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Fuzzy completeness and various operations in co-residuated lattices

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Abstract. In this paper, we introduce the notions of distance functions, Alexandrov topologies and fuzzy complete lattices an extension of Zhang's fuzzy completeness based on complete co-residuated lattices. Moreover, we investigate the properties of join (meet) preserving maps and various operations as extensions of Zadeh powerset operations. We give their examples.

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1. Introduction

As an algebraic structure for many valued logic, a complete residuated lattice is an important mathematical tool (See [1, 2, 3, 4, 5, 6, 7]). For an extension of classical rough sets introduced by Pawlak [8], many researchers [1, 2, 9, 10] developed L-lower and L-upper approximation operators in complete residuated lattices. By using the concepts of lower and upper approximation operators, fuzzy concepts, information systems and decision rules are investigated in complete residuated lattices (See [1, 2, 4, 9]).

Zhang and Fan [11] introduced the notion of fuzzy complete lattices using fuzzy partially order on a frame as generalizations of usual complete lattices. Based on residuated lattices as an extension of frame, Zhang [12], and Zhang and Xie [13] introduced the notions of partially order,join, meet and fuzzy completeness. Bělohlávek [1, 2] introduced concept lattices for information systems and fuzzy closure operators using Galois connections in a complete residuated lattice. Georgescu and Popescue [14, 15] introduced fuzzy attribute-oriented formal concept lattices for information systems and fuzzy closure operators using adjunctions in a complete residuated lattice. Rodabaugh [5] interpreted Zadeh's powersets operators from fuzzy sets to fuzzy sets as adjoint functions. Join (meet) preserving maps, fuzzy equations, fuzzy rough sets, fuzzy concepts (See [1, 2, 9, 10, 16, 17, 18]), Dedekind-MacNeille completion (See [11, 13]) and powerset operators (See [12]) are easily handle by using adjoint and Galois connection.

On the other hand, Zheng and Wang [19] introduced a complete co-residuated lattice as the generalization of t-conorm. Junsheng and Qing [4] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \vee, \wedge, \odot, \&, 0, 1)$, where $(L, \vee, \wedge, \&, 0, 1)$ is a complete residuated lattice and $(L, \vee, \wedge, \odot, 0, 1)$ is complete co-residuated lattice in a sense of Zheng and Wang [19]. Kim and Ko et al. [20] studied preserving maps and approximation operators in complete co-residuated lattices.

In this paper, we study the distance functions instead of fuzzy partially ordered sets. We define Alexandrov topologies, join preserving maps as the sense of a distance function.

This paper is organized as follows. In Section 2, we recall the definitions of complete co-residuated lattices and distance spaces. Moreover, we give their examples and properties. In Section 3, using distance functions, Alexandrov topologies and fuzzy complete lattices as an extension of Zhang's complete residuated lattices are introduced. In Theorem 3.6, we show that the Alexandrov topology induced by a distance function is a fuzzy complete lattice. In Theorem 3.8, we investigate the (dual) embedding maps and join(meet) preserving maps from sets to Alexandrov topologies. In Theorem 3.10, as extensions of Zadeh's powersets operators from fuzzy sets to fuzzy sets, four types of operations are investigated. We give their examples.

2. Preliminaries

Definition 2.1 ([4, 19, 20])**.** An algebra $(L, \wedge, \vee, \oplus, \perp, \top)$ is called a *complete co-residuated lattice* , if it satisfies the following conditions:

(C1) $L = (L, \vee, \wedge, \perp, \top)$ is a complete lattice, where $\perp$ is the bottom element and $\top$ is the top element,

(C2) $a = a \oplus \perp$, $a \oplus b = b \oplus a$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in L$,

(C3) $(\bigwedge_{i \in \Gamma} a_i) \oplus b = \bigwedge_{i \in \Gamma}(a_i \oplus b)$.

Let $(L, \leq, \oplus)$ be a complete co-residuated lattice. For each $x, y \in L$, we define

$$x \ominus y = \bigwedge \{z \in L \mid y \oplus z \geq x\}.$$

Then $(x \oplus y) \geq z$ iff $x \geq (z \ominus y)$.

For $\alpha \in L,\ A \in L^X$, we denote $(\alpha \ominus A), (\alpha \oplus A), \alpha_X \in L^X$ as $(\alpha \ominus A)(x) = \alpha \ominus A(x),\ (\alpha \oplus A)(x) = \alpha \oplus A(x),\ \alpha_X(x) = \alpha$.

Put $n(x) = \top \ominus x$. The condition $n(n(x)) = x$ for each $x \in L$ is called a *double negative law.*

Remark 2.2 ([20])**.** (1) An infinitely distributive lattice $(L, \leq, \vee, \wedge, \oplus = \vee, \perp, \top)$ is a complete co-residuated lattice. In particular, the unit interval $([0, 1], \leq, \vee, \wedge, \oplus =$

$\vee, 0, 1)$ is a complete co-residuated lattice, where

$$x \ominus y = \bigwedge\{z \in L \mid y \vee z \geq x\}$$
$$= \begin{cases} 0, & \text{if } y \geq x, \\ x, & \text{if } y \not\geq x. \end{cases}$$

(2) The unit interval with a right-continuous t-conorm $\oplus$ [6], $([0,1], \leq, \oplus)$, is a complete co-residuated lattice.

(3) $([1,\infty], \leq, \vee, \oplus = \cdot, \wedge, 1, \infty)$ is a complete co-residuated lattice, where

$$x \ominus y = \bigwedge\{z \in [1,\infty] \mid yz \geq x\}$$
$$= \begin{cases} 1, & \text{if } y \geq x, \\ \frac{x}{y}, & \text{if } y \not\geq x. \end{cases}$$

$$\infty \cdot a = a \cdot \infty = \infty, \forall a \in [1,\infty], \infty \ominus \infty = 1.$$

(4) $([0,\infty], \leq, \vee, \oplus = +, \wedge, 0, \infty)$ is a complete co-residuated lattice where

$$y \ominus x = \bigwedge\{z \in [0,\infty] \mid x + z \geq y\}$$
$$= \bigwedge\{z \in [0,\infty] \mid z \geq -x + y\} = (y - x) \vee 0,$$
$$\infty + a = a + \infty = \infty, \forall a \in [0,\infty], \infty \ominus \infty = 0.$$

(5) $([0,1], \leq, \vee, \oplus, \wedge, 0, 1)$ is a complete co-residuated lattice, where

$$x \oplus y = (x^p + y^p)^{\frac{1}{p}} \wedge 1, \ \ 1 \leq p < \infty,$$
$$x \ominus y = \bigwedge\{z \in [0,1] \mid (z^p + y^p)^{\frac{1}{p}} \geq x\}$$
$$= \bigwedge\{z \in [0,1] \mid z \geq (x^p - y^p)^{\frac{1}{p}}\} = (x^p - y^p)^{\frac{1}{p}} \vee 0,$$

(6) Let $P(X)$ be the collection of all subsets of X. Then $(P(X), \subset, \cup, \cap, \oplus = \cup, \varnothing, X)$ is a complete co-residuated lattice.

Lemma 2.3 ([20])**.** *Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $x \oplus y \leq x \oplus z$, $y \ominus x \leq z \ominus x$ and $x \ominus z \leq x \ominus y$.*

(2) *$(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$ and $x \ominus (\bigwedge_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$.*

(3) *$(\bigwedge_{i\in\Gamma} x_i) \ominus y \leq \bigwedge_{i\in\Gamma}(x_i \ominus y)$.*

(4) *$x \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \ominus y_i)$.*

(5) *$x \ominus x = \bot$, $x \ominus \bot = x$ and $\bot \ominus x = \bot$. Moreover, $x \ominus y = \bot$ iff $x \leq y$.*

(6) *$y \oplus (x \ominus y) \geq x$, $y \geq x \ominus (x \ominus y)$ and $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.*

(7) *$x \ominus (y \oplus z) = (x \ominus y) \ominus z = (x \ominus z) \ominus y$.*

(8) *$x \ominus y \geq (x \oplus z) \ominus (y \oplus z)$, $x \ominus y \geq (x \ominus z) \ominus (y \ominus z)$, $y \ominus x \geq (z \ominus x) \ominus (z \ominus y)$ and $(x \oplus y) \ominus (z \oplus w) \leq (x \ominus z) \oplus (y \ominus w)$.*

(9) *$x \oplus y = \bot$ iff $x = \bot$ and $y = \bot$.*

(10) *$(x \oplus y) \ominus z \leq x \oplus (y \ominus z)$ and $(x \ominus y) \oplus z \geq x \ominus (y \ominus z)$.*

(11) *If L satisfies a double negative law and $n(x) = \top \ominus x$, then $n(x \oplus y) = n(x) \ominus y = n(y) \ominus x$ and $x \ominus y = n(y) \ominus n(x)$.*

Definition 2.4 ([20])**.** Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. Let X be a set. A function $d_X : X \times X \to L$ is called a *distance function*, if it satisfies the following conditions:

(M1) $d_X(x,x) = \bot$ for all $x \in X$,

(M2) $d_X(x,y) \oplus d_X(y,z) \geq d_X(x,z)$ for all $x,\ y,\ z \in X$,

(M3) if $d_X(x,y) = d_X(y,x) = \bot$, then $x = y$.
The pair (X, d_X) is called a *distance space*.

Remark 2.5 ([20]). (1) We define a distance function $d_X : X \times X \to [0, \infty]$. Then (X, d_X) is called a *pseudo-quasi-metric space*.

(2) Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \to L$ as $d_L(x,y) = x \ominus y$. By Lemma 2.3 (5) and (6), (L, d_L) is a distance space. For $\tau \subset L^X$, we define a function $d_\tau : \tau \times \tau \to L$ as

$$d_\tau(A,B) = \bigvee_{x \in X} (A(x) \ominus B(x)).$$

Then (τ, d_τ) is a distance space.

3. Fuzzy completeness and various operations in co-residuated lattices

In this section, we assume $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ is a complete co-residuated lattice.

Definition 3.1 ([20]). Let (X, d_X) be a distance space and $A \in L^X$.
(i) A point x_0 is called a *fuzzy join* of A, denoted by $x_0 = \sqcup_X A$, if it satisfies
(J1) $A(x) \geq d_X(x, x_0)$,
(J2) $\bigvee_{x \in X}(d_X(x,y) \ominus A(x)) \geq d_X(x_0, y)$.
The pair (X, d_X) is called *fuzzy join complete* , if $\sqcup_X A$ exists for each $A \in L^X$.
(ii) A point x_1 is called a *fuzzy meet* of A, denoted by $x_1 = \sqcap_X A$, if it satisfies
(M1) $A(x) \geq d_X(x_1, x)$,
(M2) $\bigvee_{x \in X}(d_X(y,x) \ominus A(x)) \geq d_X(y, x_1)$.
The pair (X, d_X) is called *fuzzy meet complete*, if $\sqcap_X A$ exists for each $A \in L^X$.
The pair (X, d_X) is called *fuzzy complete*, if $\sqcap_X A$ and $\sqcup_X A$ exists for each $A \in L^X$.

Theorem 3.2 ([20]). *Let (X, d_X) be a distance space and $\Phi \in L^X$.*
(1) *A point x_0 is a fuzzy join of Φ iff $\bigvee_{x \in X}(d_X(x,y) \ominus \Phi(x)) = d_X(x_0, y)$.*
(2) *A point x_1 is a fuzzy meet of Φ iff $\bigvee_{x \in X}(d_X(y,x) \ominus \Phi(x)) = d_X(y, x_1)$.*
(3) *If $\sqcup_X \Phi$ is a fuzzy join of $\Phi \in L^X$, then it is unique. Moreover, if $\sqcap_X \Phi$ is a fuzzy meet of $\Phi \in L^X$, then it is unique.*

Example 3.3. Let (X, d_X) be a distance spaces and $A \in L^X$.
(1) Since $\bigvee_{x \in X}(d_X(x,y) \ominus d_X(x,z)) = d(z,y)$, by Theorem 3.2,

$$z = \sqcup_X (d_X)^z,$$

where $(d_X)^z(x) = d_X(x,z)$.
(2) Since $\bigvee_{x \in X}(d_X(y,x) \ominus d_X(z,x)) = d_X(y,z)$, by Theorem 3.3,

$$z = \sqcap_X (d_X)_z,$$

where $(d_X)_z(x) = d_X(z,x)$.

Remark 3.4. Let (L^X, d_{L^X}) be a function space and $\Phi \in L^{L^X}$.
(1) Since $\sqcup_{L^X} \Phi$ is a join of Φ, we see that for all $\Phi \in L^{L^X}$,

$$\begin{aligned} d_{L^X}(\sqcup_{L^X}\Phi, B) &= \bigvee_{A \in L^X}(d_{L^X}(A,B) \ominus \Phi(A)) \\ &= \bigvee_{A \in L^X} d_{L^X}(A \ominus \Phi(A), B) \end{aligned}$$

$= d_{L^X}(\bigvee_{A\in L^X}(A\ominus\Phi(A)),B).$

Then by Theorem 3.3, we have

$$\sqcup_{L^X}\Phi = \bigvee_{A\in L^X}(A\ominus\Phi(A)).$$

(2) Since $\sqcap_{L^X}\Phi$ is a meet of Φ, it follows that for all $\Phi\in L^{L^X}$,

$$\begin{aligned} d_{L^X}(B,\sqcap_{L^X}\Phi) &= \bigvee_{A\in L^X}(d_{L^X}(B,A)\ominus\Phi(A)) \\ &= \bigvee_{A\in L^X} d_{L^X}(B,\Phi(A)\oplus A) \\ &= d_{L^X}(B,\bigwedge_{A\in L^X}(\Phi(A)\oplus A)). \end{aligned}$$

Then by Theorem 3.3, $\sqcap_{L^X}\Phi = \bigwedge_{A\in L^X}(\Phi(A)\oplus A)$.

Definition 3.5. A subset $\tau\subset L^X$ is called an *Alexandrov topology* on X, if it satisfies the following conditions:

(A1) if $A_i\in\tau$ for all $i\in I$, then $\bigvee_{i\in I}A_i, \bigwedge_{i\in I}A_i\in\tau$,

(A2) if $A\in\tau$ and $\alpha\in L$, then $\alpha_X, A\ominus\alpha, A\oplus\alpha\in\tau$.

The pair (X,τ) is called an *Alexandrov topological space* on X.

Theorem 3.6. *Let (X,d_X) be a distance space. We define*

$$\begin{aligned} \tau_{d_X} &= \{A\in L^X \mid A(x)\oplus d_X(x,y)\geq A(y)\}, \\ \tau_{d_X^{-1}} &= \{A\in L^X \mid A(x)\oplus d_X(y,x)\geq A(y)\}. \end{aligned}$$

(1) τ_{d_X} *and* $\tau_{d_X^{-1}}$ *are Alexandrov topologies.*

(2) $(\tau_{d_X}, d_{\tau_{d_X}})$ *is a complete lattice.*

(3) $(\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ *is a complete lattice.*

(4) $\tau_{d_X} = \{\bigvee_{x\in X}A(x)\oplus d_X(x,-)\mid A\in L^X\}$.

(5) $\tau_{d_X^{-1}} = \{\bigvee_{x\in X}A(x)\oplus d_X(-,x)\mid A\in L^X\}$.

Proof. (1) For $A\in\tau$, $\alpha_X, \alpha\oplus A, A\ominus\alpha, (d_X)_z\in\tau$ from:

$$\begin{aligned} &(\alpha\oplus A(x))\oplus d_X(x,y)\geq(\alpha\oplus A(y)) \\ &(A(x)\ominus\alpha)\oplus d_X(x,y)\geq(A(y)\ominus\alpha) \\ &(d_X)_z(x)\oplus d_X(x,y)\leq(d_X)_z(y). \end{aligned}$$

Then τ is an Alexandrov topology with $(d_X)_z\in\tau$.

(2) For $\Phi\in L^{\tau_{d_X}}$, let $A_0=\bigwedge_{A\in\tau_{d_X}}(\Phi(A)\oplus A)$ and $A_1=\bigvee_{A\in\tau_{d_X}}(A\ominus\Phi(A))$. Then $A_0\in\tau_{d_X}$ and $A_1\in\tau_{d_X}$. Thus $A_1=\sqcup_{\tau_{d_X}}\Phi=\bigvee_{A\in\tau_{d_X}}(A\ominus\Phi(A))$ from:

$$\begin{aligned} d_{\tau_{d_X}}(A_1,B) &= d_{\tau_{d_X}}(\bigvee_{A\in\tau_{d_X}}(A\ominus\Phi(A)),B) \\ &= \bigvee_{A\in\tau_{d_X}} d_{\tau_{d_X}}(A\ominus\Phi(A),B) \\ &= \bigvee_{A\in\tau_{d_X}}(d_{\tau_{d_X}}(A,B)\ominus\Phi(A)) \\ &= d_{\tau_{d_X}}(\sqcup_{\tau_{d_X}}\Phi,B). \end{aligned}$$

Moreover, $A_0=\bigwedge_{A\in\tau_{d_X}}(A\oplus\Phi(A))=\sqcap_{\tau_{d_X}}\Phi$ from:

$$\begin{aligned} d_{\tau_{d_X}}(B,A_0) &= d_{\tau_{d_X}}(B,\bigwedge_{A\in\tau_{d_X}}(\Phi(A)\oplus A)) \\ &= \bigvee_{A\in\tau_{d_X}}(d_{\tau_{d_X}}(B,A)\ominus\Phi(A)) \\ &= d_{\tau_{d_X}}(B,\sqcap_{\tau_{d_X}}\Phi). \end{aligned}$$

So $(\tau_{d_X}, d_{\tau_{d_X}})$ is a complete lattice.

(3) It is similarly proved as (1).

(4) $W(X)=\{\bigwedge_{x\in X}A(x)\oplus d_X(x,-)\mid A\in L^X\}$.
Let $A\in\tau_{d_X}$. Then we have

$$\bigwedge_{x\in X}A(x)\oplus d_X(x,y)\geq A(y) \text{ and } \bigwedge_{x\in X}A(x)\oplus d_X(x,y)\leq A(y)\oplus d_X(y,y)=A(y).$$

Thus $A=\bigwedge_{x\in X}A(x)\oplus d_X(x,-)\in W(X)$.
Let $\bigwedge_{x\in X}(A(x)\oplus d_X(x,-)\in W(X)$. Then we get

$$\bigwedge_{y\in X}(\bigwedge_{x\in X}A(x)\oplus d_X(x,y))\oplus d_X(y,z)\geq\bigwedge_{x\in X}A(x)\oplus d_X(x,z).$$

Thus $\bigwedge_{x\in X}(A(x)\oplus d_X(x,-)\in\tau_{d_X}$.
(5) It is similarly proved as (4). □

Definition 3.7. Let (X,d_X) and (Y,d_Y) be distance spaces and $f:X\to Y$ be a map. Define $f^*:L^X\to L^Y$ as

$$f^*(A)(y)=\begin{cases}\top & \text{if } f^{-1}(\{y\})=\varnothing\\ \bigwedge A(x) & \text{if } x\in f^{-1}(\{y\}).\end{cases}$$

(i) f is called a *join preserving map*, if $f(\sqcup_X A)=\sqcup_{L^X}f^*(A)$ for each $A\in L^X$ with $\sqcup_X A$ exists.
(ii) f is called a *meet preserving map*, if $f(\sqcap_X A)=\sqcap_{L^X}f^*(A)$ for each $A\in L^X$ with $\sqcap_X A$ exists.
(iii) f is called a *join-meet preserving map*, if $f(\sqcup_X A)=\sqcap_{L^X}f^*(A)$ for each $A\in L^X$ with $\sqcup_X A$ exists.
(iv) f is called a *meet-join preserving map*, if $f(\sqcap_X A)=\sqcup_{L^X}f^*(A)$ for each $A\in L^X$ with $\sqcup_X A$ exists.
(v) f is called an *embedding map*, if f is injective an $d_X(x,y)=d_X(f(x),f(y)))$ for each $x,\ y\in X$.
(vi) f is called a *dual embedding map*, if f is injective an $d_X(x,y)=d_X(f(y),f(x)))$ for each $x,\ y\in X$.
Define $f^{\oplus},f^{s\oplus}:L^X\to L^Y$ and $f^{\leftarrow}_{\oplus},f^{s\leftarrow}_{\oplus}:L^X\to L^Y$ as

$$f^{\oplus}(A)(y)=\bigwedge_{x\in X}(A(x)\oplus d_Y(f(x),y)),$$
$$f^{s\oplus}(A)(y)=\bigwedge_{x\in X}(A(x)\oplus d_Y(y,f(x))),$$

$$f^{\leftarrow}_{\oplus}(B)(x)=\bigwedge_{z\in X}(B(f(z))\oplus d_X(z,x)),$$
$$f^{s\leftarrow}_{\oplus}(B)(x)=\bigwedge_{z\in X}(B(f(z))\oplus d_X(x,z)).$$

Theorem 3.8. *Let (X,d_X) be a distance space.*
(1) *Define $f:(X,d_X)\to(\tau_{d_X},d_{\tau_{d_X}})$ as $f(x)=(d_X)_x$. Then f is an embedding map. Moreover, if $\sqcup_X A$ exists, then*

$$\sqcup_{\tau_{d_X}}f^*(A)=\bigvee_{x\in X}(d_X(x,-))\ominus A(x))=f(\sqcup_X A),$$
$$\sqcap_{\tau_{d_X}}f^*(A)=\bigwedge_{z\in X}(A(z)\oplus d_X(z,-)).$$

If $A\in\tau_{d_X}$, then $\sqcap_{\tau_{d_X}}f^*(A)=A$.

(2) *Define* $g : (X, d_X) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ *as* $g(x) = (d_X)^x$. *Then* g *is a dual embedding map. Moreover, if* $\sqcap_X A$ *exists, then*

$$\sqcup_{\tau_{d_X^{-1}}} g^*(A) = \bigvee_{x\in X}(d_X(-,x)) \ominus A(x)) = g(\sqcap_X A),$$
$$\sqcap_{\tau_{d_X^{-1}}} g^*(A) = \bigwedge_{z\in X}(A(z) \oplus d_X(-,z)).$$

If $A \in \tau_{d_X^{-1}}$, *then* $\sqcap_{\tau_{d_X^{-1}}} g^*(A) = A$.

Proof. (1) Since $(d_X)_x(z) \oplus d_X(z,w) \geq (d_X)_x(w)$, $f(x) \in \tau_{d_X}$. Then f is well-defined. If $f(x) = f(y) = (d_X)_x = (d_X)_y$, then $d_X(x,z) = d_X(y,z)$ for each $z \in X$ implies $x = y$. Thus f is injective. Moreover, for all $x, y \in X$,

$$\begin{aligned} d_{\tau_{d_X}}(f(x), f(y)) &= d_{\tau_{d_X}}((d_X)_x, (d_X)_y) \\ &= \bigvee_{z\in X}((d_X)_x(z) \ominus (d_X)_y(z)) \\ &= \bigvee_{z\in X}(d_X(x,z) \ominus d_X(y,z)) \\ &= d_X(x,y). \end{aligned}$$

So f is an embedding map. Since $\bigvee_{x\in X}(f(x) \ominus A(x)) \in \tau_{d_X}$, for $C \in \tau_{d_X}$,

$$\begin{aligned} d_{\tau_{d_X}}(\sqcup_{\tau_{d_X}} f^*(A), C) &= \bigvee_{D\in\tau_{d_X}}(d_{\tau_{d_X}}(D,C) \ominus f^*(A)(D)) \\ &= \bigvee_{D\in\tau_{d_X}}(d_{\tau_{d_X}}(D,C) \ominus \bigwedge_{x\in f^{-1}(D)} A(x)) \\ &= \bigvee_{D\in\tau_{d_X}} \bigvee_{x\in f^{-1}(D)}(d_{\tau_{d_X}}(f(x),C) \ominus A(x)) \\ &= d_{\tau_{d_X}}(\bigvee_{x\in X}(f(x) \ominus A(x)), C). \end{aligned}$$

Hence $\sqcup_{\tau_{d_X}} f^*(A) = \bigvee_{x\in X}(f(x) \ominus A(x)) = \bigvee_{x\in X}(d_X(x,-) \ominus A(x)) = d_X(\sqcup_X A, -)$.

If $\sqcup_X A$ exists, then $\sqcup_{\tau_{d_X}} f^*(A) = d_X(\sqcup_X A, -) = f(\sqcup_X A)$. Since $f(x) \in \tau_{d_X}$, by (A1) and (A2), $\bigwedge_{x\in X}(f(x) \oplus A(x)) \in \tau_{d_X}$. For $C \in \tau_{d_X}$,

$$\begin{aligned} d_{\tau_{d_X}}(C, \sqcap_{\tau_{d_X}} f^*(A)) &= \bigvee_{D\in\tau_{d_X}}(d_{\tau_{d_X}}(C,D) \ominus f^*(A)(D)) \\ &= \bigvee_{D\in\tau_{d_X}}(d_{\tau_{d_X}}(C,D) \ominus \bigwedge_{x\in f^{-1}(D)} A(x)) \\ &= \bigvee_{D\in\tau_{d_X}} \bigvee_{x\in f^{-1}(D)}(d_{\tau_{d_X}}(C,f(x)) \ominus A(x)) \\ &= \bigvee_{D\in\tau_{d_X}} \bigvee_{x\in f^{-1}(D)} d_{\tau_{d_X}}(C, f(x) \oplus A(x)) \\ &= d_{\tau_{d_X}}(C, \bigwedge_{x\in X}(f(x) \oplus A(x))). \end{aligned}$$

Then $\sqcap_{\tau_{d_X}} f^*(A) = \bigwedge_{x\in X}(f(x) \oplus A(x))$. If $A \in \tau_{d_X}$, $\sqcap_{\tau_{d_X}} f^*(A) = A$.

(2) Since $(d_X)^x(z) \oplus d_X(w,z) \geq (d_X)^x(w)$, $g(x) \in \tau_{d_X^{-1}}$. Then g is well-defined. If $g(x) = g(y) = (d_X)^x = (d_X)^y$, then $d_X(z,x) = d_X(z,y)$ for each $z \in X$ implies $x = y$. Thus g is injective. Moreover, for all $x, y \in X$,

$$\begin{aligned} d_{\tau_{d_X^{-1}}}(g(x), g(y)) &= d_{\tau_{d_X^{-1}}}((d_X)^x, (d_X)^y) \\ &= \bigvee_{z\in X}((d_X)^x(z) \ominus (d_X)^y(z)) \\ &= \bigvee_{z\in X}(d_X(z,x) \ominus d_X(z,y)) \\ &= d_X(y,x). \end{aligned}$$

Thus g is a dual embedding map. Since $\bigvee_{x\in X}(g(x) \ominus A(x)) \in \tau_{d_X^{-1}}$, for $C \in \tau_{d_X^{-1}}$,

$$\begin{aligned} d_{\tau_{d_X^{-1}}}(\sqcup g^*(A), C) &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(D,C) \ominus g^*(A)(D)) \\ &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(D,C) \ominus \bigwedge_{x\in g^{-1}(D)} A(x)) \\ &= \bigvee_{D\in\tau_{d_X^{-1}}} \bigvee_{x\in g^{-1}(D)}(d_{\tau_{d_X^{-1}}}(g(x),C) \ominus A(x)) \\ &= d_{\tau_{d_X^{-1}}}(\bigvee_{x\in X}(g(x) \ominus A(x)), C). \end{aligned}$$

So $\sqcup_{\tau_{d_X^{-1}}} g^*(A) = \bigvee_{x\in X}(g(x) \ominus A(x)) = \bigvee_{x\in X}(d_X(-,x) \ominus A(x)) = d_X(-, \sqcap_X A)$.

If $\sqcap_X A$ exists, then $\sqcup_{\tau_{d_X^{-1}}} g^*(A) = d_X(-, \sqcap_X A) = g(\sqcap_X A)$. Since $g(x) \in \tau_{d_X^{-1}}$, by (A1) and (A2), $\bigwedge_{x\in X}(g(x) \ominus A(x)) \in \tau_{d_X^{-1}}$. For $C \in \tau_{d_X^{-1}}$,

$$\begin{aligned} d_{\tau_{d_X^{-1}}}(C, \sqcap_{\tau_{d_X^{-1}}} g^*(A)) &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(C,D) \ominus g^*(A)(D)) \\ &= \bigvee_{D\in\tau_{d_X^{-1}}}(d_{\tau_{d_X^{-1}}}(C,D) \ominus \bigwedge_{x\in g^{-1}(D)} A(x)) \\ &= \bigvee_{D\in\tau_{d_X^{-1}}}\bigvee_{x\in g^{-1}(D)}(d_{\tau_{d_X^{-1}}}(C,g(x)) \ominus A(x)) \\ &= \bigvee_{D\in\tau_{d_X^{-1}}}\bigvee_{x\in g^{-1}(D)} d_{\tau_{d_X^{-1}}}(C,g(x) \oplus A(x)) \\ &= d_{\tau_{d_X^{-1}}}(C, \bigwedge_{x\in X}(g(x) \oplus A(x))). \end{aligned}$$

Thus $\sqcap_{\tau_{d_X^{-1}}} g^*(A) = \bigwedge_{x\in X}(g(x) \oplus A(x))$. If $A \in \tau_{d_X^{-1}}$, then $\sqcap_{\tau_{d_X^{-1}}} g^*(A) = A$. □

Lemma 3.9. *Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice. For each $x_i, y_i \in L$, we have the following properties.*

(1) $(\bigvee_{i\in\Gamma} x_i) \ominus (\bigvee_{i\in\Gamma} y_i) \le \bigvee_{i\in\Gamma}(x_i \ominus y_i)$.

(2) $(\bigwedge_{i\in\Gamma} x_i) \ominus (\bigwedge_{i\in\Gamma} y_i) \le \bigvee_{i\in\Gamma}(x_i \ominus y_i)$.

Proof. (1) Since $x_i \le (x_i \ominus y_i) \oplus y_i \le \bigvee_{i\in\Gamma}(x_i \ominus y_i) \oplus (\bigvee_{i\in\Gamma} y_i)$, we conclude that

$$\bigvee_{i\in\Gamma} x_i \le \bigvee_{i\in\Gamma}(x_i \ominus y_i) \oplus (\bigvee_{i\in\Gamma} y_i).$$

Then $(\bigvee_{i\in\Gamma} x_i) \ominus (\bigvee_{i\in\Gamma} y_i) \le \bigvee_{i\in\Gamma}(x_i \ominus y_i)$.

(2) Since $x_i \le (x_i \ominus y_i) \oplus y_i$, we conclude that

$$\bigwedge_{i\in\Gamma} x_i \le (x_i \ominus y_i) \oplus y_i \text{ iff } x_i \ominus y_i \ge \bigwedge_{i\in\Gamma} x_i \ominus y_i.$$

Then $\bigvee_{i\in\Gamma}(x_i \ominus y_i) \ge \bigvee_{i\in\Gamma}(\bigwedge_{i\in\Gamma} x_i \ominus y_i) = (\bigwedge_{i\in\Gamma} x_i) \ominus (\bigwedge_{i\in\Gamma} y_i)$. □

Theorem 3.10. *Let (X, d_X) and (Y, d_Y) be distance spaces. Then the following properties hold.*

(1) *If $f : (X, d_X) \to (Y, d_Y)$ is a map with $d_X(x, y) \ge d_Y(f(x), f(y))$ for each $x, y \in X$, then $d_Y(\sqcup_Y f^{s\oplus}(A), f(\sqcup_X A)) = \bot$ and $d_Y(f(\sqcap_X A), \sqcap_Y f^{\oplus}(A)) = \bot$, for each $A \in L^X$.*

(2) *$d_{L^X}(B, A) \ge d_{L^Y}(f^{\oplus}(B), f^{\oplus}(A))$ and $d_{L^X}(B, A) \ge d_{L^Y}(f^{s\oplus}(B), f^{s\oplus}(A))$.*

(3) *$d_{L^Y}(C, D) \ge d_{L^X}(f^{\leftarrow}_{\oplus}(C), f^{\leftarrow}_{\oplus}(E))$ and $d_{L^Y}(C, D) \ge d_{L^X}(f^{s\leftarrow}_{\oplus}(C), f^{s\leftarrow}_{\oplus}(E))$.*

(4) *$f^{\oplus}(A) \in \tau_{d_Y}$ and $f^{s\oplus}(A) \in \tau_{d_Y^{-1}}$.*

(5) *$f^{\leftarrow}_{\oplus}(A) \in \tau_{d_X}$ and $f^{s\leftarrow}_{\oplus}(A) \in \tau_{d_X^{-1}}$.*

Proof. (1) For each $A \in L^X$,

$$\begin{aligned} d_Y(\sqcup_Y f^{s\oplus}(A), f(\sqcup_X A)) &= \bigvee_{y\in Y}(d_Y(y, f(\sqcup_X A)) \ominus f^{s\oplus}(A)(y)) \\ &= \bigwedge_{y\in X}(d_Y(y, f(\sqcup_X A)) \ominus \bigwedge_{x\in X}(A(x) \oplus d_Y(y, f(x))) \\ &= \bigvee_{y\in X}\bigvee_{x\in X}((d_Y(y, f(\sqcup_X A)) \ominus d_Y(y, f(x))) \ominus A(x)) \\ &= \bigvee_{x\in X}(d_Y(f(x), f(\sqcup_X A)) \ominus A(x)) \\ &\le \bigvee_{x\in X}(d_X(x, \sqcup_X A) \ominus A(x)) \\ &= d_X(\sqcup_X A, \sqcup_X A) = \bot. \end{aligned}$$

For $A \in L^X$,

$$\begin{aligned} d_Y(f(\sqcap_X A), \sqcap_Y f^{\oplus}(A)) &= \bigvee_{y\in Y}(d_Y(f(\sqcap_X A), y) \ominus f^{\oplus}(A)(y)) \\ &= \bigwedge_{y\in X}(d_Y(f(\sqcap_X A), y) \ominus \bigwedge_{x\in X}(A(x) \oplus d_Y(f(x), y)) \end{aligned}$$

$$\begin{aligned}&= \bigvee_{y\in X}\bigvee_{x\in X}((d_Y(f(\sqcap_X A),y)\ominus d_Y(f(x),y))\ominus A(x))\\ &= \bigvee_{x\in X}(d_Y(f(\sqcap_X A),f(x))\ominus A(x))\\ &\leq \bigvee_{x\in X}(d_X(\sqcap_X A,x)\ominus A(x))\\ &= d_X(\sqcap_X A,\sqcap_X A)=\bot.\end{aligned}$$

(2) Since $d_Y(y,f(x))\oplus A(x)\oplus(B(x)\ominus A(x))\geq d_Y(y,f(x))\oplus B(x)$, we conclude that

$$(B(x)\ominus A(x))\geq(d_Y(y,f(x))\oplus B(x))\ominus(d_Y(y,f(x))\oplus A(x)).$$

By Lemma 3.9,

$$\begin{aligned}d_{L^X}(B,A)&=\bigvee_{x\in X}(B(x)\ominus A(x))\\ &\geq \bigwedge_{x\in X}(d_Y(y,f(x))\oplus B(x))\ominus\bigwedge_{x\in X}(d_Y(y,f(x))\oplus A(x))\\ &= d_{L^Y}(f^{\oplus}(B),f^{\oplus}(A)).\end{aligned}$$

(3) $d_{L^Y}(C,E)\geq d_{L^X}f^{\leftarrow}_{\oplus}(C),f^{\leftarrow}_{\oplus}(E))$ from:

$$\begin{aligned}f^{\leftarrow}_{\oplus}(C)(x)\ominus f^{\leftarrow}_{\oplus}(E)(x)&=\bigwedge_{z\in X}(d_X(x,z)\oplus C(f(x)))\ominus\bigwedge_{z\in X}(d_X(x,z)\oplus E(f(x)))\\ &\leq \bigvee_{z\in X}(C(f(x))\ominus E(f(x)))\\ &\leq d_{L^Y}(C,E).\ \text{[By Lemma 2.3 (8) and 3.9]}\end{aligned}$$

Similarly, $d_{L^Y}(C,E)\geq d_{L^X}f^{s\leftarrow}_{\oplus}(C),f^{s\leftarrow}_{\oplus}(E))$.

(4) By Theorem 3.6, $f^{\oplus}(A)\in\tau_{d_Y}$ from:

$$\begin{aligned}f^{\oplus}(A)(y)\odot d_Y(y,w)&=\bigwedge_{y\in X}(A(x)\oplus d_Y(f(x),y)\oplus d_Y(y,w)\\ &\geq \bigwedge_{y\in X}(A(x)\oplus d_Y(f(x),w)\\ &= f^{\oplus}(A)(w).\end{aligned}$$

Also by Theorem 3.6, $f^{s\oplus}(A)\in\tau_{d_Y^{-1}}$ from:

$$\begin{aligned}f^{s\oplus}(A)(y)\oplus d_Y(w,y)&=\bigwedge_{y\in X}(A(x)\oplus d_Y(y,f(x))\oplus d_Y(w,y)\\ &\geq \bigwedge_{y\in X}(A(x)\oplus d_Y(w,f(x))\\ &= f^{s\oplus}(A)(w).\end{aligned}$$

□

Example 3.11. Let $([0,1],\leq,\vee,\wedge,\oplus,\ominus,0,1)$ be a complete co-residuated lattice defined as $n(x)=1-x$,

$$x\oplus y=(x+y)\wedge 1,\quad x\ominus y=(x-y)\vee 0.$$

Let $X=\{a,b,c\}$ be a set and $A,B\in[0,1]^X$ with

$$A(x)=0.3, A(y)=0.2, A(z)=0.5, B(x)=0.6, B(y)=0.3, B(z)=0.5.$$

Define $d_X\in L^{X\times X}$ as

$$d_X=\begin{pmatrix}0 & 0.5 & 0.8\\ 0.7 & 0 & 0.6\\ 0.4 & 0.6 & 0\end{pmatrix}$$

We easily show that d_X is a distance function. Moreover,

$$\begin{aligned}A&=\bigwedge_{x\in X}(A(x)\oplus d_X(x,-))=\bigwedge_{x\in X}(A(x)\oplus d_X(-,x))\\ B&=\bigwedge_{x\in X}(B(x)\oplus d_X(x,-))=\bigwedge_{x\in X}(B(x)\oplus d_X(-,x)).\end{aligned}$$

By Theorem 3.6, $A,B\in\tau_{d_X}, A,B\in\tau_{d_X^{-1}}$.

(1) Define $f : (X, d_X) \to (\tau_{d_X}, d_{\tau_{d_X}})$ as $f(x) = (d_X)_x$. Then $d_X(x,y) = d_{\tau_{d_X}}(f(x), f(y))$ for all $x, y \in X$. Since $\sqcup_X (d_X)^z = z$ from Example 3.3,

$$\begin{aligned}\sqcup f^*((d_X)^z) &= \bigvee_{x\in X}(d_X(x,-)) \ominus (d_X)^z(x)) = d_X(z,-)\\ &= f(\sqcup_X (d_X)^z) = f(z) = (0.4, 0.6, 0),\\ \sqcap_{\tau_{d_X}} f^*((d_X)^z) &= \bigwedge_{y\in X}((d_X)^z(y) \oplus d_X(y,-)) = (0.4, 0.6, 0).\end{aligned}$$

Since $\bigvee_{x\in X}(d_X(x,-)) \ominus A(x)) \neq d_X(\sqcup_X A, -)$ for $A = (0.3, 0.2, 0.5)$, $\sqcup_X A$ does not exist.

$$\begin{aligned}\sqcup_{\tau_{d_X}} f^*(A) &= \bigvee_{x\in X}(d_X(x,-)) \ominus A(x)) = (0.5, 0.2, 0.5),\\ \sqcap_{\tau_{d_X}} f^*(A) &= \bigwedge_{y\in X}((d_X)^z(y) \oplus d_X(y,-)) = (0.3, 0.2, 0.5).\end{aligned}$$

(2) Define $g : (X, d_X) \to (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ as $g(x) = (d_X)^x$. Then $d_X(x,y) = d_{\tau_{d_X^{-1}}}(g(y), g(x))$ for all $x, y \in X$. Since $\sqcap(d_X)_z = z$ from Example 3.3,

$$\begin{aligned}\sqcup_{\tau_{d_X^{-1}}} g^*((d_X)_z) &= \bigvee_{x\in X}(d_X(-,x)) \ominus (d_X)_z(x)) = d_X(-,z)\\ &= d_X(-, \sqcap_X (d_X)_z) = g(\sqcap_X (d_X)_z) = (0.8, 0.6, 0),\\ \sqcap_{\tau_{d_X^{-1}}} g^*((d_X)_z) &= \bigwedge_{y\in X}((d_X)_z(y) \oplus d_X(-,y)) = (0.4, 0.6, 0).\end{aligned}$$

Since $\bigvee_{x\in X}(d_X(-,x) \ominus A(x)) \neq d_X(-, \sqcap_X A)$ for $A = (0.3, 0.2, 0.5)$, $\sqcap_X A$ does not exist.

$$\begin{aligned}\sqcup_{\tau_{d_X^{-1}}} g^*(A) &= \bigvee_{x\in X}(d_X(-,x)) \ominus A(x)) = (0.3, 0.4, 0.4),\\ \sqcap_{\tau_{d_X^{-1}}} g^*(A) &= \bigwedge_{z\in X}(A(z) \oplus d_X(-,z)) = (0.3, 0.2, 0.5).\end{aligned}$$

4. Conclusion

Using distance functions, we discuss the notions of fuzzy completeness and Alexandrov topologies on co-residuated lattices. In particular, we investigate join (meet) preserving maps between various operations based on co-residuated lattices.

In the future, by using the concepts of fuzzy completeness and various operators, information systems and decision rules are investigated in co-residuated lattices.

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