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Commutative quantales and commutative quantale frame

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Abstract. We introduce the notion of commutative quantale frames as a viewpoint of relational semantics for a fuzzy logic. We investigate the relations between commutative quantale frames and commutative quantales. We study their properties and give their examples.

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1. Introduction

Mulvey (See [1, 2]) introduced Quantales as the non-commutative generalization of the lattice of open sets in topological spaces. In fact, a commutative unital quantale coincides with a complete residuated lattice. Höhle (See [3, 4]) developed the algebraic structures and many valued topologies in a sense of quantales and cqm-lattices.

Discrete duality is a duality between a class of algebras and an associated class of relational systems (referred to as frames) without a topology. Dualities are developed between algebras and logical relational systems such as Boolean algebras and classical propositional logic; MV-algebra and Lukasiewicz logic; Heyting algebra and intuitionistic logic; BL-algebra and basic fuzzy logics; Monoidal t-norm logic and MTL-algebras (See [5, 6, 7, 8, 9, 10]). A duality leads in a natural way to relational semantics for a logic which supports part of foundation of theoretic computer science. As a duality between algebras and logical relational systems, Oh and Kim (See [11, 12]) introduced the notion of residuated connections and residuated frames in fuzzy logics.

In this paper, we introduce the notions of commutative quantales and commutative quantale frames as a duality. Let (X, e_X) be a fuzzy partially ordered set. In

Theorem 3.7, $\tau_{e_X} = \{A \in L^X \mid A(x) \odot e_X(x,y) \leq A(y)\}$ is an Alexandrov topology. In Theorem 3.8, if (X, e_X, P) is a commutative quantale frame, we show that $(\tau_{e_X}, \vee, \wedge, \otimes, \nearrow, 0_X, \infty_X)$ is a commutative quantale where, $A, B \in \tau_{e_X}$,

$$(A \otimes B)(z) = \bigvee_{x,y \in X} P(x,y,z) \odot A(x) \odot B(y).$$

$$(A \Rightarrow B)(x) = \bigwedge_{y,z \in X} (P(x,y,z) \odot A(y) \to B(z)).$$

In Theorem 3.14, if $(X, \wedge, \vee, \oplus, \nearrow, 0, 1)$ is a commutative quantale and e_X is a r-fuzzy poset on X, we show that (L^X, e_{L^X}, P) is a commutative quantale frame where $P : L^X \times L^X \times L^X \to L$ as

$$P(A, B, C) = \bigwedge_{x \in X} ((A \otimes B)(x) \to C(x))$$

and $(A \otimes B)(x) = \bigvee_{y,z \in X} (A(y) \odot B(z) \odot e_X(y \odot z, x))$. We give their examples.

2. Preliminaries

Definition 2.1 ([1, 2, 3])**.** A triple $(L, \leq, \odot)$ is called a *commutative quantale* (*c-quantale*, for short), if it satisfies the following conditions:

(Q1) $L = (L, \leq, \vee, \wedge, 0, 1)$ is a complete lattice, where 1 is the universal upper bound and 0 denotes the universal lower bound,

(Q2) $a \odot b = b \odot a$ and $a \odot (b \odot c) = (a \odot b) \odot c$ for all $a, b, c \in L$,

(Q3) $\odot$ is distributive over arbitrary joins, i.e.,

$$(\bigvee_{i \in \Gamma} a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).$$

A c-quantale is called *unital*, if $a = a \odot e$, for each $a \in L$. A unital c-quantale is called a *strictly two-sided, commutative quantale* (*sc-quantale*, for short), if $e = 1$.

Remark 2.2 ([1, 2, 3])**.** (1) A completely distributive lattice is an sc-quantale. In particular, the unit interval $([0,1], \leq, \vee, \odot = \wedge, 0, 1)$ is an sc-quantale.

(2) The unit interval with a left-continuous t-norm $\odot$, $([0,1], \leq, \odot)$, is an sc-quantale.

(3) Let $(L, \leq, \odot)$ be a c-quantale. For each $x, y \in L$, we define

$$x \to y = \bigvee \{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, i.e.,

$(x \odot y) \leq z$ iff $x \leq (y \to z)$.

(4) $([0,\infty], \leq_{op}, \vee_{op}, +, \wedge_{op}, \infty, 0)$ is a sc-quantale, where $\leq_{op} = \geq, \vee_{op} = \wedge, \wedge_{op} = \vee$ and

$$\begin{aligned} x \to y &= \bigvee\nolimits_{op} \{z \in [0,\infty] \mid x + z \leq_{op} y\} \\ &= \bigwedge \{z \in [0,\infty] \mid z \geq -x + y\} = (y - x) \vee 0. \end{aligned}$$

In this paper, we assume $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a c-quantale. For $\alpha \in L, A, B \in L^X$, we denote $(\alpha \to A), (A \to B), (\alpha \odot A), \alpha_X \in L^X$ as $(\alpha \to A)(x) = \alpha \to A(x), (A \to B)(x) = (A(x) \to B(x)), (\alpha \odot A)(x) = \alpha \odot A(x), (A \odot B)(x) = A(x) \odot B(x)$, and $\alpha_X(x) = \alpha$.

Lemma 2.3 ([3, 4, 13, 14]). *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \le z$, $(x \odot y) \le (x \odot z)$, $x \to y \le x \to z$ and $z \to x \le y \to x$.*
(2) $x \odot y \le x \wedge y \le x \vee y$.
(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.
(4) $x \to (\bigvee_{i\in\Gamma} y_i) \ge \bigvee_{i\in\Gamma}(x \to y_i)$.
(5) $(\bigwedge_{i\in\Gamma} x_i) \to y \ge \bigvee_{i\in\Gamma}(x_i \to y)$.
(6) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(7) $x \odot (x \to y) \le y$ *and* $x \to y \le (y \to z) \to (x \to z)$.
(8) $y \odot z \le x \to (x \odot y \odot z)$ *and* $x \odot (x \odot y \to z) \le y \to z$.
(9) $x \to y \le (x \odot z) \to (y \odot z)$.
(10) $(x \to y) \odot (y \to z) \le x \to z$.
(11) $x \to y = 1$ *iff* $x \le y$.
(12) $x = \bigwedge_{y\in L}((x \to y) \to y)$.

Definition 2.4 ([3, 4, 13, 14]). Let X be a set. A function $e_X : X \times X \to L$ is called:

(E1) *reflexive*, if $e_X(x, x) = 1$ for all $x \in X$,
(E2) *transitive*, if $e_X(x, y) \odot e_X(y, z) \le e_X(x, z)$, for all $x, y, z \in X$,
(E3) *anti − symmetric*, if $e_X(x, y) = e_X(y, x) = \top$, then $x = y$.

If e_X satisfies (E1) and (E2), (X, e_X) is a fuzzy preordered set. If e_X satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially ordered set (simply, fuzzy poset).

Example 2.5. (1) We define a function $e_{L^X} : L^X \times L^X \to L$ as

$$e_{L^X}(A, B) = \bigwedge_{x\in X}(A(x) \to B(x)).$$

Then by Lemma 2.3 (10) and (11), (L^X, e_{L^X}) is a fuzzy poset.

(2) If (X, e_X) is a fuzzy poset and we define a function $e_X^{-1}(x, y) = e_X(y, x)$, then (X, e_X^{-1}) is a fuzzy poset.

3. Commutative quantales and commutative quantale frame

As an extension of MTL-frames [9], we define a commutative quantale frame.

Definition 3.1. Let (X, e_X) be a fuzzy poset. The triple (X, e_X, P) is called a *commutative quantale frame* (*cq-frame*, for short), if $P : X \times X \times X \to L$ satisfies the following conditions:

(P1) $P(x, y, z) \odot e_X(x', x) \odot e_X(y', y) \odot e_X(z, z') \le P(x', y', z')$,
(P2) $P(x, y, z) \le P(y, x, z)$,
(P3) $P(x, y, z) \odot P(z, y', z') \le \bigvee_{u\in X}(P(y, y', u) \odot P(x, u, z'))$,
(P4) $P(x, y, z) \odot P(x', z, z') \le \bigvee_{u\in X}(P(x', x, u) \odot P(u, y, z'))$.

A cq-frame (X, e_X, P) is called an *scq–frame*, if satisfies

(P5) $\bigvee_{u\in X} P(u, x, x) = 1$.
(P6) $P(x, y, z) \le e_X(x, z) \wedge e_X(y, z)$.

Definition 3.2. Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale. A fuzzy poset (X, e_X) is an *r-fuzzy poset*, if (R) $e_X(a * b, c) = e_X(a, b \nearrow c)$ for each $a, b, c \in X$.

Theorem 3.3. *Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale. A pair (X, e_X) is an r-fuzzy poset. We define*

$$P(x, y, z) = e_X(x * y, z).$$

Then the following properties hold.

(1) *(X, e_X, P) is a cq-frame.*

(2) *If it is an sc-quantale, $e_X(x * y, z) \leq e_X(x, z) \wedge e_X(y, z)$ for each $x, y, z \in X$, then (X, e_X, P) is an scq-frame.*

Proof. (1) (P1) Since $e_X(x', y \nearrow z) = e_X(y, x' \nearrow z)$, we have

$$\begin{aligned}
&P(x, y, z) \odot e_X(x', x) \odot e_X(y', y) \odot e_X(z, z') \\
&= e_X(x * y, z) \odot e_X(x', x) \odot e_X(y', y) \odot e_X(z, z') \\
&= e_X(x, y \nearrow z) \odot e_X(x', x) \odot e_X(y', y) \odot e_X(z, z') \\
&\leq e_X(x', y \nearrow z) \odot e_X(y', y) \odot e_X(z, z') \\
&= e_X(y, x' \nearrow z) \odot e_X(y', y) \odot e_X(z, z') \\
&\leq e_X(y', x' \nearrow z) \odot e_X(z, z') \\
&\leq e_X(x' * y', z) \odot e_X(z, z') \\
&\leq e_X(x' * y', z') = P(x', y', z').
\end{aligned}$$

(P2) For each $x, y, z \in X$, $P(x, y, z) = e_X(x * y, z) = e_X(y * x, z) = P(y, x, z)$.

(P3)

$$\begin{aligned}
&P(x, y, z) \odot P(z, y', z') = e_X(x * y, z) \odot e_X(z * y', z') \\
&\leq e_X(x * y, y' \nearrow z') = e_X(y * y', x \nearrow z') \\
&= e_X(y * y', x \nearrow z') \odot e_X(x * (x \nearrow z'), z')) \\
&\leq \bigvee e_X(y * y', u) \odot e_X(x * u, z') \\
&= \bigvee P(y, y', u) \odot P(x * u, z').
\end{aligned}$$

(P4)

$$\begin{aligned}
&P(x, y, z) \odot P(x', z, z') = e_X(x * y, z) \odot e_X(x' * z, z') \\
&\leq e_X(x * y, x' \nearrow z') = e_X(x * x', y \nearrow z') \\
&= e_X(x' * x, y \nearrow z') \odot e_X(y \nearrow z', y \nearrow z') \\
&= e_X(x' * x, y \nearrow z') \odot e_X((y \nearrow z') * y, z') \\
&\leq \bigvee_{u \in X} (e_X(x' * x, u) \odot e_X(u * y, z')) \\
&= \bigvee_{u \in X} (P(x', x, u) \odot P(u * y, z')).
\end{aligned}$$

(2) (P5) $\bigvee_{u \in X} P(u, x, x) \geq e_X(1 * x, x) = e_X(x, x) = 1$.

(P6) $P(x, y, z) = e_X(x * y, z) \leq e_X(y, z)$ and $P(x, y, z) = e_X(x * y, z) \leq e_X(x, z)$. □

Example 3.4. Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a c-quantale.

(1) Define $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$ and $P(x, y, z) = (x \odot y) \to z = e_L(x \odot y, z)$. Since $P(x, y, z) = (x \odot y) \to z = e_L(x \odot y, z) = e_L(x, y \to z)$, (L, e_L) is an r-fuzzy poset. By Theorem 3.3 (1), (L, e_L, P) is a cq- frame.

(2) Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale. Define $e_X : X \times X \to L$ as

$$e_X(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x \not\leq y, \end{cases}$$

Since $x * y \leq z$ iff $x \leq y \nearrow z$, $e_X(x * y, z) = e_X(x, y \nearrow z)$ for each $x, y, z \in X$. Hence (X, e_X) is an r-fuzzy poset. Furthermore, define $P(x, y, z) = e_X(x * y, z)$. Then (X, e_X, P) is an cq-frame.

(3) Let X be a set and $P(X) = \{A \mid A \subset X$. Then $(P(X), \cap, \cup, * = \cap, \nearrow, \varnothing, X)$ is an sc-quantale with

$$A \nearrow B = \bigcup\{C \in P(X) \mid A \cap C \subset B\} = A^c \cup B.$$

Define $e : P(X) \times P(X) \to L$ as

$$e(A, B) = \begin{cases} 1, & \text{if } A \subset B, \\ 0, & \text{if } A \not\subset B. \end{cases}$$

Since $A \cap B \subset C$ iff $A \subset B^c \cup C$, $e(A \cap B, C) = e(A, B \nearrow C)$ for each $A, B, C \in P(X)$. Then $(P(X), e)$ is an r-fuzzy poset. Thus by Theorem 3.3 (1), $(P(X), e, P)$ is a cq-frame with $P(A, B, C) = e(A \cap B, C)$ for each $A, B, C \in P(X)$. But it is not an scq-frame because $X = \{a, b, c\}$, $A = \{a, b\}$, $B = \{b, c\}$ and $P(A, B, A) = e(A \cap B, A) = 1 \not\leq e(A, A) \wedge e(B, A) = 0$.

(4) Let X be a set. Then $(L^X, \wedge, \vee, \odot, \to, 0_X, 1_X)$ is a c-quantale. Define $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is an r-fuzzy poset. Furthermore, define $P(A, B, C) = \bigwedge_{x \in X}(A(x) \odot B(x) \to C(x))$. Then (L^X, e_{L^X}, P) is a cq-frame.

(5) Define on $L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ as follows:

$$x \odot y = 0 \vee (x + y - 1), \ x \to y = 1 \vee (1 - x + y).$$

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is an sc-quantale. Let $X = \{0, x, y, 1\}$ be a set such that $0 < x < y < 1$. Define $*$ and $\nearrow$ as follows:

$*$	0	x	y	1
0	0	0	0	0
x	0	0	x	x
y	0	x	y	y
1	0	x	y	1

$\nearrow$	0	x	y	1
0	1	1	1	1
x	x	1	1	1
y	0	x	1	1
1	0	x	y	1

Then $(X, \wedge, \vee, *, \nearrow, 0, 1)$ is a c-quantale.

(a) Define $e_X : X \times X \to L$ as follows

e_X	0	x	y	1
0	1	1	1	1
x	$\frac{1}{2}$	1	1	1
y	$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	1

Since

$$\begin{aligned} & e_X(0, x) = e_X(0 * 0, x) = e_X(0, 0 \nearrow x) \\ & = e_X(0, 1) = e_X(0, 0 \nearrow 0) = e_X(0 * 0, 0) = e_X(0, 0) \\ & = e_X(0 * x, x) = e_X(x, 0 \nearrow x) = e_X(x, 1) \\ & = e_X(0 * y, x) = e_X(0, y \nearrow x) = e_X(0, x) = e_X(y, 1) \\ & = e_X(0 * 1, x) = e_X(1, 0 \nearrow x) = e_X(1, 1) \\ & = e_X(1, x \nearrow x) = e_X(1 * x, x) = e_X(x, x) \\ & = e_X(1, y \nearrow y) = e_X(1 * y, y) = e_X(y, y) = 1, \end{aligned}$$

$$\begin{aligned}&e_X(x,y)=e_X(y*x,y)=e_X(x,y\nearrow y)=e_X(x,1)\\&=e_X(x,0\nearrow x)=e_X(x*0,x)=e_X(0,x)\\&=e_X(x*y,y)=e_X(y,x\nearrow y)=e_X(y,1)\\&=e_X(y,0\nearrow y)=e_X(0*y,y)=e_X(0,y)=1,\end{aligned}$$

$$\begin{aligned}&e_X(x,0)=e_X(x*y,0)=e_X(x,y\nearrow 0)\\&=e_X(y*x,0)=e_X(y,x\nearrow 0)=e_X(y,x)\\&=e_X(1*x,0)=e_X(1,x\nearrow 0)=e_X(1,x)\\&=e_X(x*1,0)=e_X(x,1\nearrow 0)=\tfrac{1}{2}.\end{aligned}$$

Similarly, $e_X(y,0)=e_X(1,0)=\frac{1}{2}$. Then we get for all $x,y,z\in X$,

$$e_X(x,y)\odot e_X(y,z)\le e_X(x,z) \text{ and } e_X(x*y,z)=e_X(x,y\nearrow z)\le e_X(x,z).$$

Thus(X,e_X) is an r-fuzzy poset. Put $P(x,y,z)=e_X(x*y,z)$. Then (X,e_X,P) is an scq-frame.

(b) Define $e_X : X\times X\to L$ as follows as

$$e(x,y)=\begin{cases}1, & \text{if } x=y,\\ 0, & \text{if } x\neq y.\end{cases}$$

Since $1=e_X(x,x)=e_X(y*x,x)\neq e_X(y,x\to x)=e_X(y,1)=0$, (X,e_X) is not an r-fuzzy poset.

Definition 3.5. (1) A subset $\tau_X\subset L^X$ is called an *Alexandrov topology* on X, if it satisfies the following conditions:

(O1) $\alpha_X\in\tau_X$,

(O2) if $A_i\in\tau_X$ for all $i\in I$, then $\bigvee_{i\in I}A_i,\bigwedge_{i\in I}A_i\in\tau_X$,

(O3) if $A\in\tau_X$ and $\alpha\in L$, then $\alpha\odot A,\alpha\to A\in\tau_X$.

The pair (X,τ_X) is called an *Alexandrov topological space.*

Remark 3.6. Let $\tau_X\subset L^X$. Define $e_{\tau_X}:\tau_X\times\tau_X\to L$ as $e_{\tau_X}(A,B)=\bigwedge_{x\in X}(A(x)\to B(x))$. Then (τ_X,e_{τ_X}) is a fuzzy poset.

Theorem 3.7. *Let (X,e_X) be a fuzzy poset. Define*

$$\tau_{e_X}=\{A\in L^X\mid A(x)\odot e_X(x,y)\le A(y)\}.$$

Then τ_{e_X} is an Alexandrov topology on X such that

$$\tau_{e_X}=\{\bigvee_{x\in X}(A(x)\odot e_X(x,-))\mid A\in L^X\}.$$

Proof. Since $\alpha_X(x)\odot e_X(x,y)\le\alpha_X(y)$, we have $\alpha_X\in\tau_{e_X}$.

If $A_i\in\tau_{e_X}$ for all $i\in I$, then

$$\begin{aligned}&(\textstyle\bigwedge_{i\in I}A_i)(x)\odot e_X(x,y)\le\bigwedge_{i\in I}(A_i\odot e_X(x,y))\le\bigwedge_{i\in I}A_i(y),\\&(\textstyle\bigvee_{i\in I}A_i)(x)\odot e_X(x,y)=\bigvee_{i\in I}(A_i(x)\odot e_X(x,y))\le\bigvee_{i\in I}A_i(y).\end{aligned}$$

Thus $\bigwedge_{i\in I}A_i,\bigvee_{i\in I}A_i\in\tau_{e_X}$.

If $A\in\tau_{e_X}$ and $\alpha\in L$, then $\alpha\odot(\alpha\to A(x))\odot e_X(x,y)\le A(x)\odot e_X(x,y)\le A(y)$. Thus $(\alpha\to A(x))\odot e_X(x,y)\le(\alpha\to A(y))$. So $\alpha\to A\in\tau_{e_X}$. Easily, $\alpha\odot A\in\tau_{e_X}$. Hence τ_{e_X} is an Alexandrov topology on X.

Put $\tau = \{\bigvee_{x\in X}(A(x)\odot e_X(x,-)) \mid A\in L^X\}$. Let $A\in\tau_{e_X}$. Then we have

$$\bigvee_{x\in X}(A(x)\odot e_X(x,-)) = A\in\tau.$$

If $\bigvee_{x\in X}(A(x)\odot e_X(x,-))\in\tau$, then we get

$$\bigvee_{x\in X}(A(x)\odot e_X(x,z))\odot e_X(z,y)\le\bigvee_{x\in X}(A(x)\odot e_X(x,y)).$$

Thus $\bigvee_{x\in X}(A(x)\odot e_X(x,-))\in\tau_{e_X}$. So $\tau_{e_X}=\tau$. □

Theorem 3.8. *Let (X,e_X,P) be a cq-frame. For $A,B\in\tau_{e_X}$, we define*

$$(A\otimes B)(z)=\bigvee_{x,y\in X}(P(x,y,z)\odot A(x)\odot B(y)),$$
$$(A\Rightarrow B)(x)=\bigwedge_{y,z\in X}((P(x,y,z)\odot A(y))\to B(z)).$$

Then we have :
(1) $(\tau_{e_X},\vee,\wedge,\otimes,\Rightarrow,0_X,1_X)$ *is a c-quantale,*
(2) *if (X,e_X,P) be an scq-frame, then it is an sc-quantale.*

Proof. (1) For $A,\ B\in\tau_{e_X}$, by (P1),

$$\begin{aligned}(A\otimes B)(z)\odot e_X(z,y) &= \bigvee_{x_0,y_0}\Big(P(x_0,y_0,z)\odot A(x_0)\odot B(y_0)\odot e_X(z,y)\Big)\\ &\le\bigvee_{x_0,y_0}\Big(P(x_0,y_0,y)\odot A(x_0)\odot B(y_0)\Big)\\ &=(A\otimes B)(y).\end{aligned}$$

Then $A\otimes B\in\tau_{e_X}$.

For $A,\ B\in\tau_{e_X}$, also by (P1),

$$\begin{aligned}(A\Rightarrow B)(x) &= \bigwedge_{y,z\in X}\Big(P(x,y,z)\odot A(y)\to B(z)\Big)\\ &\le\bigwedge_{y,z\in X}\Big(e_X(x,x_0)\odot P(x_0,y,z)\odot A(y)\to B(z)\Big)\\ &= e_X(x,x_0)\to\bigwedge_{y,z\in X}\Big(P(x_0,y,z)\odot A(y)\to B(z)\Big)\\ &= e_X(x,x_0)\to(A\Rightarrow B)(x_0).\end{aligned}$$

Thus $A\Rightarrow B\in\tau_{e_X}$.

For $A,\ B,\ C\in\tau_{e_X}$, we have

$$\begin{aligned}&((A\otimes B)\otimes C)(w)\\ &=\bigvee_{p,z\in X}\Big(P(z,p,w)\odot(A\otimes B)(z)\odot C(p)\Big)\\ &=\bigvee_{p,z\in X}\Big(P(z,p,w)\odot(\bigvee_{x,y\in X}(A(x)\odot B(y)\odot P(x,y,z))\odot C(p)\Big)\\ &=\bigvee_{p,z\in X}\bigvee_{x,y\in X}\Big(P(z,p,w)\odot P(x,y,z)\odot A(x)\odot B(y)\odot C(p)\Big)\\ &\le\bigvee_{p\in X}\bigvee_{x,y\in X}\bigvee_{u\in X}\Big(P(y,p,u)\odot P(x,u,w)\odot A(x)\odot B(y)\odot C(p)\Big)\ \text{[By (P3)]}\\ &=\bigvee_{p,y\in X}\bigvee_{x,u\in X}\Big(P(y,p,u)\odot P(x,u,w)\odot A(x)\odot B(y)\odot C(p)\Big)\\ &=\bigvee_{x,u\in X}\Big(P(x,u,w)\odot A(x)\odot\bigvee_{p,y\in X}(P(y,p,u)\odot B(y)\odot C(p))\Big)\\ &=\bigvee_{x,u\in X}\Big(P(x,u,w)\odot A(x)\odot(B\otimes C)(u)\Big)\\ &=(A\otimes(B\otimes C))(w),\end{aligned}$$

$(A\otimes(B\otimes C))(w)$

$$
\begin{aligned}
&= \bigvee_{p,z\in X} \Big(P(z,p,w)\odot A(z)\odot (B\otimes C)(p)\Big)\\
&= \bigvee_{p,z\in X} \Big(P(z,p,w)\odot A(z)\odot \bigvee_{x,y\in X}(P(x,y,p)\odot B(x)\odot C(y))\Big)\\
&= \bigvee_{p,z\in X}\bigvee_{x,y\in X} \Big(P(z,p,w)\odot P(x,y,p)\odot A(z)\odot B(x)\odot C(y)\Big)\\
&\le \bigvee_{z\in X}\bigvee_{x,y\in X}\bigvee_{u\in X} \Big(P(z,x,u)\odot P(u,y,w)\odot A(z)\odot B(x)\odot C(y)\Big) \text{ [By (P4)]}\\
&= \bigvee_{y\in X}\bigvee_{u\in X} \Big(\Big(\bigvee_{z,x\in X}(P(z,x,u)\odot A(z)\odot B(x))\Big)\odot P(u,y,w)\odot C(y)\Big)\\
&= \bigvee_{y\in X}\bigvee_{u\in X} \Big((A\otimes B)(u)\odot P(u,y,w)\odot C(y)\Big)\\
&= ((A\otimes B)\otimes C))(w),
\end{aligned}
$$

$$
\begin{aligned}
&(A\otimes B)(z)\le C(z)\\
&\Leftrightarrow \bigvee_{x,y\in X}(P(x,y,z)\odot A(x)\odot B(y))\le C(z)\\
&\Leftrightarrow A(x)\le \bigwedge_{y,z\in X}(P(x,y,z)\odot B(y))\to C(z)\\
&\Leftrightarrow A(x)\le (B\Rightarrow C)(x).
\end{aligned}
$$

Since $A\otimes B_i \le A\otimes\bigvee_{i\in\Gamma} B_i$, we have $\bigvee_{i\in\Gamma}(A\otimes B_i)\le A\otimes\bigvee_{i\in\Gamma} B_i$. Since $A\otimes B_i\le A\otimes B_i$ iff $B_i\le A\Rightarrow (A\otimes B_i)$, then

$$\bigvee_{i\in\Gamma} B_i \le \bigvee_{i\in\Gamma}(A\Rightarrow (A\otimes B_i))\le A\Rightarrow \bigvee_{i\in\Gamma}(A\otimes B_i).$$

Thus $\bigvee_{i\in\Gamma}(A\otimes B_i)\ge A\otimes\bigvee_{i\in\Gamma} B_i$. So $\bigvee_{i\in\Gamma}(A\otimes B_i) = A\otimes\bigvee_{i\in\Gamma} B_i$. Hence $(\tau_{e_X},\vee,\wedge,\otimes,\Rightarrow,0_X,1_X)$ is a c-quantale.

(2) By (P6), $P(x,y,z)\le e_X(y,z)$. For $A\in\tau_{e_X}$, since $A(x)\odot e_X(x,y)\le A(y)$,

$$
\begin{aligned}
(1_X\otimes A)(z) &= \bigvee_{x,y\in X}(P(x,y,z)\odot 1_X(x)\odot A(y))\\
&= \bigvee_{x,y\in X}(P(x,y,z)\odot A(y))\\
&\le \bigvee_{y\in X}(e_X(y,z)\odot A(y))\\
&\le A(z),
\end{aligned}
$$

$$
\begin{aligned}
A(z) &= \bigvee_{u\in X}(P(u,z,z)\odot A(z)) \text{ [By (P5)]}\\
&\le \bigvee_{u,y\in X}(P(u,y,z)\odot 1_X(u)\odot A(y))\\
&= (1_X\otimes A)(z).
\end{aligned}
$$

Then $(\tau_{e_X},\vee,\wedge,\otimes,\Rightarrow,0_X,1_X)$ is an sc-quantale. □

Theorem 3.9. *Let $(X,\wedge,\vee,*,\nearrow,0,1)$ be a c-quantale with an r-fuzzy poset (X,e_X). For $A,B\in\tau_{e_X}$, we define*

$$
\begin{aligned}
(A\otimes B)(z) &= \bigvee_{x,y\in X}(e_X(x*y,z)\odot A(x)\odot B(y)),\\
(A\Rightarrow B)(x) &= \bigwedge_{y,z\in X}((e_X(x*y,z)\odot A(y))\to B(z)).
\end{aligned}
$$

Then (1) $(\tau_{e_X},\vee,\wedge,\otimes,\Rightarrow,0_X,1_X)$ is a c-quantale, where

$$
\begin{aligned}
(A\otimes B)(z) &= \bigvee_{y\in X} A(y\nearrow z)\odot B(y) = \bigvee_{x\in X} A(x)\odot B(x\nearrow z),\\
(A\Rightarrow B)(x) &= \bigwedge_{z\in X}(A(x\nearrow z)\to B(z)) = \bigwedge_{y\in X}(A(y)\to B(x*y)).
\end{aligned}
$$

(2) *If it is an sc-quantale and $e_X(x*y,z)\le e_X(x,z)\wedge e_X(y,z)$ for each $x,y,z\in X$, then $(\tau_{e_X},\vee,\wedge,\otimes,\Rightarrow,0_X,1_X)$ is an sc-quantale.*

Proof. (1) Put $P(x,y,z)=e_X(x*y,z)$. By Theorem 3.3, (X,e_X,P) is a cq-frame. Since $\bigvee_{x\in X}(e_X(x*y,z)\odot A(x)) = \bigvee_{x\in X}(e_X(x,y\nearrow z)\odot A(x)) = A(y\nearrow z)$ for

$A \in \tau_{e_X}$ and $\bigvee_{y\in X}(e_X(x*y,z)\odot B(y)) = \bigvee_{y\in X}(e_X(y,x\nearrow z)\odot B(y)) = B(x\nearrow z)$ for $A \in \tau_{e_X}$,

$$\begin{aligned}(A\otimes B)(z) &= \bigvee_{x,y\in X} e_X(x*y,z)\odot A(x)\odot B(y)\\ &= \bigvee_{y\in X} A(y\nearrow z)\odot B(y)\\ &= \bigvee_{x\in X} A(x)\odot B(x\nearrow z).\end{aligned}$$

Since $\bigvee_{x\in X}(e_X(x*y,z)\odot A(x)) = \bigvee_{x\in X}(e_X(x,y\nearrow z)\odot A(x)) = A(y\nearrow z)$ for $A\in\tau_{e_X}$ and $\bigvee_{y\in X}(e_X(x*y,z)\odot B(y)) = \bigvee_{y\in X}(e_X(y,x\nearrow z)\odot B(y)) = B(x\nearrow z)$ for $A \in \tau_{e_X}$,

$$\begin{aligned}(A\Rightarrow B)(x) &= \bigwedge_{y,z\in X}(e_X(x*y,z)\odot A(y)\to B(z))\\ &= \bigwedge_{z\in X}(\bigvee_{y\in X}(e_X(y,x\nearrow z)\odot A(y))\to B(z))\\ &= \bigwedge_{z\in X}(A(x\nearrow z)\to B(z)),\end{aligned}$$

$$\begin{aligned}(A\Rightarrow B)(x) &= \bigwedge_{y\in X}(A(y)\to \bigwedge_{z\in X}(e_X(x*y,z)\to B(z)))\\ &= \bigwedge_{y\in X}(A(y)\to B(x*y)).\end{aligned}$$

(2) It follows from Theorem 3.8 (2). □

Example 3.10. Let $(L,\wedge,\vee,\odot,\to,0,1)$ be a c-quantale with an r-fuzzy poset (L,e_L) and $e_L(x,y)=x\to y$. For $A,B\in\tau_{e_L}$, we define

$$\begin{aligned}(A\otimes B)(z) &= \bigvee_{x,y\in L}(((x\odot y)\to z)\odot A(x)\odot B(y))\\ &= \bigvee_{y\in L} A(y\to z)\odot B(y)\\ &= \bigvee_{x\in L} A(x)\odot B(x\to z),\end{aligned}$$

$$\begin{aligned}(A\Rightarrow B)(x) &= \bigwedge_{y,z\in L}(((x\odot y)\to z)\odot A(y)\to B(z))\\ &= \bigwedge_{z\in L}(A(x\to z)\to B(z))\\ &= \bigwedge_{y\in L}(A(y)\to B(x\odot y)).\end{aligned}$$

Then $(\tau_{e_L},\vee,\wedge,\otimes,\Rightarrow,0_L,1_L)$ is a c-quantale.

Example 3.11. Let $(P(X),e,P)$ be a cq-frame from Example 3.4 (3). Let $\tau_e = \{\alpha\in L^{P(X)} \mid \alpha(A)\odot e(A,B)\le\alpha(B)\} = \{\alpha\in L^{P(X)}\mid \alpha(A)\le\alpha(B), \text{for } A\subset B\}$. For $\alpha,\beta\in\tau_e$, we define

$$\begin{aligned}(\alpha\otimes\beta)(C) &= \bigvee_{A,B\in P(X)}(P(A,B,C)\odot\alpha(A)\odot\beta(B))\\ &= \bigvee_{A\cap B\subset C}(\alpha(A)\odot\beta(B))\\ &= \bigvee_{B\in P(X)}(\alpha(B^c\cup C)\odot\beta(B))\\ &= \bigvee_{A\in P(X)}(\alpha(A)\odot\beta(A^c\cup C)),\end{aligned}$$

$$\begin{aligned}(\alpha\Rightarrow\beta)(A) &= \bigwedge_{B,C\in P(X)}(P(A,B,C)\odot\alpha(B)\to\beta(C))\\ &= \bigwedge_{A\subset B^c\cup C}(\alpha(B)\to\beta(C))\\ &= \bigwedge_{C\in P(X)}(\alpha(A^c\cup C)\to\beta(C))\\ &= \bigwedge_{B\in P(X)}(\alpha(B)\to\beta(A\cap B)).\end{aligned}$$

Then by Theorem 3.8, $(\tau_e,\vee,\wedge,\otimes,\Rightarrow,0_{P(X)},1_{P(X)})$ is a c-quantale ,where $0_{P(X)}(A) = 0, 1_{P(X)}(A)=1$ for each $A\in P(X)$.

Example 3.12. (1) Define $(A\to B)(x) = A(x)\to B(x)$ and $(A\odot B)(x) = A(x)\odot B(x)$ for each $x\in X$. Then $(L^X,\vee,\wedge,\odot,\to,0_X,1_X)$ is a sc-quantale.

(2) Let (L^X, e_{L^X}, P) be a cq-frame where $P(A,B,C) = \bigwedge_{x\in X}(A(x) \odot B(x) \to C(x))$ from Example 3.4 (4). Let $\tau_{e_{L^X}} = \{\alpha \in L^{L^X} \mid \alpha(A) \odot e_{L^X}(A,B) \le \alpha(B)\}$. For $\alpha, \beta \in \tau_{e_{L^X}}$, we define

$$\begin{aligned}(\alpha \otimes \beta)(C) &= \bigvee_{A,B\in L^X}(P(A,B,C) \odot \alpha(A) \odot \beta(B))\\ &= \bigvee_{B\in L(X)}(\alpha(B \to C) \odot \beta(B))\\ &= \bigvee_{A\in L(X)}(\alpha(A) \odot \beta(A \to C)),\end{aligned}$$

$$\begin{aligned}(\alpha \Rightarrow \beta)(A) &= \bigwedge_{B,C\in L^X}(P(A,B,C) \odot \alpha(B) \to \beta(C))\\ &= \bigwedge_{A\le B\to C}(\alpha(B) \to \beta(C))\\ &= \bigwedge_{C\in L^X}(\alpha(A \to C) \to \beta(C))\\ &= \bigwedge_{B\in L^X}(\alpha(B) \to \beta(A \odot B)).\end{aligned}$$

Then by Theorem 3.8, $(\tau_{e_{L^X}}, \vee, \wedge, \otimes, \Rightarrow, 0_{L^X}, 1_{L^X})$ is a c-quantale, where $0_{L^X}(A) = 0, 1_{L^X}(A) = 1$ for each $A \in L^X$.

Theorem 3.13. *Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a sc-quantale and e_X is an r-fuzzy poset. For $A,\ B \in L^X$, we define $A \otimes B \in L^X$ as*

$$(A \otimes B)(x) = \bigvee_{y,z\in X}(A(y) \odot B(z) \odot e_X(y * z, x)).$$

Then the following properties hold.

(1) If $A,\ B \in L^X$ with $B(1) = 1$, then $A \otimes B \ge A$ and $A \otimes E = A$ for $E(1) = 1, E(x) = 0,$ otherwise. Moreover, $(L^X, \vee, \wedge, \otimes, 0_X, 1_X)$ is a c-quantale but not an sc-quantale.

(2) If $A,\ B \in M(L^X)$ and $e_X(a,b) \odot e_X(c,d) \le e_X(a*c, b*d)$, where

$$M(L^X) = \{A \in L^X \mid A(a * b) \ge A(a) \odot A(b)\},$$

then $A \otimes B \in M(L^X)$.

(3) If $A \in M(L^X) \cap \tau_{e_X}$ and $e_X(a,b) \odot e_X(c,d) \le e_X(a*c, b*d)$, then $A \otimes A \le A$.

Proof. (1) For $A, B \in L^X$ with $B(1) = 1$,

$$\begin{aligned}(A \otimes B)(a) &= \bigvee_{x,y\in X}(A(x) \odot B(y) \odot e_X(x * y, a))\\ &\ge A(a) \odot B(1) \odot e_X(a * 1, a)\\ &= A(a).\end{aligned}$$

Moreover, $(A \otimes E)(a) = A(a) \odot E(1) \odot e_X(a * 1, a) = A(a)$ for $E(1) = 1, E(x) = 0,$ otherwise. But $E \ne 1_X$, that is, E is not a top element. Then $(L^X, \vee, \wedge, \otimes, 0_X, 1_X)$ is not an sc-quantale.

(2) For $A,\ B \in M(L^X)$,

$$\begin{aligned}&(A \otimes B)(a) \odot (A \otimes B)(b)\\ &= \Big(\bigvee_{x,y\in X} A(x) \odot B(y) \odot e_X(x * y, a)\Big) \odot \Big(\bigvee_{z,w\in X}(A(z) \odot B(w) \odot e_X(z * w, b)\Big)\\ &= \bigvee_{x,y,z,w\in X}\Big(A(x) \odot A(z) \odot B(y) \odot B(w) \odot e_X(x * y, a) \odot e_X(z * w, b)\Big)\\ &\le \bigvee_{x,y,z,w\in X}\Big(A(x * z) \odot B(y * w) \odot e_X(x * y * z * w, a * b)\Big)\\ &\le (A \otimes B)(a * b).\end{aligned}$$

(3) For $A \in M(L^X) \cap \tau_{e_X}$,

$$\begin{aligned}(A \otimes A)(a) &= \bigvee_{x,y\in X}(A(x) \odot A(y) \odot e_X(x * y, a))\\ &\le \bigvee_{x,y\in X}(A(x * y) \odot e_X(x * y, a))\end{aligned}$$

$\leq A(a).$ □

Theorem 3.14. *Let* $(X, \wedge, \vee, *, \nearrow, 0, 1)$ *be a c-quantale and* e_X *is an r-fuzzy poset on* X*. Define* $P : L^X \times L^X \times L^X \to L$ *as follows:*

$$P(A, B, C) = \bigwedge_{x \in X}((A \otimes B)(x) \to C(x)),$$
$$(A \otimes B)(x) = \bigvee_{y,z \in X}(A(y) \odot B(z) \odot e_X(y * z, x)).$$

Then

(1)(L^X, e_{L^X}, P) *is a cq-frame,*

(2) *if* $(X, \wedge, \vee, *, \nearrow, 0, 1)$ *is a sc-quantale and* $P : W(L^X) \times W(L^X) \times W(L^X) \to L$*, where* $W(L^X) = \{A \in L^X \mid A(1) = 1\}$*, then* $(W(L^X), e_{W(L^X)}, P)$ *is an scq-frame,*

(3) *if* $P : \tau_{e_X} \times \tau_{e_X} \times \tau_{e_X} \to L$*, then* $(\tau_{e_X}, e_{\tau_{e_X}}, P)$ *is a cq-frame.*

Proof. (1) (P1) For $A,\ B,\ C,\ A',\ B',\ C' \in L^X$,

$$\begin{aligned} & A'(y) \odot B'(z) \odot e_X(y * z, x) \odot (A'(y) \to A(y)) \odot (B'(z) \to B(z)) \\ & \quad \odot(A(y) \odot B(z) \odot e_X(y * z, x) \to C(x)) \odot (C(x) \to C'(x)) \\ \leq\ & A(y) \odot B(z) \odot e_X(y * z, x) \odot (A(y) \odot B(z) \\ & \quad \odot e_X(y * z, x) \to C(x)) \odot (C(x) \to C'(x)) \leq C'(x). \end{aligned}$$

Then we have

$$\begin{aligned} & (A'(y) \to A(y)) \odot (B'(z) \to B(z)) \odot (A(y) \odot B(z) \\ & \quad \odot e_X(y * z, x) \to C(x)) \odot (C(x) \to C'(x)) \\ \leq\ & (A'(y) \odot B'(z) \odot e_X(y * z, x)) \to C'(x). \end{aligned}$$

Thus we get

$$e_{L^X}(A', A) \odot e_{L^X}(B', B) \odot P(A, B, C) \odot e_{L^X}(C, C') \leq P(A', B', C').$$

(P2) For $A,\ B,\ C \in L^X$,

$$\begin{aligned} P(A, B, C) &= \bigwedge_{x \in X}((A \otimes B)(x) \to C(x)) \\ &= \bigwedge_{x \in X}((B \otimes A)(x) \to C(x)) \\ &= P(B, A, C). \end{aligned}$$

(P3) For $A,\ B,\ C,\ B',\ C' \in L^X$,

$$\begin{aligned} & P(A, B, C) \odot P(C, B', C') \\ \leq\ & \Big(A(a) \odot B(b) \odot e_X(a * b, c) \to C(c)\Big) \odot \Big(C(c) \odot B'(b') \odot e_X(c * b', c') \to C'(c')\Big) \\ =\ & \Big(A(a) \odot B(b) \odot e_X(a * b, c) \to C(c)\Big) \odot \Big(C(c) \to (B'(b') \odot e_X(c * b', c') \to C'(c')\Big) \\ =\ & \Big(A(a) \odot B(b) \odot e_X(a * b, c)\Big) \odot \Big((B'(b') \odot e_X(c * b', c') \to C'(c')\Big) \\ \leq\ & A(a) \odot B(b) \odot e_X(a * b, c) \odot B'(b') \odot e_X(c * b', c') \to C'(c'). \end{aligned}$$

Since

$$\begin{aligned} e_X(a * w, c') \odot e_X(b * b', w) &= e_X(w, a \nearrow c') \odot e_X(b * b', w) \\ &\leq e_X(b * b', a \nearrow c') \\ &= e_X(a * b, b' \nearrow c') \odot e_X(b' * (b' \nearrow c'), c') \\ &\leq \bigvee_{c \in X}(e_X(a * b, c) \odot e_X(c * b', c')), \end{aligned}$$

$$\begin{aligned} & P(A, B, C) \odot P(C, B', C') \\ \leq\ & \bigvee_{c \in X} \Big(A(a) \odot B(b) \odot e_X(a * b, c) \odot B'(b') \odot e_X(c * b', c')\Big) \to C'(c') \\ \leq\ & A(a) \odot B(b) \odot e_X(a * w, c') \odot B'(b') \odot e_X(b * b', w) \to C'(c'). \end{aligned}$$

Then we get

$P(A,B,C)\odot P(C,B',C')$
$\leq \bigwedge_{b,b'}\Big(A(a)\odot B(b)\odot e_X(a*w,c')\odot B'(b')\odot e_X(b*b',w)\to C'(c')\Big)$
$= A(a)\odot e_X(a*w,c')\odot\Big(\bigvee_{b,b'} B(b)\odot B'(b')\odot e_X(b*b',w)\Big)\to C'(c')\Big).$

Put $U(w)=(B\otimes B')(w)=\bigvee_{b,b'} B(b)\odot B'(b')\odot e_X(b*b',w)$. Then
$P(A,B,C)\odot P(C,B',C')$
$\leq \Big(\bigwedge_w(U(w)\to U(w))\odot\bigwedge_{c'}\Big(\bigvee_{a,w} A(a)\odot e_X(a*w,c')\odot U(w)\to C'(c')\Big)$
$\leq \bigvee_U P(B,B',U)\odot P(A,U,C')$.
(P4) For $A,\ B,\ C,\ A',\ C'\in L^X$,
$P(A,B,C)\odot P(A',C,C')$
$\leq \Big(A(a)\odot B(b)\odot e_X(a*b,c)\to C(c)\Big)\odot\Big(A'(a')\odot C(c)\odot e_X(a'*c,c')\to C'(c')\Big)$
$= \Big(A(a)\odot B(b)\odot e_X(a*b,c)\to C(c)\Big)\odot\Big(C(c)\to(A'(a')\odot e_X(a'*c,c')\to C'(c')\Big)$
$\leq A(a)\odot B(b)\odot e_X(a*b,c)\odot A'(a')\odot e_X(a'*c,c')\to C'(c')$.
Since

$$\begin{aligned}
e_X(a*a',w)\odot e_X(w*b,c') &= e_X(a*a',w)\odot e_X(w,b\nearrow c')\\
&\leq e_X(a*a',b\nearrow c')\\
&= e_X(a*b,a'\nearrow c')\odot e_X(a'*(a'\nearrow c'),c')\\
&\leq \bigvee_{c\in X}(e_X(a*b,c)\odot e_X(a'*c,c')),
\end{aligned}$$

$P(A,B,C)\odot P(A',C,C')$
$\leq \bigvee_{c\in X}\Big(A(a)\odot B(b)\odot e_X(a*b,c)\odot A'(a')\odot e_X(a'*c,c')\Big)\to C'(c')$
$\leq A(a)\odot A'(a')\odot e_X(a*a',w)\odot B(b)\odot e_X(w*b,c')\to C'(c')$.
Then we get
$P(A,B,C)\odot P(A',C,C')$
$\leq \bigwedge_{a,a'}\Big(A(a)\odot A'(a')\odot e_X(a*a',w)\odot B(b)\odot e_X(w*b,c')\to C'(c')\Big)$
$= (\bigvee_{a,a'}\Big(A(a)\odot A'(a')\odot e_X(a*a',w)\Big)\odot B(b)\odot e_X(w*b,c')\to C'(c')$.
Put $D(w)=(A\otimes A')(w)=\bigvee_{a,a'}(A(a)\odot A'(a')\odot e_X(a*a',w))$. Then
$P(A,B,C)\odot P(A',C,C')$
$\leq \bigwedge_w(D(w)\to D(w))\odot\bigwedge_{c'}(\bigvee_{b,w}(D(w)\odot B(b)\odot e_X(w*b,c'))\to C'(c'))$
$\leq \bigvee_D P(A',A,D)\odot P(D,B,C')$.
(2) (P5) Since $E\otimes B=B$ for $B\in W(L^X)$ and $E(1)=1, E(x)=0,$ otherwise,

$$\begin{aligned}
\bigvee_A P(A,B,B) &= \bigvee_A\Big(\bigwedge_{x\in X}((A\otimes B)(x)\to B(x))\Big)\\
&\geq \bigwedge_{x\in X}((E\otimes B)(x)\to B(x))=1.
\end{aligned}$$

(P6) Since $A\otimes B\geq A$ and $A\otimes B\geq B$ for $A,\ B\in W(L^X)$ from Theorem 3.13(1), we have

$$P(A,B,C)\leq e_{W(L^X)}(A,C)\text{ and }P(A,B,C)\leq e_{W(L^X)}(B,C).$$

Then $P(A,B,C)\leq e_{W(L^X)}(A,C)\wedge e_{W(L^X)}(B,C)$. □

Example 3.15. Let $(L,\wedge,\vee,\odot,\to,0,1)$ be a c-quantale and $(X,\vee,\wedge,*,\nearrow,0,1)$ be an sc-quantale with a fuzzy partially order e_X and $P(x,y,z)=e_X(x*y,z)$ for each $x,y,z\in X$ in Example 3.4 (2).
(1) By Example 3.4 (2), (X,e_X,P) is a cq-frame.

(2) $\tau_{e_X} = \{A \in L^X \mid A(x) \odot e_X(x,y) \le A(y)\}$
$= \{A \in L^X \mid \bigvee_{x \le y} A(x) \le A(y)\}$
$= \{A \in L^X \mid A(x) \le A(y), x \le y\}.$

(3) By Theorem 3.9 (1), for $A,\ B \in \tau_{e_X}$,

$$\begin{aligned}(A \otimes B)(z) &= \bigvee_{x,y \in X}(P(x,y,z) \odot A(x) \odot B(y)) \\ &= \bigvee_{x*y \le z}(A(x) \odot B(y)) \\ &= \bigvee_{y \in X}(A(y \nearrow z) \odot B(y)) \\ &= \bigvee_{x \in X}(A(x) \odot B(x \nearrow z)),\end{aligned}$$

$$\begin{aligned}(A \Rightarrow B)(x) &= \bigwedge_{y,z \in X}(P(x,y,z) \odot A(y) \to B(z)) \\ &= \bigwedge_{x \le (y \nearrow z)}(A(y) \to B(z)) \\ &= \bigwedge_{z \in X}(A(x \nearrow z) \to B(z)) \\ &= \bigwedge_{y \in X}(A(y) \to B(x*y)).\end{aligned}$$

Then $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a c-quantale.

(4) If $A, B \in \tau_{e_X}$ with $A(1) = B(1) = 1$, then

$$(A \otimes B)(z) = \bigvee_{x*y \le z}(A(x) \odot B(y)) \ge A(z) \odot A(1) = A(z),$$

that is, $A \otimes B \ge A$, $A \otimes B \ge B$ and, for $E(1) = 1, E(x) = 0,$ otherwise, $A \otimes E = A$ with $E \in \tau_{e_X}$.

(5) Let $A, B \in M(L^X)$, where $M(L^X) = \{A \in M(L^X) \mid A(a*b) \ge A(a) \odot A(b)\}$. Since $a \le b$ and $c \le d$, $a*c \le b*d$. Then $e_X(a,b) \odot e_X(c,d) \le e_X(a*c, b*d)$. Thus $A \otimes B \in M(L^X)$.

(6) If $A \in M(L^X) \cap \tau_{e_X}$, $e_X(a,b) \odot e_X(c,d) \le e_X(a*c, b*d)$, then $A \otimes A \le A$ from:

$$\begin{aligned}(A \otimes A)(a) &= \bigvee_{x*y \le a}(A(x) \odot A(y)) \ge A(a) \odot A(1) = A(a), \\ (A \otimes A)(a) &= \bigvee_{x,y \in X}(A(x) \odot A(y) \odot e_X(x*y, a)) \\ &\le \bigvee_{x,y \in X}(A(x*y) \odot e_X(x*y, a)) \le A(a).\end{aligned}$$

If $A(1) = 1$, then $A \otimes A \ge A$ from:

$$(A \otimes A)(z) = \bigvee_{x*y \le z}(A(x) \odot A(y)) \ge A(z) \odot A(1) = A(z).$$

(7) Put $W(L^X) = \{A \in L^X \mid A(1) = 1\}$. Define $P : W(L^X) \times W(L^X) \times W(L^X) \to L$ as follows:

$$P(A,B,C) = \bigwedge_{x \in X}((A \otimes B)(x) \to C(x)).$$

(P5) Since $E \otimes B = B$ from (4),

$$\begin{aligned}\bigvee_A P(A,B,B) &= \bigvee_A \Big((\bigwedge_{x \in X}((A \otimes B)(x) \to B(x)) \Big) \\ &\ge \bigwedge_{x \in X}((E \otimes B)(x) \to B(x)) = 1.\end{aligned}$$

(P6) Since $A \otimes B \ge A$ and $A \otimes B \ge B$, we have

$$P(A,B,C) \le e_{W(L^X)}(A,C) \text{ and } P(A,B,C) \le e_{W(L^X)}(B,C).$$

Then $P(A,B,C) \le e_{W(L^X)}(A,C) \wedge e_{W(L^X)}(B,C)$. Thus $(W(L^X), e_{W(L^X)}, P)$ is an scq-frame.

Example 3.16. Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ on $L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be two sc-quantales with an r-fuzzy poset (X, e_X) as in Example 3.4 (5) .

For $A = (0, \frac{1}{4}, \frac{1}{2}, 0)$, since $\frac{1}{2} = A(y) \odot e_X(y,1) \not\leq A(1) = 0$, $A \notin \tau_{e_X}$. Put $\overline{A} = \bigvee_{x\in X}(A(x) \odot e_X(x,y)) = (0, \frac{1}{4}, \frac{1}{2}, \frac{1}{2})$. Then $\overline{A} \in \tau_{e_X}$.

For $B = (\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4})$, Since $\frac{3}{4} = B(y) \odot e_X(y,1) \not\leq B(1) = \frac{1}{4}$, $B \notin \tau_{e_X}$. Put $\overline{B} = \bigvee_{x\in X}(B(x) \odot e_X(x,y)) = (\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4})$. Then $\overline{B} \in \tau_{e_X}$.

For $\overline{A},\ \overline{B} \in \tau_{e_X}$,

$$\begin{aligned}(\overline{A} \otimes \overline{B})(z) &= \bigvee_{x\in X}(\overline{A}(x) \odot \overline{B}(x \nearrow z))\\ &= \bigvee_{y\in X}(\overline{A}(y \nearrow z) \odot \overline{B}(y))\\ &= (0, 0, \tfrac{1}{4}, \tfrac{1}{4}),\end{aligned}$$

$$\begin{aligned}(\overline{A} \Rightarrow \overline{B})(x) &= \bigwedge_{y\in X}(\overline{A}(y) \to \overline{B}(x \odot y))\\ &= \bigwedge_{z\in X}(\overline{A}(x \nearrow z) \to \overline{B}(z))\\ &= (1, 1, 1, 1),\end{aligned}$$

$$\begin{aligned}(\overline{B} \Rightarrow \overline{A})(x) &= \bigwedge_{y\in X}(\overline{B}(y) \to \overline{A}(x \odot y))\\ &= \bigwedge_{z\in X}(\overline{B}(x \nearrow z) \to \overline{A}(z))\\ &= (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{3}{4}).\end{aligned}$$

In Theorem 3.9 (2), $1 = e_X(x * x, 0) \not\leq e_X(x,0) \wedge e_X(x,0) = \frac{1}{2}$. For $1_X \in \tau_{e_X}$,

$$\overline{A} \otimes 1_X = \bigvee_{x\in X}(\overline{A}(x) \odot 1_X(x \nearrow z)) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \geq \overline{A},$$

$$\overline{B} \otimes 1_X = \bigvee_{x\in X}(\overline{B}(x) \odot 1_X(x \nearrow z)) = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}) \geq \overline{B}.$$

For $E = (1, 0, 0, 0)$, $\overline{E} = \bigvee_{x\in X}(E(x) \odot e_X(x,y)) = 1_X$, $\overline{B} \otimes E = \overline{B}$.

Then $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a c-quantale but not a sc-quantale.

Example 3.17. Let $(X = [0,\infty], \leq_{op}, \vee_{op}, +, \wedge_{op}, \infty, 0)$ be an sc-quantale where $\leq_{op} = \geq, \vee_{op} = \wedge, \wedge_{op} = \vee$ and

$$\begin{aligned}x \nearrow y &= \bigvee\nolimits_{op}\{z \in [0,\infty] \mid x + z \leq_{op} y\}\\ &= \bigwedge\{z \in [0,\infty] \mid z \geq -x + y\}\\ &= (y - x) \vee 0,\end{aligned}$$

$$e_X^0(x,y) = \begin{cases} 0 & \text{if } x \geq y \\ \infty & \text{if } x < y. \end{cases}$$

Define $P(x,y,z) = e_X^0(x+y,z)$ for each $x,y,z \in [0,\infty]$.

(1) By Theorem 3.3, $([0,\infty], e_X^0, P)$ is a cq-frame with (P5)

$$\bigvee_{x\in X} P(x,y,y) = (\bigvee_{op})_{x\in X} e_X^0(x+y,y) \geq_{op} e_X^{\infty}(0+y,y) = 0.$$

But $0 = e_X^0(2+3,4) \not\leq_{op} e_X^0(2,4) \wedge e_X^0(3,4) = \infty$. Then $([0,\infty], e_X^0, P)$ is not an scq-frame.

(2) $$\begin{aligned}\tau_{e_X^0} &= \{A \in [0,\infty]^{[0,\infty]} \mid A(x) + e_X^0(x,y) \leq_{op} A(y)\}\\ &= \{A \in [0,\infty]^{[0,\infty]} \mid \bigwedge\nolimits_{x\geq y} A(x) \leq_{op} A(y)\}\\ &= \{A \in [0,\infty]^{[0,\infty]} \mid A(x) \geq A(y), x \geq y\},\end{aligned}$$

because $A(x) \geq \bigwedge_{x\geq y} A(x) \geq A(y)$ for $y \leq x$.

(3) For $A,\ B\in\tau_{e^0_X}$, we define

$$\begin{aligned}(A\otimes B)(z) &= (\bigvee_{op})_{x,y\in X}(P(x,y,z)+A(x)+B(y))\\ &= \bigwedge_{x+y\geq z}(A(x)+B(y))\\ &= \bigwedge_{y\in X}(A((z-y)\vee 0)+B(y))\\ &= \bigwedge_{x\in X}(A(x)+B((z-x)\vee 0)),\end{aligned}$$

$$\begin{aligned}(A\Rightarrow B)(x) &= (\bigwedge_{op})_{y,z\in X}(P(x,y,z)+A(y))\nearrow B(z))\\ &= \bigvee_{x\geq(-y+z)\vee 0}((B(z)-A(y))\vee 0)\\ &= \bigvee_{z\in X}((B(z)-A((z-x)\vee 0))\vee 0)\\ &= \bigvee_{y\in X}((B(x+y)-A(y))\vee 0).\end{aligned}$$

Then $(\tau_{e^0_X},\leq_{op}=\geq,\vee_{op},\wedge_{op},\otimes,\Rightarrow,\infty_X,0_X)$ is a c-quantale where $0_X(x)=0$ and $\infty_X(x)=\infty$ for each $x\in[0,\infty]$.

(4) If $A,B\in\tau_{e^0_X}$ with $A(0)=B(0)=0$, then

$$(A\otimes B)(z)=\bigwedge_{x+y\geq z}(A(x)+B(y))\leq A(z)+A(0)=A(z).$$

Thus $A\otimes B\geq_{op}A$, $A\otimes B\geq_{op}B$ and, for $E(0)=0, E(x)=\infty,$ otherwise, $A\otimes E=A$ with $E\in\tau_{e^0_X}$.

(5) Let $A,B\in M([0,\infty]^{[0,\infty]})$, where

$$M([0,\infty]^{[0,\infty]})=\{A\in[0,\infty]^{[0,\infty]}\mid A(a+b)\geq_{op}A(a)+A(b)\}.$$

Since $a\geq b$ and $c\geq d$, $a+c\geq b+d$. Then $e^0_X(a,b)+e^0_X(c,d)\leq_{op}e^0_X(a+c,b+d)$. Thus $A\otimes B\in M([0,\infty]^{[0,\infty]})$.

(6) Let $A\in M([0,\infty]^{[0,\infty]})\cap\tau_{e^0_X}$. Since $e^0_X(a,b)+e^0_X(c,d)\leq_{op}e^0_X(a+c,b+d)$, $A\otimes A\geq A$ from:

$$\begin{aligned}(A\otimes A)(a) &= (\bigvee_{op})_{x,y\in X}(A(x)+A(y)+e^0_X(x+y,a))\\ &\leq_{op}(\bigvee_{op})_{x,y\in X}(A(x+y)+e^0_X(x+y,a))\\ &\leq_{op}A(a).\end{aligned}$$

If $A(0)=0$, then $A\otimes A\leq A$ from:

$$(A\otimes A)(a)=\bigwedge_{x+y\geq z}(A(x)+A(y))\leq A(a)+A(0)=A(a).$$

(7) Put $W([0,\infty]^{[0,\infty]})=\{A\in[0,\infty]^{[0,\infty]}\mid A(0)=0\}$. Define

$$P:W([0,\infty]^{[0,\infty]})\times W([0,\infty]^{[0,\infty]})\times W([0,\infty]^{[0,\infty]})\to[0,\infty]$$

as follows:

$$P(A,B,C)=(\bigwedge_{op})_{x\in X}((A\otimes B)(x)\nearrow C(x)).$$

(P5) Since $E\otimes B=B$, from (4),

$$\begin{aligned}(\bigvee_{op})_A P(A,B,B) &= (\bigvee_{op})_A\Big((\bigwedge_{op})_{x\in X}((A\otimes B)(x)\nearrow B(x))\Big)\\ &\geq_{op}(\bigwedge_{op})_{x\in X}((E\otimes B)(x)\nearrow B(x))=0.\end{aligned}$$

(P6) Since $A\otimes B\geq_{op}A$ and $A\otimes B\geq_{op}B$, we have

$$P(A,B,C)\leq_{op}e_{W([0,\infty]^{[0,\infty]})}(A,C)\text{ and }P(A,B,C)\leq_{op}e_{W([0,\infty]^{[0,\infty]})}(B,C).$$

Then we get

$$P(A,B,C)\leq_{op}e_{W([0,\infty]^{[0,\infty]})}(A,C)\wedge_{op}e_{W([0,\infty]^{[0,\infty]})}(B,C).$$

Thus $(W([0,\infty]^{[0,\infty]}), e_{W([0,\infty]^{[0,\infty]})}, P)$ is an scq-frame.

Example 3.18. Let $(P(X), \cap, \cup, \odot = \cap, \nearrow, \varnothing, X)$ be an sc-quantale, where

$$A \nearrow B = \bigcup\{C \in P(X) \mid A \cap C \subset B\} = A^c \cup B.$$

(1) Define a map $e^1 : P(X) \times P(X) \to P(X)$ as

$$e^1(A,B) = A \nearrow B = A^c \cup B.$$

Then $(P(X), e^1)$ is a fuzzy poset.

(2) $(P(X), e^1, P^1)$ is a cq-frame, where $P^1(A,B,C) = e^1(A \cap B, C)$ for each $A,\ B,\ C \in P(X)$ from Theorem 3.3. But it is not an sc-quantale frame, because $X = \{a,b,c\}, A = \{a,b\}, B = \{b,c\}$ and $P^1(A,B,A) = e^1(A \cap B, A) = X \not\subset e^1(A,A) \cap e^1(B,A) = \{a,b\}$.

(3) Let $\tau_{e^1} = \{\alpha \in P(X)^{P(X)} \mid \alpha(A) \cap e^1(A,B) \subset \alpha(B)\}$. For $\alpha, \beta \in \tau_{e^1}$, we define

$$\begin{aligned}(\alpha \otimes \beta)(C) &= \bigvee_{A,B \in P(X)}(e^1(A \cap B, C) \cap \alpha(A) \cap \beta(B))\\ &= \bigvee_{B \in P(X)}(\alpha(B^c \cup C) \cap \beta(B))\\ &= \bigvee_{A \in P(X)}(\alpha(A) \cap \beta(A^c \cup C)),\end{aligned}$$

$$\begin{aligned}(\alpha \Rightarrow \beta)(A) &= \bigwedge_{B,C \in P(X)}(e^1(A \cap B, C) \cap \alpha(B) \nearrow \beta(C))\\ &= \bigwedge_{B \in P(X)}(\alpha(A^c \cup B) \nearrow \beta(B))\\ &= \bigwedge_{C \in P(X)}(\alpha(C) \nearrow \beta(A \cap C)).\end{aligned}$$

Then $(\tau_{d^1}, \vee, \wedge, \otimes, \Rightarrow, 0_{P(X)}, 1_{P(X)})$ is a commutative co-quantale, where $0_{P(X)}(A) = \varnothing, 1_{P(X)}(A) = X$ for each $A \in P(X)$.

(4) If $\alpha, \beta \in \tau_{e^1}$ with $\alpha(X) = \beta(X) = X$, then

$$(\alpha \otimes \beta)(C) \supset e^1(C \cap X, C) \cap \alpha(C) \cap \beta(X) = \alpha(C).$$

For $\gamma(X) = X, \gamma(A) = \varnothing$, otherwise, we have $\gamma \in \tau_{e^1}$,

$$(\alpha \otimes \gamma)(C) = e^1(C \cap X, C) \cap \alpha(C) \cap \gamma(X) = \alpha(C).$$

Thus $\alpha \otimes \beta \supset \alpha$, $\alpha \otimes \beta \supset \beta$ and $\alpha \otimes \gamma = \alpha$.

(5) Let $\alpha,\ \beta \in M(P(X)^{P(X)})$, where

$$M(P(X)^{P(X)}) = \{\alpha \in P(X)^{P(X)} \mid \alpha(A \cap B) \supset \alpha(A) \cap \alpha(B)\}.$$

Then by Theorem 3.13 (2), $\alpha \otimes \beta \in M(P(X)^{P(X)})$.

(6) If $\alpha \in M(P(X)^{P(X)}) \cap \tau_{e^1}$ and $\alpha(X) = X$, then $\alpha \otimes \alpha = \alpha$ from:

$$(\alpha \otimes \alpha)(C) \supset e^1(C \cap X, C) \cap \alpha(C) \cap \alpha(X) = \alpha(C),$$

$$\begin{aligned}(\alpha \otimes \alpha)(C) &= \bigvee_{A,B \in P(X)}(e^1(A \cap B, C) \cap \alpha(A) \cap \alpha(B))\\ &\subset \bigvee_{A,B \in P(X)}(e^1(A \cap B, C) \cap \alpha(A \cap B)))\\ &\subset \alpha(C).\end{aligned}$$

(7) Put $W(\tau_{e^1}) = \{\alpha \in \tau_{e^1} \mid \alpha(X) = X\}$. Define $e_W : W(\tau_{e^1}) \times W(\tau_{e^1}) \to P(X)$ and $P : W(\tau_{e^1}) \times W(\tau_{e^1}) \times W(\tau_{e^1}) \to P(X)$ as follows:

$$\begin{aligned}e_W(\alpha,\beta) &= \bigwedge_{A \in P(X)}((\alpha(A) \nearrow \beta(A)),\\ P(\alpha,\beta,\eta) &= \bigwedge_{A \in P(X)}((\alpha \otimes \beta)(A) \nearrow \eta(A)) = e_W(\alpha \otimes \beta, \eta).\end{aligned}$$

(P5) Since $\beta \otimes \gamma = \beta$, from (4),

$$\bigvee_\alpha P(\alpha,\beta,\beta) = \bigvee_\alpha \Big(\bigwedge_{A\in P(X)}((\alpha\otimes\beta)(A) \nearrow \beta(A))\Big)$$
$$\geq \bigwedge_{A\in P(X)}((\gamma\otimes\beta)(A) \nearrow \beta(A)) = X.$$

(P6) Since $\alpha\otimes\beta \supset \alpha$ and $\alpha\otimes\beta\supset\beta$, we have

$$P(\alpha,\beta,\eta)\subset e_W(\alpha,\eta) \text{ and } P(\alpha,\beta,\eta)\subset e_W(\alpha,\eta).$$

Then $P(\alpha,\beta,\eta)\subset e_W(\alpha,\eta)\cap e_W(\beta,\eta)$. Thus $(W(P(X)^{P(X)}), e_W, P)$ is an scq-frame.

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