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New concepts on R_1 fuzzy soft topological spaces

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Abstract. In this paper, we have introduced and studied some new notions of R_1 separation axiom in fuzzy soft topological spaces by using quasi-coincident relation for fuzzy soft points. We have observed that all these notions satisfy good extension property. We have shown that these notions are preserved under the one-one, onto and FSP continuous mapping. Moreover, we have obtained some other properties of this new concept.

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1. Introduction

In 1999, the Russian researcher Molodtsov [1] introduced the concept of a soft set and pointed out several directions, e.g., game theory, Riemann integration, theory of measurement, smoothness of functions and so on. Maji et al. [2] presented some new definitions on soft sets and discussed in detail the application of soft set theory in decision making problems. Chen et al. [3] studied the parametrization reduction of soft sets. Ahmat and Kharal [4] presented some more properties of fuzzy soft sets and introduced the notion of a mapping on fuzzy soft sets. Şenel [5, 6] represented the relation between soft topological space and soft ditopological space and also introduced a new approach to Hausdorff space theory via the soft sets. Aktas and Cagman [7] defined the notion of soft groups and derived some properties. In 2010, Nazmul and Samanta [8] defined soft topological groups, normal soft topological groups and homomorphisms. Furthermore, Shabir and Naz [9] introduced the concept of soft topological space and studied neighborhoods and separation axioms. B. Pazar Varol et al. [10] interpreted categories related to categories of topological spaces as special categories of soft sets. Also, Tripathy [11], Acharjee [12] and Debnath [13] developed fuzzy soft bitopological spaces. In 2011, Tanay et al. [14] gave the topological structure of fuzzy soft sets.

2. Preliminaries

Now we recall some definitions and concepts which will be used in our work.

Definition 2.1 ([1])**.** A pair (F, E) is said to be a *soft set* over the initial universe X, if F is a mapping from E to $P(X)$, where $P(X)$ is the collection of subsets of X.

Definition 2.2 ([1])**.** Let $A \subseteq E$. Then a pair (F, A) is called a *soft set* over X, provided that F is a mapping given by $F : A \longrightarrow P(X)$ such that $F(e) = \varnothing$, if $e \notin A$ and $F(e) \neq \varnothing$, if $e \in A$, where X is an initial universe set and E be the set of parameters, $P(X)$ be the set of all subsets of X. Here F is called an *approximate function* of the soft set (F, A) and the value $F(e)$ is a set called e-element of the soft set. In other words, the soft set is a parameterized family of subsets of the set X.

Definition 2.3 ([15])**.** Let X be an initial universe set and let E be a set of parameters. Let $I^X (I = [0, 1])$ denotes the set of all fuzzy sets of X. Let $A \subseteq E$. A pair (F, A) is called a *fuzzy soft set* over X, provided that F is a mapping given by $F : A \longrightarrow I^X$ such that $F(e) = O_X$, if $e \notin A$ and $F(e) = O_x$, if $e \in A$, where $O_x = 0$ for all $x \in X$. Here F is called approximate function of the fuzzy soft set (F, A) and the value $F(e)$ is a fuzzy set called e-element of the fuzzy soft set (F, A). Thus a fuzzy soft set (F, A) over X can be represented by the set of ordered pairs $(F, A) = \{(e, F(e)) : e \in A, F(e) \in I^X\}$. In other words, the fuzzy soft set is a parameterized family of fuzzy subsets of the set X.

Definition 2.4 ([16])**.** A soft set (F, E) is said to be a *soft point* over X, if there exist $t \in E$ and $x \in X$ such that

$$F(t) = \begin{cases} \{x\} \; if \; t = e \\ \varnothing \quad if \; t \in E - \{e\}. \end{cases}$$

In this case, x is called the *support point* of the soft point, x is called the *support set* of the soft point and e is called the *expressive parameter*.

Definition 2.5 ([17])**.** Let $f \in S$.

(i) If $f(e) = \varnothing$ for all $e \in E$, then f is called the *null soft point* and denoted by e_Φ.

(ii) If $f(e) = U$ for all $e \in E$, then f is called the *universal soft point* and denoted by $e_{\tilde{E}}$.

(iii) If there is only one parameter $e \in E$ in f, then f is denoted by e_{f_i}.

(iv) If there is only one parameter $e \in E$ in f and $f(e) = \{u\}$, then f is denoted by e_f.

Definition 2.6 ([15])**.** The fuzzy soft set $f_A \in FSS(X, E)$ is called a *fuzzy soft point*, if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha$ $(0 \leq \alpha \leq 1)$ and $\mu^e_{f_A}(y) = 0 \forall y \in X - \{x\}$, and this fuzzy soft point is denoted by x^e_α or f_e. The class of all fuzzy soft points of X is denoted by $FSP(X, E)$.

Definition 2.7 ([18])**.** A fuzzy soft point x^e_α over X is a fuzzy soft set over X defined as follows:

$$x^e_\alpha(\acute{e}) = \begin{cases} x_\alpha \; if \acute{e} = e \\ 0 \quad if \acute{e} \in E - \{e\}. \end{cases}$$

Where, x_α is the fuzzy point in X with support x and value $\alpha, \alpha \in (0,1]$. The set of all fuzzy soft points in X is denoted by $FSP(X,E)$.

The fuzzy soft point x^e_α is said to *belong to* a fuzzy soft set f_E, denoted by $x^e_\alpha \in f_E$, if $\alpha \leq f(e)(x)$. Every non-null fuzzy soft set f_E can be expressed as the union of all the fuzzy soft points belonging to f_E. The *complement* of a fuzzy soft point x^e_α is a fuzzy soft set over X.

Definition 2.8 ([2])**.** Let (F,A) and (G,B) be two soft sets over a common universe X and over a common parameter E. Then the *union* of two soft sets (F,A) and (G,B) over X is the soft set (H,C), where $C = A \cup B$, is defined by:

$$H(e) = \begin{cases} F(e) & if \; e \in A - B \\ G(e) & if \; e \in B - A \\ F(e) \cup G(e) & if \; e \in B \cap A, \end{cases}$$

for each $e \in C$. It is denoted by $(H,C) = (F,A) \cup (G,B)$.

Definition 2.9 ([2])**.** Let (F,A) and (G,B) be two soft sets over a common universe X and over a common parameter E. The *intersection* of two soft sets (F,A) and (G,B) over X is the soft set (H,C), where $C = A \cap B$, is defined by $H(e) = F(e) \cap G(e)$, for each $e \in C$. It is denoted by $(H,C) = (F,A) \cap (G,B)$.

Definition 2.10 ([14])**.** The *fuzzy soft complement* of a fuzzy soft set (F,A), denoted by $(F,A)^c$ and is defined as $(F,A)^c = (F^c,A)$, where $F^c(e) = 1 - F(e)$, for every $e \in A$. Clearly $((F,A)^c)^c = (F,A)$ and $(1_E)^c = 0_E$ and $(0_E)^c = 1_E$.

Definition 2.11 ([19])**.** The fuzzy soft sets (F,E) and (G,E) in (X,E) are said to be *fuzzy soft quasi-coincident*, denoted by $(F,E)q(G,E)$, if there exist $e \in E,\; x \in X$ such that $F(e)(x) + G(e)(x) > 1$.

If (F,E) is not fuzzy soft quasi-coincident with (G,E), then we write $(F,E)\bar{q}(G,E)$, i.e., $(F,E)\bar{q}(G,E)$ if and only if $F(e)(x) + G(e)(x) \leq 1$, i.e., $F(e)(x) \leq G^c(e)(x)$ for all $x \in X$ and $e \in E$.

A fuzzy soft point x^e_α is said to be *soft quasi-coincident with* fuzzy soft set (F,E), denoted by $x^e_\alpha q(F,E)$, if there exist $e \in E, x \in X$ such that $\alpha + F(e)(x) > 1$ and if $x^e_\alpha \bar{q}(F,E)$, then $\alpha + F(e)(x) \leq 1$.

Definition 2.12 ([15])**.** A *fuzzy soft topology* τ on (X,E) is a family of fuzzy soft sets over (X,E), satisfying the following properties:
(i) $0_E, 1_E \in \tau$,
(ii) if $(F,A),\; (G,B) \in \tau$, then $(F,A) \cap (G,B) \in \tau$,
(iii) if $(F,A)_\alpha \in \tau \; \forall \alpha \in \lambda$, then $\bigcup_{\alpha \in \Lambda} (F,A)_\alpha \in \tau$.

If τ is a fuzzy soft topology on (X,E), the triple (X,τ,E) is called a *fuzzy soft topological space*. Each member of τ is called a *fuzzy soft open set* in (X,τ,E). A fuzzy soft set (F,E) over X is called a *fuzzy soft closed*, if $(F,E)^c \in \tau$.

Definition 2.13 ([20])**.** The *Cartesian product* of two fuzzy soft sets (F, A) and (G, B) is defined as a fuzzy soft set $(H, C) = (F, A) \times (G, B)$, where $C = A \times B$ and $H : C \longrightarrow (X, E)$ is defined by $H(e, \acute{e}) = F(e) \times G(\acute{e})$ for all $(e, \acute{e}) \in C$, where

$$F(e) \times G(\acute{e}) = \{x/ \min\{F(e)(x), G(\acute{e})(x)\} : x \in X\}.$$

Definition 2.14 ([21])**.** Let $F_A \in FSS(X, E)$ and $G_B \in FSS(Y, K)$. Then, the *fuzzy soft product* of F_A and G_B, denoted by $F_A \times G_B$, is a fuzzy soft set over $X \times Y$ and is defined by $(F_A \times G_B)(e, k) = F_A(e) \times G_B(k)\ \forall (e, k) \in E \times K$ and for all $(x, y) \in X \times Y$, we have

$$(F_A \times G_B)(e, k)(x, y) = (F_A(e) \times G_B(k))(x, y) = \min\ \{F_A(e)(x), G_B(k)(y)\}.$$

Definition 2.15. Let $\{(X_i, E_i), i \in \Lambda\}$ be any family of soft sets and let X and E denote the Cartesian product of these soft sets, i.e., $X = \Pi_{i\in\Lambda} X_i$ and $E = \Pi_{i\in\Lambda} E_i$. Note that (X, E) consists of all soft points $P = \langle (x_i)^{e_i}_{\alpha}, i \in \Lambda\ and\ \alpha \in (0, 1)\rangle$, where $x_i \in X$ and $e_i \in E_i$. Recall that for each $j_0 \in \Lambda$, we define the *projection* $(P_q)_{j_0}$ from the product soft set (X, E) to the soft co-ordinate space (X_{j_0}, E_{j_0}), i.e., $(P_q)_{j_0} : (X, E) \longrightarrow (X_{j_0}, E_{j_0})$ by $(P_q)_{j_0}((x_i)^{e_i}_{\alpha}) = (x_{j_0})^{e_{j_0}}_{\alpha}$. These projections are used to define the soft product topology.

Definition 2.16 ([22])**.** The soft mappings $(P_q)_i, i \in \{1, 2\}$ is called a *soft projection mapping* from $FSS(X_1, A_1) \times FSS(X_2, A_2)$ to $FSS(X_i, A_i)$ and is defined by $(P_q)_i((F_1, A_1) \times (F_2, A_2)) = P_i(F_1 \times F_2)_{q_i(A_1\times A_2)} = (F_i, A_i)$, where $(F_1, A_1) \in FSS(X_1, A_1)$ and $(F_2, A_2) \in FSS(X_2, A_2)$ and also $P_i : X_1 \times X_2 \longrightarrow X_i$ and $q_i : A_1 \times A_2 \longrightarrow A_i$ are projection mapping is classical meaning.

Definition 2.17 ([22])**.** Let $FSS(X, E)$ and $FSS(Y, K)$ be the collection of all the fuzzy soft sets over X and Y, respectively and $E,\ K$ be the parameters sets for X and Y, respectively. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be two maps. Then the *fuzzy soft mapping from X to Y* is a f_{up} and is denoted by $f_{up} : FSS(X, E) \longrightarrow FSS(Y, K)$.

(i) Let $(F, A) \in FSS(X, E)$. Then the *image* of (F, A) under the fuzzy soft mapping f_{up} is a fuzzy soft set over Y, denoted by $f_{up}(F, A)$ and is defined as

$$f_{up}(F, A)(k)(y) = \begin{cases} \sup\{u(x) = y\}\ \sup\{p(e) = k\}\ F_A(e)(x) \\ \qquad\qquad\qquad\quad \text{if } u^{-1}(y) \neq \varnothing\ and\ p^{-1}(k) \neq \varnothing \\ 0 \qquad\qquad\qquad\quad \text{otherwise,} \end{cases}$$

$\forall y \in Y, k \in K$.

(ii) Let $(G, B) \in FSS(Y, K)$. Then the *inverse image* of (G, B) under the fuzzy soft mapping f_{up} is a fuzzy soft set over X, denoted by $f^{-1}_{up}(G, B)$ and is defined as

$$f^{-1}_{up}(G, B)(e)(x) = (G, B)(p(e))(u(x))\ \forall e \in E,\ x \in X.$$

Definition 2.18 ([23])**.** Let $(f, \tilde{\tau})$ be a soft topological space and $g \tilde{\subseteq} f$. Then the collection

$$\tilde{\tau}_g = \{h \tilde{\cap} g : h \in \tilde{\tau}\}$$

is called a *soft subspace topology* on g and $(g, \tilde{\tau}_g)$ is called a *soft topological subspace* of $(f, \tilde{\tau})$.

Definition 2.19 ([23])**.** A soft topological property is said to be *hereditary*, if whenever a soft topological space $(f, \tilde{\tau})$ has that property, then so does every soft topological subspace of it.

Definition 2.20. Let X be a non-empty set and T be a soft topology on (X, E), where E is a parameters set. Let $\tau = \omega(T)$ be the set of all fuzzy soft lower semi-continuous mappings from (X, T, E) to I^X (The family of all fuzzy sets in X). Then

$$\omega(T) = \{(G, E) \in FSS(X, E) : (G, E)^{-1}(\alpha, 1] \in T\} \text{ for each } \alpha \in I_1.$$

It can be shown that $\omega(T)$ is a fuzzy soft topology on (X, E).

Let P be the property of a soft topological space (X, T, E) and FSP be its topological analogue. Then FSP is called a *ôgood extensionö* of P, if the statement (X, T, E) has P, i.e., $(X, \omega(T), E)$ has FSP holds good for every soft topological space (X, T, E).

Definition 2.21 ([21])**.** Let $\{(X_i, \tau_i), i \in \Lambda\}$ be a family of fuzzy soft topological spaces relative to the parameters sets Ei respectively. Then their product is defined as the fuzzy soft topological space (X, τ, E); where $X = \Pi_{i\in\Lambda} Xi, E = \Pi_{i\in\Lambda} Ei$ and τ is the fuzzy soft topology over X which is initial with respect to the family $\{(PXi, qEi), i \in \Lambda\}, PXi : \Pi_{i\in\Lambda} Xi \longrightarrow Xi$ and $qEi : \Pi_{i\in\Lambda} Ei \longrightarrow Ei, i \in \Lambda$ are the projection maps i.e τ is generated by

$$\{(PXi, qEi)^{-1}(F, Ai) : i \in \Lambda, (F, Ai) \in \tau_i\}.$$

Definition 2.22. Let $\{(X_i, \tau_i), i \in \Lambda\}$ be a family of fuzzy soft topological spaces relative to the parameters sets E_i respectively, X be a non-empty set with parameters set E and for each $i \in \Lambda, (f_{up})_i : (X_i, \tau_i) \longrightarrow X$ be a soft mappings. Then the fuzzy soft topology τ over X is said to be *final with respect to* the family $\{(f_{up})_i; i \in \Lambda\}$, if τ has as subbase the set

$$S = \{(f_{up})_i(F, Ai) : i \in \Lambda, (F, Ai) \in \tau_i\},$$

i.e., the fuzzy soft topology τ over X is generated by S.

Definition 2.23 ([22])**.** Let $f_p : FSS(X, A) \longrightarrow FSS(Y, B)$ and $g_q : FSS(Y, B) \longrightarrow FSS(Z, C)$ be two fuzzy soft mappings.Then the composition of f_p and g_q is denoted by $f_p o g_q$ and defined by $f_p o g_q = f o g_{poq}$.

Definition 2.24 ([24])**.** A fuzzy soft topological space (X, τ, E) is said to be fuzzy soft R_1 (FSR_1,for short) iff for every $x^e_\alpha, y^e_\beta \in FSP(X, E)$with $x^e_\alpha \bar{q} \bar{y}^e_\beta$ implies there exist $O_{x^e_\alpha}, O_{y^e_\beta} \in \tau$ such that $O_{x^e_\alpha} \bar{q} O_{y^e_\beta}$.

3. Main results

Definition 3.1. A fuzzy soft topological space (X, τ, E) is called a:

(i) FSR_1(i) space ,if for any pair of fuzzy soft points x^e_r, y^e_s in (X, E) with $x \neq y$, whenever there exists $(H, E) \in \tau$ with $H(e)(x) \neq H(e)(y)$ $\forall e \in E$, then there exist $(F, E), (G, E) \in \tau$ such that $x^e_r q(F, E)$, $y^e_s q(G, E)$ and $(F, E) \cap (G, E) = \varnothing$.

(ii) FSR_1(ii) space, if for any pair of fuzzy soft points x^e_r, y^e_s in (X, E) with $x \neq y$, whenever there exists $(H, E) \in \tau$ with $H(e)(x) \neq H(e)(y)$ $\forall e \in E$, then there exist $(F, E), (G, E) \in \tau$ such that $x^e_r \in (F, E)$, $y^e_s \in (G, E)$ and $(F, E) \bar{q} (G, E)$.

(iii) FSR_1(iii) space, if for any pair of fuzzy soft points x^e_r, y^e_s in (X, E) with $x \neq y$, whenever there exists $(H, E) \in \tau$ with $H(e)(x) \neq H(e)(y)$ $\forall e \in E$, then there exist $(F, E), (G, E) \in \tau$ such that $x^e_r q(F, E)$, $y^e_s q(G, E)$ and $(F, E) \bar{q} (G, E)$.

3.1. **Subspaces in fuzzy soft $\mathbf{R}_1$ topological spaces.** In this section, we show that our notions satisfy hereditary property.

Theorem 3.2. *Let (X, τ, E) be a fuzzy soft topological space, $A \subseteq X$, $t_A = \{(F_A, E) = (F, E) \cap A : (F, E) \in \tau\}$, then (X, τ, E) is $FSR_1(j) \Rightarrow (A, t_A, E)$ is $FSR_1(j)$, where $j = i,\ ii,\ iii$.*

Proof. Let (X, τ, E) be a fuzzy soft topological space and (X, τ, E) is $\text{FSR}_1(j)$. We have to prove that (A, t_A, E) is $\text{FSR}_1(j)$. Let x_r^e, y_s^e be fuzzy soft points in (A, E) with $x \neq y$. Then x_r^e, y_s^e are also fuzzy soft points in (X, E) as $A \subseteq X$ with $x \neq y$. Consider $(M, E) \in t_A$ with $M(e)(x) \neq M(e)(y)$. Here (M, E) can be written as $(F_A, E) = (F, E) \cap A$, where $(F, E) \in \tau$. Then $F(e)(x) \neq F(e)(y)$. Since (X, τ, E) is $\text{FSR}_1(j)$ fuzzy soft topological space, there exist $(G, E), (H, E) \in \tau$ such that $x_r^e q(G, E)$, $y_s^e q(H, E)$ and $(G, E) \cap (H, E) = \varnothing$. From the definition of t_A, we obtain $(G_A, E) = ((G, E) \cap A), (H_A, E) = ((H, E) \cap A) \in t_A$.

Now, we have the following implications:

$$\begin{aligned} y_s^e q(H, E) &\Rightarrow H(e)(y) + \text{s} > 1\ \forall y \in X, e \in E \\ &\Rightarrow H(e)(y) \cap A(y) + s > 1\ y \in A \subseteq X \\ &\Rightarrow ((H, E) \cap A)(e)(y) + s > 1 \\ &\Rightarrow (H_A, E)(e)(y) + s > 1 \\ &\Rightarrow y_s^e q(H_A, E). \end{aligned}$$

Also, we obtain the following implications:

$$\begin{aligned} x_r^e q(G, E) &\Rightarrow G(e)(x) + r > 1,\ x \in X,\ e \in E \\ &\Rightarrow G(e)(x) \cap A(x) + r > 1,\ x \in A \subseteq X \\ &\Rightarrow ((G, E) \cap A)(e)(x) + r > 1 \\ &\Rightarrow (G_A, E)(e)(x) + r > 1 \\ &\Rightarrow x_r^e q(G_A, E). \end{aligned}$$

Further, we have

$$\begin{aligned} &(G, E) \cap (H, E) = \varnothing \\ &\Rightarrow (G \cap H)(e)(x) = 0\ \forall x \in X,\ e \in E \\ &\Rightarrow \min\ (G(e)(x), H(e)(x)) = 0 \\ &\Rightarrow \min\ (((G, E) \cap A)(e)(x), ((H, E) \cap A)(e)(x)) = 0\ x \in A \subseteq X \\ &\Rightarrow \min\ ((G_A, E)(e)(x), (H_A, E)(e)(x)) = 0 \\ &\Rightarrow ((G_A, E) \cap (H_A, E))(e)(x) = 0 \\ &\Rightarrow (G_A, E) \cap (H_A, E) = \varnothing. \end{aligned}$$

Thus It follows that there exists $(G_A, E), (H_A, E) \in t_A$ such that

$$x_r^e q(G_A, E), y_s^e q(H_A, E) \text{ and } (G_A, E) \cap (H_A, E) = \varnothing.$$

So (A, t_A, E) is $\text{FSR}_1(j)$. Hence the prove is complete. □

3.2. **Productivity andpProjectivity in fuzzy soft $\mathbf{R}_1$ topological spaces.** In this section, we show that our notions satisfy productive and projective properties.

Theorem 3.3. *Let $(X_i, \tau_i, E_i), i \in \Lambda$ be a fuzzy soft topological spaces and $X = \Pi_{i \in \Lambda} X_i$, $E = \Pi_{i \in \Lambda} E_i$ and τ be the fuzzy soft topology on (X, E). Then for all $i \in \Lambda$, (X_i, τ_i, E_i) is $FSR_1(j)$ if and only if (X, τ, E) is $FSR_1(j)$, where $j = i,\ ii,\ iii$.*

Proof. Let (X_i, τ_i, E_i) be $\text{FSR}_1(j)$ space for all $i \in \Lambda$. We have to prove that (X, τ, E) is $\text{FSR}_1(j)$. Let x_r^e, y_s^e be fuzzy soft points in (X, E) with $x \neq y$ and $(H, E) \in \tau$

with $H(e)(x) \neq H(e)(y)$. But we have $H(e)(x) = \min\ \{H_i(e_i)(x_i) : i \in \Lambda\}$, $H(e)(y) = \min\ \{H_i(e_i)(y_i) : i \in \Lambda\}$. Then there is at least one $(H_i, E_i) \in \tau_i$ and $(x_i)_r^{ei}, (y_i)_s^{ei}$ are fuzzy soft points with $x_i \neq y_i$ for some $i \in \Lambda$ with $H_i(e_i)(x_i) \neq H_i(e_i)(y_i)$. Since (X_i, τ_i, E_i) is $\mathrm{FSR}_1(j)$, there exist $(F_i, E_i)(G_i, E_i) \in \tau_i$ such that $(x_i)_r^{e_i} q(F_i, E_i), (y_i)_s^{e_i} q(G_i, E_i)$ and $(F_i, E_i) \cap (G_i, E_i) = \varnothing$. But we have

$$PX_i(x) = x_i,$$

$$PX_i(y) = y_i,$$

$$qE_i(e) = e_i.$$

Now, we have the following implications:

$$\begin{aligned}(x_i)_r^{e_i} q(F_i, E_i) &\Rightarrow F_i(e_i)(x_i) + r > 1\ \forall x_i \in X_i, e_i \in E_i\\ &\Rightarrow F_i(qE_i(e))(PX_i(x)) + r > 1\ \forall x \in X,\ e \in E\\ &\Rightarrow (F_i oqE_i)(e)(PX_i(x)) + r > 1\\ &\Rightarrow (F_i oqE_i oPX_i)(e)(x) + r > 1\\ &\Rightarrow x_r^e q(F_i oqE_i oPX_i, E).\end{aligned}$$

Further, we have

$$\begin{aligned}(y_i)_s^{e_i} q(G_i, E_i) &\Rightarrow G_i(e_i)(y_i) + s > 1\ \forall y_i \in X_i,\ e_i \in E_i\\ &\Rightarrow G_i(qE_i(e))(PX_i(y)) + s > 1\ \forall y \in X,\ e \in E\\ &\Rightarrow (G_i oqE_i)(e)(PX_i(y)) + s > 1\\ &\Rightarrow (G_i oqE_i oPX_i)(e)(y) + s > 1\\ &\Rightarrow y_s^e q(G_i oqE_i oPX_i, E).\end{aligned}$$

Also, we obtain the following implications:

$$\begin{aligned}&(F_i, E_i) \cap (G_i, E_i) = \varnothing\\ &\Rightarrow (F_i \cap G_i)(e_i)(x_i) = 0\ \forall x_i \in X_i,\ e_i \in E_i\\ &\Rightarrow \min\ (F_i(e_i)(x_i), G_i(e_i)(x_i)) = 0\\ &\Rightarrow \min\ (F_i(qE_i(e))(PX_i(x)), G_i(qE_i(e))(PX_i(x))) = 0\ \forall x \in X,\ e \in E\\ &\Rightarrow \min\ ((F_i oqE_i oPX_i)(e)(x), (G_i oqE_i oPX_i)(e)(x)) = 0\\ &\Rightarrow ((F_i oqE_i oPX_i) \cap (G_i oqE_i oPX_i))(e)(x) = 0\\ &\Rightarrow (F_i oqE_i oPX_i, E_i) \cap (G_i oqE_i oPX_i, E_i) = \varnothing.\end{aligned}$$

Thus it follows that there exist $(F_i oqE_i oPX_i, E), (G_i oqE_i oPX_i, E) \in \tau_i$ such that

$$x_r^e q(F_i oqE_i oPX_i, E), y_s^e q(G_i oqE_i oPX_i, E)$$

and

$$(F_i oqE_i oPX_i, E_i) \cap (G_i oqE_i oPX_i, E_i) = \varnothing.$$

So (X, τ, E) is $\mathrm{FSR}_1(j)$.

Conversely, let (X, τ, E) be a fuzzy soft topological space and (X, τ, E) is $\mathrm{FSR}_1(j)$. We have to prove that $(X_i, \tau_i, E_i), i \in \Lambda$ is $\mathrm{FSR}_1(j)$. Let a_i be a fixed element in X_i. Let $A_i = \{x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j \text{for some } i \neq j\}$. Then $A_i$is a subset of X. Thus (A_i, τ_{A_i}, E_i) is a subspace of (X, τ, E). Since (X, τ, E) is FSR_1(i), (A_i, τ_{A_i}, E_i) is $\mathrm{FSR}_1(j)$. Now we have (A_i, E_i) is homeomorphic image of (X_i, E_i). So it is clear that for all $i \in \Lambda$, (X_i, τ_i, E_i) is $\mathrm{FSR}_1(j)$ space. Hence the proof is complete. □

3.3. **Mappings in fuzzy soft $\mathbf{R}_1$ topological spaces.** In this section, we show that our notions satisfy order preserving property.

Theorem 3.4. *Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces. Let $u : X \longrightarrow Y$,$p : E \longrightarrow K$ be one-one, onto, fuzzy soft open and fuzzy soft continuous maps and hence a mapping $f_{up} : FSS(X, E) \longrightarrow FSS(Y, K)$ be a one-one, onto and fuzzy soft open and continuous map, then (X, τ_1, E) is $FSR_1(j) \Rightarrow (Y, \tau_2, K)$ is $FSR_1(j)$, where $j = i,\ ii,\ iii$.*

Proof. Let (X, τ_1, E) be a fuzzy soft topological space and (X, τ_1, E) is $\mathrm{FSR}_1(j)$. We have to prove that (Y, τ_2, K) is $\mathrm{FSR}_1(j)$. Let $\acute{x}^k_r$, $\acute{y}^k_s$ be fuzzy soft points in (Y, K) with $\acute{x} \neq \acute{y}$ and let $(H, K) \in \tau_2, e \in E$ with $H(k)(\acute{x}) \neq H(k)(\acute{y})$. Since f_{up} is onto and so u,p are onto, then there exist fuzzy soft points x^e_r, y^e_s in (X, E) with $f_{up}(x^e_r) = \acute{x}^e_r$, $f_{up}(y^e_s) = \acute{y}^e_s$ and $\acute{x}^e_r \neq \acute{y}^e_s$ as f_{up} is one-one. Since f_{up} is soft continuous, $f^{-1}_{up}(H, K) \in \tau_1$ with $(f^{-1}_{up}(H, K))(k)(x) \neq (f^{-1}_{up}(H, K))(k)(y)$. Again, since (X, τ_1, E) is $FSR_1(i)$, then there exist $(F, E), (G, E) \in \tau_1$ such that

$$x^e_r q(F, E), y^e_s q(G, E) \text{ and } (F, E) \cap (G, E) = \varnothing.$$

As f_{up} is open, $f_{up}(F, E) \in \tau_2$.

Now, we have the following implication:

$$\begin{aligned} & f_{up}(F, E)(e)(\acute{x}) = \sup\{u(x) = \acute{x}\} \sup\{p(\acute{e}) = e\} F(e)(x) \\ \Rightarrow\ & f_{up}(F, E)(e)(\acute{x}) = F(e)(x) \text{ for some } x \end{aligned}$$

and

$$\begin{aligned} & f_{up}(G, E)(e)(\acute{y}) = \sup\{u(y) = \acute{y}\} \sup\{p(\acute{e}) = e\} G(e)(y) \\ \Rightarrow\ & f_{up}(G, E)(e)(\acute{y}) = G(e)(y) \text{ for some } y. \end{aligned}$$

Now, we get the following implications:

$$\begin{aligned} x^e_r q(F, E) & \Rightarrow F(e)(x) + r > 1\ \forall x \in X,\ e \in E \\ & \Rightarrow f_{up}(F, E)(e)(\acute{x}) + r > 1 \\ & \Rightarrow \acute{x}^e_r q f_{up}(F, E). \end{aligned}$$

Further, we have the following implications:

$$\begin{aligned} y^e_s q(G, E) & \Rightarrow G(e)(y) + s > 1\ \forall y \in X,\ e \in E \\ & \Rightarrow f_{up}(G, E)(e)(\acute{y}) + s > 1 \\ & \Rightarrow \acute{y}^e_s q f_{up}(G, E). \end{aligned}$$

Also, we obtain the following implications:

$$\begin{aligned} (F, E) \cap (G, E) = \varnothing & \Rightarrow \min\ (F(e)(x), G(e)(y)) = 0\ \forall x,\ y \in X,\ e \in E \\ & \Rightarrow \min\ (f_{up}(F, E)(e)(\acute{x}), f_{up}(G, E)(e)(\acute{y})) = 0 \\ & \Rightarrow f_{up}(F, E)(e)(\acute{x}) \cap f_{up}(G, E)(e)(\acute{y}) = 0 \\ & \Rightarrow f_{up}(F, E) \cap f_{up}(G, E) = \varnothing. \end{aligned}$$

Thus it follows that there exist $f_{up}(F, E), f_{up}(G, E) \in \tau_2$ such that

$$\acute{x}^e_r q f_{up}(F, E), \acute{y}^e_s q f_{up}(G, E) \text{ and } f_{up}(F, E) \cap f_{up}(G, E) = \varnothing.$$

So (Y, τ_2, K) is $\mathrm{FSR}_1(j)$ space. Hence the proof is thus complete. □

Theorem 3.5. *Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces. Let $u : X \longrightarrow Y$, $p : E \longrightarrow K$ be one-one, onto, soft open and soft continuous maps, i.e., a fuzzy soft mapping $f_{up} : FSS(X, E) \longrightarrow FSS(Y, K)$ be a one-one, onto, soft open and fuzzy soft continuous map. If (Y, τ_2, K) is $FSR_1(j)$, then (X, τ_1, E) is $FSR_1(j)$, where $j = i,\ ii,\ iii$.*

Proof. Let (Y, τ_2, K) be a fuzzy soft topological space and (Y, τ_2, K) is $\mathrm{FSR}_1(j)$. We have to prove that (X, τ_1, E) is $\mathrm{FSR}_1(j)$. Let x^e_r, y^e_s be fuzzy soft points in (X, E)

with $x \neq y$ and let $(H, E) \in \tau_1$ with $H(e)(x) \neq H(e)(y)$ $\forall e \in E$. Since f_{up} is one-one, i.e., u, p are one-one, there exist fuzzy soft points x_r^e, y_s^e in (X, E) with $f_{up}(x_r^e) = \acute{x}_r^e$, $f_{up}(y_s^e) = \acute{y}_s^e$ and $\acute{x}_r^e \neq \acute{y}_s^e$ as f_{up} is one-one. Then $\acute{x}_r^k$, $\acute{y}_s^k$ are fuzzy soft points in (Y, K). As f_{up} is soft open, $f_{up}(H, E) \in \tau_2$ with $(f_{up}(H, E))(e)(\acute{x}) \neq (f_{up}(H, E))(e)(\acute{y})$. Again, since (Y, τ_2, K) is $FSR_1(i)$, there exist $(F, K), (G, K) \in \tau_2$ such that $\acute{x}_r^k q(F, K), \acute{y}_s^k q(G, K)$ and $(F, K) \cap (G, K) = \varnothing$. As f_{up} is fuzzy soft continuous, $f_{up}^{-1}(F, K), f_{up}^{-1}(G, K) \in \tau_1$.

Now, by definition of fuzzy soft inverse mapping, we have

$$f_{up}^{-1}(F, K)(e)(x) = (F, K)(p(e))(u(x))$$

and

$$f_{up}^{-1}(G, K)(e)(y) = (G, K)(p(e))(u(y)),$$

$\forall x,\ y \in X,\ e \in E$, where $p(e) = k$ $\forall k \in K$ and $u(x) = \acute{x},\ u(y) = \acute{y}$ $\forall \acute{x},\ \acute{y} \in Y$.

Now, we have the following implications:

$$\begin{aligned} \acute{x}_r^k q(F, K) &\Rightarrow F(k)(\acute{x}) + r > 1\ \forall \acute{x} \in Y,\ k \in K \\ &\Rightarrow (F, K)(p(e))(u(x)) + r > 1\ \forall x \in X,\ e \in E \\ &\Rightarrow f_{up}^{-1}(F, K)(e)(x) + r > 1 \\ &\Rightarrow x_r^e q f_{up}^{-1}(F, K). \end{aligned}$$

Further, we get the following implications:

$$\begin{aligned} \acute{y}_s^k q(G, K) &\Rightarrow G(k)(\acute{y}) + s > 1\ \forall \acute{y} \in Y,\ k \in K \\ &\Rightarrow (G, K)(p(e))(u(y)) + s > 1\ \forall y \in X,\ e \in E \\ &\Rightarrow f_{up}^{-1}(G, K)(e)(y) + s > 1 \\ &\Rightarrow y_s^e q f_{up}^{-1}(G, K). \end{aligned}$$

Also, we obtain following implications:

$$\begin{aligned} &(F, K) \cap (G, K) = \varnothing \\ \Rightarrow &\min\ (F(k)(\acute{x}), G(k)(\acute{y})) = 0\ \forall \acute{x},\ \acute{y} \in Y,\ k \in K \\ \Rightarrow &\min\ ((F, K)(p(e))(u(x)), (G, K)(p(e))(u(y))) = 0\ \forall x,\ y \in X,\ e \in E \\ \Rightarrow &\min\ (f_{up}^{-1}(F, K)(e)(x), f_{up}^{-1}(G, K)(e)(y)) = 0 \\ \Rightarrow &f_{up}^{-1}(F, K)(e)(x) \cap f_{up}^{-1}(G, K)(e)(y) = 0 \\ \Rightarrow &f_{up}^{-1}(F, K) \cap f_{up}^{-1}(G, K) = \varnothing. \end{aligned}$$

Thus it follows that there exist $f_{up}^{-1}(F, K), f_{up}^{-1}(G, K) \in \tau_1$ such that

$$x_r^e q f_{up}^{-1}(F, K), y_s^e q f_{up}^{-1}(G, K) \text{ and } f_{up}^{-1}(F, K) \cap f_{up}^{-1}(G, K) = \varnothing.$$

So (X, τ_1, E) is $FSR_1(j)$ space. Hence the proof is complete. □

4. Conclusion

The main result of this paper is introducing some new concepts of fuzzy soft R_1 topological spaces using quasi-coincidence sense. We discuss some features of this concepts and present their subspaces and hereditary properties. We hope that interested members of the scientific community will find useful applications such as decision making problem, game theory, artificial intelligence for these theories in near future.

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