\} \succcurlyeq \mathrm{rmin}\{[\alpha, \beta], [\alpha, \beta]\} = [\alpha, \beta]$ and $\xi(a \rightsquigarrow b) \leq \max\{\xi(a), \xi(b)\} \leq \max\{t, t\} = t$. Thus $x \rightsquigarrow y \in \ell(X, f, [\alpha, \beta])$ and $a \rightsquigarrow b \in \ell(X, \xi, t)$. So $\ell(X, f, [\alpha, \beta])$ and $\ell(X, \xi, t)$ are subalgebras of X. □

Corollary 4.6. *If $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X, then its nonempty level set $\ell\left(X, \mathcal{C}_{(f,\xi)}, [\alpha, \beta], t\right)$ is a subalgebra of X for all $[\alpha, \beta] \in [[0, 1]]$ and $t \in [-1, 0]$.*

Theorem 4.7. *Let $\left(X, \mathcal{C}_{(f,\xi)}\right)$ be a crossing cubic structure on X in which its nonempty f-level set and ξ-level set are subalgebras of X for all level indices. Then $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X.*

Proof. Assume that $\ell(X, f, [\alpha, \beta])$ and $\ell(X, \xi, t)$ are nonempty subalgebras of X for all level indices $[\alpha, \beta] \in [[0, 1]]$ and $t \in [-1, 0]$. Suppose that there exist $x, y, a, b \in X$ such that $f(x \rightsquigarrow y) \prec \mathrm{rmin}\{f(x), f(y)\}$ and $\xi(a \rightsquigarrow b) > \max\{\xi(a), \xi(b)\}$. Taking $[\alpha_x, \beta_y] := \mathrm{rmin}\{f(x), f(y)\}$ and $t_{a \rightsquigarrow b} := \max\{\xi(a), \xi(b)\}$ induces $x, y \in \ell(X, f, [\alpha_x, \beta_y])$ and $a, b \in \ell(X, \xi, t_{a \rightsquigarrow b})$. But $x \rightsquigarrow y \notin \ell(X, f, [\alpha_x, \beta_y])$ and $a \rightsquigarrow b \notin \ell(X, \xi, t_{a \rightsquigarrow b})$. This is a contradiction, and then $f(x \rightsquigarrow y) \succcurlyeq \mathrm{rmin}\{f(x), f(y)\}$ and $\xi(x \rightsquigarrow y) \leq \max\{\xi(x), \xi(y)\}$ for all $x, y \in X$. Thus $\left(X, \mathcal{C}_{(f,\xi)}\right)$ is a crossing cubic subalgebra of X. □

Theorem 4.8. *Given a subset L of X, we define a crossing cubic structure $\big(X, \mathcal{C}_{(f,\xi)}\big)$ as follows:*

$$f : X \to [[0,1]],\ x \mapsto \begin{cases} [\alpha,\beta] & \text{if } x \in L, \\ [0,0] & \text{otherwise,} \end{cases}$$

$$\xi : X \to [-1,0],\ x \mapsto \begin{cases} t & \text{if } x \in L, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in (0,1]$ with $\alpha < \beta$ and $t \in [-1,0)$. Then L is a subalgebra of X if and only if $\big(X, \mathcal{C}_{(f,\xi)}\big)$ is a crossing cubic subalgebra of X.

Proof. We know that $\ell(X, f, [\alpha,\beta]) = L$, $\ell(X, f, [0,0]) = X$, $\ell(X, \xi, t) = L$ and $\ell(X, \xi, 0) = X$. Using Theorems 4.5 and 4.7, we have the desired result. □

Theorem 4.9. *If $\big(X, \mathcal{C}_{(f,\xi)}\big)$ is a crossing cubic subalgebra of X, then the set*

$$X_{\left(X,\mathcal{C}_{(f,\xi)}\right)} := \{x \in X \mid f(x) = f(0),\ \xi(x) = \xi(0)\}$$

is a subalgebra of X.

Proof. Let $x, y \in X_{\left(X,\mathcal{C}_{(f,\xi)}\right)}$. Then $f(x) = f(0) = f(y)$ and $\xi(x) = \xi(0) = \xi(y)$. Thus

$$\begin{aligned} &f(x \rightsquigarrow y) \succcurlyeq \text{rmin}\{f(x), f(y)\} = \text{rmin}\{f(0), f(0)\} = f(0), \\ &\xi(x \rightsquigarrow y) \leq \max\{\xi(x), \xi(y)\} = \max\{\xi(0), \xi(0)\} = \xi(0). \end{aligned} \tag{4.3}$$

We get $f(x \rightsquigarrow y) = f(0)$ and $\xi(x \rightsquigarrow y) = \xi(0)$ by combining Proposition 4.3 and (4.3). Thus $x \rightsquigarrow y \in X_{\left(X,\mathcal{C}_{(f,\xi)}\right)}$. So $X_{\left(X,\mathcal{C}_{(f,\xi)}\right)}$ is a subalgebra of X. □

The following theorem describes how to create a new crossing cubic subalgebra from a given crossing cubic subalgebra in BCI-algebras.

Theorem 4.10. *Let $\big(X, \mathcal{C}_{(f,\xi)}\big)$ be a crossing cubic subalgebra on a BCI-algebra X and let $\big(X, \mathcal{C}_{(f^{\rightsquigarrow},\xi^{\rightsquigarrow})}\big)$ be a crossing cubic structure on X in which*

$$f^{\rightsquigarrow} : X \to [[0,1]],\ x \mapsto f(0 \rightsquigarrow x) \text{ and } \xi^{\rightsquigarrow} : X \to [-1,0],\ x \mapsto \xi(0 \rightsquigarrow x). \tag{4.4}$$

Then $\big(X, \mathcal{C}_{(f^{\rightsquigarrow},\xi^{\rightsquigarrow})}\big)$ is a crossing cubic subalgebra of X.

Proof. Note that every BCI-algebra X satisfies:

$$(\forall x, y \in X)(0 \rightsquigarrow (x \rightsquigarrow y) = (0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y)).$$

It follows from (4.1) and (4.4) that

$$\begin{aligned} f^{\rightsquigarrow}(x \rightsquigarrow y) &= f(0 \rightsquigarrow (x \rightsquigarrow y)) = f((0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y)) \\ &\succcurlyeq \text{rmin}\{f(0 \rightsquigarrow x), f(0 \rightsquigarrow y)\} = \text{rmin}\{f^{\rightsquigarrow}(x), f^{\rightsquigarrow}(y)\} \end{aligned}$$

and

$$\begin{aligned} \xi^{\rightsquigarrow}(x \rightsquigarrow y) &= \xi(0 \rightsquigarrow (x \rightsquigarrow y)) = \xi(0 \rightsquigarrow x) \rightsquigarrow (0 \rightsquigarrow y)) \\ &\leq \max\{\xi(0 \rightsquigarrow x), \xi(0 \rightsquigarrow y)\} = \max\{\xi^{\rightsquigarrow}(x), \xi^{\rightsquigarrow}(y)\}. \end{aligned}$$

Therefore $\big(X, \mathcal{C}_{(f^{\rightsquigarrow},\xi^{\rightsquigarrow})}\big)$ is a crossing cubic subalgebra of X. □

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