

LM-fuzzy bitopological spaces

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ABSTRACT. In this paper, we introduce a new class of topology called an LM-fuzzy bitopological spaces. Also we introduces the closure operator $\tau_1\tau_2 - Cl(A, r)$, and interior operator $\tau_1\tau_2 - Int(A, r)$ in LM-fuzzy bitopological spaces as an extension of them in mated fuzzy bitopological spaces and establishes the relationship between $\tau_1\tau_2 - Cl(A, r)$ and the smooth closure $\tau_1\tau_2 - cl(A, r)$. Furthermore, we introduce different concepts of closed sets and open sets in double fuzzy bitopological spaces are extended to LM-fuzzy bitopological spaces and the algebraic structures associated with the families of these sets are investigated. As a result, certain monoids (denoted by $\tau_1\tau_2 - O_r$, $\tau_1\tau_2 - C_r$, $\tau_1\tau_2 - L_r$, $\tau_1\tau_2 - RO_r$, $\tau_1\tau_2 - RC_r$, $\tau_1\tau_2 - SO_r$, $\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bO_r$, $\tau_1\tau_2 - bC_r$, $\tau_1\tau_2 - GO_r$, $\tau_1\tau_2 - GC_r$, $\tau_1\tau_2 - RGO_r$, $\tau_1\tau_2 - RGC_r$, $\tau_1\tau_2 - GSO_r$, $\tau_1\tau_2 - GSC_r$, $\tau_1\tau_2 - GbO_r$ and $\tau_1\tau_2 - GbC_r$) contained in L^X are identified. Finally, the study associates a lattice of monoids to each element of M, which is associative, complemented but not modular.

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1. INTRODUCTION

The concept of fuzzy sets was first presented in 1965 by Zadeh [1]. After that, Chang in 1968 [2] introduced the concept of fuzzy topology. The concept of LM-fuzzy topology was introduced by Kubiak [3] and Sostak [4], since that many authors have contributed to the development of the theory of LM-fuzzy topological spaces. Kelly first proposed the concept of a bitopological space in 1963 [5]. In an ideal topological space, he also introduced local function. In 1995, Kandil et al. [6] introduced a concept of fuzzy bitopological spaces.

In this paper, we introduce a new class of topology called an *LM-fuzzy bitopological spaces*. Also we introduces the closure operator $\tau_1\tau_2 - Cl(A, r)$, and interior operator

$\tau_1\tau_2 - Int(A, r)$ in LM-fuzzy bitopological spaces as an extension of them in mated fuzzy bitopological spaces and establishes the relationship between $\tau_1\tau_2 - Cl(A, r)$ and the smooth closure $\tau_1\tau_2 - cl(A, r)$. Furthermore, we introduce different concepts of closed sets and open sets in double fuzzy bitopological spaces are extended to LM-fuzzy bitopological spaces and the algebraic structures associated with the families of these sets are investigated. As a result, certain monoids (denoted by $\tau_1\tau_2 - O_r$, $\tau_1\tau_2 - C_r$, $\tau_1\tau_2 - L_r$, $\tau_1\tau_2 - RO_r$, $\tau_1\tau_2 - RC_r$, $\tau_1\tau_2 - SO_r$, $\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bO_r$, $\tau_1\tau_2 - bC_r$, $\tau_1\tau_2 - GO_r$, $\tau_1\tau_2 - GC_r$, $\tau_1\tau_2 - RGO_r$, $\tau_1\tau_2 - RGC_r$, $\tau_1\tau_2 - GSO_r$, $\tau_1\tau_2 - GSC_r$, $\tau_1\tau_2 - GbO_r$ and $\tau_1\tau_2 - GbC_r$) contained in L^X are identified. Finally, the study associates a lattice of monoids to each element of M, which is associative, complemented but not modular.

2. PRELIMINARIES

Throughout this paper, X denotes a non-empty set, $I = [0, 1]$. The constant L-set having the value α is denoted by $\underline{\alpha}$.

Definition 2.1 ([7]). Let L be a poset. Then L is called

- (i) a *join-semilattice*, if $a \vee b \in L$ for every $a, b \in L$,
- (ii) a *meet-semilattice*, if $a \wedge b \in L$ for every $a, b \in L$.

L is called a *lattice*, if it is both a join-semilattice and a meet-semilattice.

Definition 2.2 ([8]). Let L be a lattice. Then $a \in L$ is called

- (i) a *minimal element* of L, if $\nexists b \in L$ such that $b \leq a$ and $b \neq a$,
- (ii) a *maximal element* of L, if $\nexists b \in L$ such that $b \geq a$ and $b \neq a$,
- (iii) an *atom* of L, if a is a minimal element in $L \setminus \{0\}$,
- (iv) a *dual atom* of L, if a is a maximal element in $L \setminus \{1\}$.

Definition 2.3 ([8]). A completely distributive lattice L is a called a *F-lattice*, if L has an order reversing involution $' : L \rightarrow L$.

Definition 2.4 ([8]). Let L be a poset. Then L is called

- (i) a *complete join-semilattice*, if every join for an arbitrary subset of L exists,
- (ii) a *complete meet-semilattice*, if every meet for an arbitrary subset of L exists,
- (iii) a *complete lattice*, if it is both a complete join-semilattice and a complete meet-semilattice.

Definition 2.5. A *monoid* is a set X with a binary operation $* : X \times X \rightarrow X$ which is associative and has an identity element.

Definition 2.6 ([7]). A *DeMorgan algebra* is a structure $A = (A, \vee, \wedge, 0, 1, ')$ such that

- (i) $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice,
- (ii) $'$ is a De Morgan involution: $(a \wedge b)' = a' \vee b'$ and $(a')' = a$.

Definition 2.7 ([8]). For every $A \in L^X$, A' is defined by $A'(x) = (A(x))'$ for every $x \in X$.

Definition 2.8 ([8]). An *L-fuzzy point* x_α is an L-fuzzy set $A \in L^X$ such that $A(x) = \alpha \neq 0$ and $A(y) = 0$ for $x \neq y$. The set of all L-fuzzy points x_α is denoted by $pt(L^X)$.

Definition 2.9 ([8]). Let $A, B \in L^X$. Then A quasi coincides with B at x , if $A(x) \not\leq B(x)$. Also, A is said to be quasi coincident with B , if A quasi coincides with B at some $x \in X$ and is denoted by $A\bar{q}B$.

Definition 2.10 ([3, 4]). Let L be an Lattice and M is a completely distributive lattice. Then an LM -fuzzy topology on a set X is defined to be a mapping $\tau : L^X \rightarrow M$ satisfying:

- (i) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,
- (ii) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$ for every $A, B \in L^X$,
- (iii) $\tau(\bigvee_{j \in J} A_j) \geq \bigwedge_{j \in J} \tau(A_j)$ for every $\{A_j\}_{j \in J} \in L^X$.

The pair (X, τ) is called an LM -fuzzy topological space (LM-fts, for short). The elements of τ are called LM -fuzzy open sets and the complement of LM -fuzzy open sets are called LM -fuzzy closed sets.

Definition 2.11 ([9]). Let τ be a map from L^X to M . Then for $r \in M$, the r -level decomposition of τ is defined as $\{A \in L^X : \tau(A) \geq r\}$.

Definition 2.12 ([10]). Let (X, τ) be an LM-fts. The closure of A denoted by $cl(A)$ is defined as

$$cl(A) = \bigwedge \{F \in L^X : A \leq F, \tau(F') > 0\}.$$

Definition 2.13 ([10]). Let (X, τ) be an LM-fts. The interior of A denoted by $int(A)$ is defined as

$$int(A) = \bigvee \{U \in L^X : A \geq U, \tau(U) > 0\}.$$

3. LM-FUZZY BITOPOLOGICAL SPACES

In this section, we define the concept of an LM -fuzzy bitopological space on set X and we define closure and interior operators in LM -fuzzy topological spaces. Also we define a base and subbase of $\tau_i, i \in \{1, 2\}$.

Definition 3.1. Let (X, τ_1) and (X, τ_2) be a two LM -fuzzy topological spaces. Then the triple (X, τ_1, τ_2) is called an LM -fuzzy bitopological space on set X (LM-fbts, for short).

Definition 3.2. Let (X, τ_1, τ_2) be an LM-fbts and let $A \in L^X$. Then A is called $\tau_1\tau_2$ - LM -fuzzy open set, if $A \subseteq U \cup V$ for every $U \in \tau_1$ and $V \in \tau_2$. The complement $\tau_1\tau_2$ - LM -fuzzy open set is called $\tau_1\tau_2$ - LM -fuzzy closed set.

Remark 3.3. (1) The union of any family of $\tau_1\tau_2$ - LM -fuzzy open sets is a $\tau_1\tau_2$ - LM -fuzzy open set.

(2) The intersection of any family of $\tau_1\tau_2$ - LM -fuzzy open sets is a $\tau_1\tau_2$ - LM -fuzzy open set.

Definition 3.4. Let (X, τ_1, τ_2) be an LM-fbts and let $A \in L^X$. Then $\tau_1\tau_2$ -closure of A denoted by $\tau_1\tau_2 - cl(A)$ is defined as

$$\tau_1\tau_2 - cl(A) = \bigwedge \{F \in L^X : A \leq F, \tau_i(F') > 0, i \in \{1, 2\}\}.$$

Definition 3.5. Let (X, τ_1, τ_2) be an LM-fbts and let $A \in L^X$. Then $\tau_1\tau_2$ -interior of A denoted by $\tau_1\tau_2 - int(A)$ is defined as

$$\tau_1\tau_2 - int(A) = \bigvee \{U \in L^X : A \geq U, \tau_i(U') > 0, i \in \{1, 2\}\}.$$

Definition 3.6. Let (X, τ_1, τ_2) be an LM-fbts. For $r \in M$ the operator $\tau_1\tau_2 - Cl : L^X \times M \rightarrow L^X$ defined by

$$\tau_1\tau_2 - Cl(A, r) = \bigwedge \{F \in L^X : A \leq F, \tau_i(F') \geq r, i \in \{1, 2\}\}$$

is called the $\tau_1\tau_2$ -LM-fuzzy closure operator on (X, τ_1, τ_2) .

Definition 3.7. Let (X, τ_1, τ_2) be an LM-fbts. For $r \in M$ the operator $\tau_1\tau_2 - Int : L^X \times M \rightarrow L^X$ defined by

$$\tau_1\tau_2 - Int(A, r) = \bigvee \{U \in L^X : A \geq U, \tau_i(U) \geq r, i \in \{1, 2\}\}$$

is called the $\tau_1\tau_2$ -LM-fuzzy interior operator on (X, τ_1, τ_2) .

Theorem 3.8. Let (X, τ_1, τ_2) be an LM-fbts, let $A, B \in L^X$ and let $r, s \in M$. Then

- (1) $\tau_1\tau_2 - Cl(A, 0) = A = \tau_1\tau_2 - Int(A, 0)$,
- (2) $A \leq \tau_1\tau_2 - cl(A) \leq \tau_1\tau_2 - Cl(A, r)$,
- (3) $\tau_1\tau_2 - Cl(A \vee B, r) = \tau_1\tau_2 - Cl(A, r) \vee \tau_1\tau_2 - Cl(B, r)$,
- (4) $\tau_1\tau_2 - Cl(A, r) \leq \tau_1\tau_2 - Cl(A, s)$, if $r \leq s$,
- (5) $\tau_1\tau_2 - Cl(\tau_1\tau_2 - Cl(A, r), r) = \tau_1\tau_2 - Cl(A, r)$,
- (6) $\tau_1\tau_2 - Int(A', r) = (\tau_1\tau_2 - Cl(A, r))'$,
- (7) $\tau_1\tau_2 - Int(A, r) \leq \tau_1\tau_2 - int(A) \leq A$,
- (8) $\tau_1\tau_2 - Int(A \wedge B, r) = \tau_1\tau_2 - Int(A, r) \wedge \tau_1\tau_2 - Int(B, r)$,
- (9) $\tau_1\tau_2 - Int(A, r) \geq \tau_1\tau_2 - Int(A, s)$, if $r \leq s$,
- (10) $\tau_1\tau_2 - Int(\tau_1\tau_2 - Int(A, r), r) = \tau_1\tau_2 - Int(A, r)$,
- (11) if $A = \tau_1\tau_2 - Int(\tau_1\tau_2 - Cl(A, r), r)$, then $\tau_1\tau_2 - Cl(\tau_1\tau_2 - Int(A', r), r) = A'$.

Definition 3.9. Let (X, τ_1, τ_2) be an LM-fbts and let $\beta : L^X \rightarrow M$ with $\beta \leq \tau_1 \vee \tau_2$. Then β is called a base of τ_i , if

$$\forall A \in L^X, \tau_i(A) = \bigvee_{\alpha \in \Lambda} \bigwedge_{B_\alpha = A} \beta(B_\alpha),$$

where

$$\bigvee_{\alpha \in \Lambda} \bigwedge_{B_\alpha = A} \beta(B_\alpha)$$

will be denoted by $\beta^{(\sqcup)}(A)$.

Definition 3.10. Let (X, τ_1, τ_2) be an LM-fbts and let $\phi : L^X \rightarrow M$, with $\phi \leq \tau_1 \vee \tau_2$. Then ϕ is called a subbase of τ_i if $\phi^{(\cap)} : L^X \rightarrow M$ is a base of τ_i where

$$\phi^{(\cap)}(A) = \bigvee_{(\cap)_{\alpha \in J} B_\alpha = A} \bigwedge \phi(B_\alpha)$$

for every $A \in L^X$ with (\cap) standing for finite intersection.

Lemma 3.11. Let (X, τ_1, τ_2) be an LM-fbts. Then $\phi : L^X \rightarrow M$ with $\phi \leq \tau_1 \vee \tau_2$ is the base of τ_i iff $\phi^{(\sqcup)}(1_X) = 1$.

4. DIFFERENT TYPES OF $\tau_1\tau_2$ -OPEN SETS AND $\tau_1\tau_2$ -CLOSED SETS

In this section, we introduces different types of closed and open sets in an LM-fuzzy bitopological space and studies the interrelations between them. Also, the algebraic structures associated with the collections of these sets are investigated.

Definition 4.1. Let (X, τ_1, τ_2) be an LM-fbts. For $r \in M$, a fuzzy set A is called

- (i) $\tau_1\tau_2$ - r -fuzzy open, if $\tau_i(A) \geq r, i \in \{1, 2\}$,
- (ii) $\tau_1\tau_2$ - r -fuzzy closed, if $\tau_i(A'), i \in \{1, 2\}$. is $\tau_1\tau_2$ - r -fuzzy open.

The family of all $\tau_1\tau_2$ - r -fuzzy open sets are denoted by $\tau_1\tau_2 - O_r$ and the family of all $\tau_1\tau_2$ - r -fuzzy closed sets are denoted by $\tau_1\tau_2 - C_r$.

Theorem 4.2. Let $\tau_i : L^X \rightarrow M, i \in \{1, 2\}$ be a function. Then τ_i is an LM-fuzzy bitopology on X iff $\tau_1\tau_2 - O_r$ is an L-bitopology on X for every $r \in M$.

Remark 4.3. Clearly, $\tau_1\tau_2 - Cl(A, r)$ gives the $\tau_1\tau_2$ -closure of A in the L-bitopology $\tau_1\tau_2 - O_r$.

Theorem 4.4. Let (X, τ_1, τ_2) be an LM-fbts. Then

- (1) $\tau_1\tau_2 - O_r$ and $\tau_1\tau_2 - C_r$ are lattices,
- (2) $\tau_1\tau_2 - O_r$ is a complete join-semilattice and $\tau_1\tau_2 - C_r$ is a complete meet-semilattice,
- (3) $\tau_1\tau_2 - O_r$ and $\tau_1\tau_2 - C_r$ are monoids.

Remark 4.5. However, $\tau_1\tau_2 - O_r$ need not be a complete meet-semilattice and $\tau_1\tau_2 - C_r$ need not be a complete join-semilattice. Let $X = L = I, M = I \times I$ with partial ordering defined by $r_1 \leq r_2$ and $s_1 \leq s_2$ and $\mathbb{V} = \{\underline{\alpha} : \alpha = \frac{1}{4} + \frac{1}{n}, n \in \mathbb{N} \setminus \{1\}\}$.

Now, consider the LM-fuzzy bitopology τ_i defined by

$$\tau_1(B) = \tau_2(B) = \begin{cases} (1, 1) & \text{if } B \in \{\underline{0}, \underline{1}\} \\ (\alpha, \frac{1}{4}) & \text{if } B = \underline{\alpha} \in \mathbb{V} \\ 0 & \text{otherwise} \end{cases}$$

Then for $r = (\frac{1}{4}, \frac{1}{4})$, $\tau_1\tau_2 - O_r = \mathbb{V} \cup \{\underline{0}, \underline{1}\}$ which is not closed under arbitrary meet since $\bigwedge_{B \in \mathbb{V}} B = (\frac{1}{4}) \notin \tau_1\tau_2 - O_r$. Hence, $\tau_1\tau_2 - O_r$ is not a complete meet-semilattice and consequently, $\tau_1\tau_2 - C_r$ is not a complete join-semilattice.

For any $r \in M$, let $\tau_1\tau_2 - L_r = \tau_1\tau_2 - O_r \cap \tau_1\tau_2 - C_r$. Then, $\tau_1\tau_2 - L_r$ is a lattice containing $\underline{0}$ and $\underline{1}$ and if $A \in \tau_1\tau_2 - L_r$ then $A' \in \tau_1\tau_2 - L_r$.

Theorem 4.6. Let (X, τ_1, τ_2) be an LM-fbts. Then for any $r \in M$, $\tau_1\tau_2 - L_r = \tau_1\tau_2 - O_r \cap \tau_1\tau_2 - C_r$ is a De Morgan algebra.

Theorem 4.7. Let (X, τ_1, τ_2) be an LM-fbts where L is a chain. Then $\tau_1\tau_2 - L_r$ is a Boolean algebra iff $\tau_1\tau_2 - L_r \subseteq \{\chi_G : G \subseteq X\}$.

Proof. Suppose $\tau_1\tau_2 - L_r$ is a Boolean algebra and let $A \in \tau_1\tau_2 - L_r$. If there exists $x \in X$ such that $0 < A(x) < 1$, then for every $B \in L^X$, either $(A \vee B) = \underline{1}$ or $(A \wedge B) = \underline{0}$ which contradicts the existence of complement for A in $\tau_1\tau_2 - L_r$. Consequently, $\tau_1\tau_2 - L_r \subseteq \{\chi_G : G \subseteq X\}$.

Conversely, suppose $\tau_1\tau_2 - L_r \subseteq \{\chi_G : G \subseteq X\}$ By Theorem 4.3, $\tau_1\tau_2 - L_r$ is a De-Morgan algebra with an order reversing involution $'$. Also, $\underline{0}, \underline{1} \in \tau_1\tau_2 - L_r$ with $A \wedge \underline{1} = A$ and $A \vee \underline{0} = A$ for every $A \in \tau_1\tau_2 - L_r$. \square

Remark 4.8. Though $\tau_1\tau_2 - L_r$ is a lattice, it is neither atomic nor dual atomic. Besides, it need not be a complete lattice.

Theorem 4.9. Let (X, τ_1, τ_2) be an LM-fbts. If $\tau_1\tau_2 - O_r$ is closed under the order reversing involution defined in L^X then $\tau_1\tau_2 - L_r$ is a complete lattice and hence an L -bitopology.

Remark 4.10. The converse of the above Theorem 4.5 is not true.

Theorem 4.11. Let $G \subseteq L^X$ be a complete De Morgan algebra. Then for any non-trivial completely distributive lattice M , there exists an LM-fuzzy bitopology τ_i on X such that $G = \tau_1\tau_2 - L_r$ for some non-zero $r \in M$.

Proof. Let M be any non-trivial completely distributive lattice and $0 \neq r \in M$. Then, define an LM-fuzzy bitopology τ_i on X as follows:

$$\tau_1(A) = \tau_2(A) = \begin{cases} 1 & \text{if } A \in \{\underline{0}, \underline{1}\} \\ r & \text{if } B \in A \setminus \{\underline{0}, \underline{1}\} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $\tau_1\tau_2 - O_r = \tau_1\tau_2 - C_r = \tau_1\tau_2 - L_r = A$. \square

Definition 4.12. Let (X, τ_1, τ_2) be an LM-fbts, $A \in L^X$ and $r \in M$. Then A is called

- (i) $\tau_1\tau_2$ - r -regular fuzzy open (or $\tau_1\tau_2$ - r -rfo) set, if $A = \tau_1 - Int(\tau_2 - Cl(A, r), r)$,
- (ii) $\tau_1\tau_2$ - r -regular fuzzy closed (or $\tau_1\tau_2$ - r -rfc) set, if A' is $\tau_1\tau_2$ - r -rfo,
- (iii) $\tau_1\tau_2$ - r -semi fuzzy open (or $\tau_1\tau_2$ - r -sfo) set, if $A \leq \tau_1 - Cl(\tau_2 - Int(A, r), r)$,
- (iv) $\tau_1\tau_2$ - r -semi fuzzy closed (or $\tau_1\tau_2$ - r -sfc) set, if A' is $\tau_1\tau_2$ - r -sfo,
- (v) $\tau_1\tau_2$ - r -fuzzy b-open (or $\tau_1\tau_2$ - r -fbo) set, if $A \leq (\tau_1 - Int(\tau_2 - Cl(A, r), r)) \vee (\tau_1 - Cl(\tau_2 - Int(A, r), r))$,
- (vi) $\tau_1\tau_2$ - r -fuzzy b-closed (or $\tau_1\tau_2$ - r -fbc) set, if A' is $\tau_1\tau_2$ - r -fbo set.

The family of all $\tau_1\tau_2$ - r -regular fuzzy open (resp. $\tau_1\tau_2$ - r -semi fuzzy open, $\tau_1\tau_2$ - r -fuzzy b-open) are denoted by $\tau_1\tau_2 - RO_r$ (resp. $\tau_1\tau_2 - SO_r$, $\tau_1\tau_2 - bO_r$) and the family of all $\tau_1\tau_2$ - r -regular fuzzy closed (resp. $\tau_1\tau_2$ - r -semi fuzzy closed, $\tau_1\tau_2$ - r -fuzzy b-closed) are denoted by $\tau_1\tau_2 - RC_r$ (resp. $\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bC_r$).

Theorem 4.13. Let (X, τ_1, τ_2) be an LM-fbts, $A \in L^X$ and $r \in M$. Then the following statements hold:

- (1) every $\tau_1\tau_2$ - r -open is $\tau_1\tau_2$ - r -semi fuzzy open,
- (2) every $\tau_1\tau_2$ - r -regular fuzzy closed is $\tau_1\tau_2$ - r -semi fuzzy open,
- (3) every $\tau_1\tau_2$ - r -semi fuzzy open is $\tau_1\tau_2$ - r -b fuzzy open.

Proof. (1) Let A be an $\tau_1\tau_2$ - r -open set. Then $A = \tau_1\tau_2 - Int(A) \leq \tau_1 - Cl(\tau_2 - Int(A, r), r)$. Thus A is $\tau_1\tau_2$ - r -semi fuzzy open.

(2) Let A be an $\tau_1\tau_2$ - r -regular fuzzy closed set. Then $A = \tau_1 - Cl(\tau_2 - Int(A, r), r)$, implies $A \leq \tau_1 - Cl(\tau_2 - Int(A, r), r)$. Thus A is $\tau_1\tau_2$ - r -semi fuzzy open.

(3) Let A be an $\tau_1\tau_2$ - r -semi fuzzy open. Then $A \leq \tau_1 - Cl(\tau_2 - Int(A, r), r)$, implies $A \leq \tau_1 - Cl(\tau_2 - Int(A, r), r) \vee \tau_1 - Int(\tau_2 - Cl(A, r), r)$. Thus A is $\tau_1\tau_2$ - r -b fuzzy open. \square

Remark 4.14. $\tau_1\tau_2 - C_r \subseteq \tau_1\tau_2 - SC_r \subseteq \tau_1\tau_2 - bC_r$ and $\tau_1\tau_2 - O_r \subseteq \tau_1\tau_2 - SO_r \subseteq \tau_1\tau_2 - bO_r$.

Theorem 4.15. Let (X, τ_1, τ_2) be an LM-fbts. Then

- (1) $\tau_1\tau_2 - RO_0 = \tau_1\tau_2 - RC_0 = L^X$,
- (2) $\underline{0}, \underline{1} \in \tau_1\tau_2 - RO_r \cap \tau_1\tau_2 - RC_r$.

Theorem 4.16. Let (X, τ_1, τ_2) be an LM-fbts.

- (1) If $A \in \tau_1\tau_2 - C_r$, then $\tau_1\tau_2 - Int(A, r) \in \tau_1\tau_2 - RO_r$.
- (2) If $A \in \tau_1\tau_2 - O_r$, then $\tau_1\tau_2 - Cl(A, r) \in \tau_1\tau_2 - RC_r$.

Theorem 4.17. Let (X, τ_1, τ_2) be an LM-fbts. Then $\tau_1\tau_2 - RO_r \cap \tau_1\tau_2 - RC_r = \tau_1\tau_2 - L_r$ for every $r \in M$.

Proof. We have $\tau_1\tau_2 - RO_r \subseteq \tau_1\tau_2 - O_r$ and $\tau_1\tau_2 - RC_r \subseteq \tau_i - C_r$. Then $\tau_1\tau_2 - RO_r \cap \tau_1\tau_2 - RC_r \subseteq \tau_1\tau_2 - L_r$. For the reverse implication, let $A \in \tau_1\tau_2 - L_r$. Then $\tau_1 - Int(\tau_2 - Cl(A, r), r) = \tau_1 - Int(A, r) = A$. Again, $\tau_1 - Cl(\tau_2 - Int(A, r), r) = \tau_1 - Cl(A, r) = A$. Thus $A \in \tau_1\tau_2 - RC_r$. \square

Theorem 4.18. Let (X, τ_1, τ_2) be an LM-fbts. Then

- (1) $\tau_1\tau_2 - RC_r \cup \tau_1\tau_2 - RO_r \subseteq \tau_1\tau_2 - SC_r \cap \tau_1\tau_2 - SO_r$,
- (2) $\tau_1\tau_2 - RC_r \cup \tau_1\tau_2 - RO_r \subseteq \tau_1\tau_2 - bC_r \cap \tau_1\tau_2 - bO_r$,
- (3) $\tau_1\tau_2 - SC_r \cup \tau_1\tau_2 - SO_r \subseteq \tau_1\tau_2 - bC_r \cap \tau_1\tau_2 - bO_r$.

Proof. (2) Let $A \in \tau_1\tau_2 - RO_r$. Then $A = \tau_1 - Int(\tau_2 - Cl(A, r), r) = A$ implies $(\tau_1 - Int(\tau_2 - Cl(A, r), r)) \wedge (\tau_1 - Cl(\tau_2 - Int(A, r), r)) \leq A$. Thus $A \in \tau_1\tau_2 - bO_r$. Again, if $A \in \tau_1\tau_2 - RO_r$, then $A' \in \tau_1\tau_2 - RC_r \subseteq \tau_1\tau_2 - C_r$. Thus $\tau_1\tau_2 - Cl(A', r) = A'$. Consequently, $\tau_1 - Int(\tau_2 - Cl(A', r), r) \leq A'$ and $(\tau_1 - Int(\tau_2 - Cl(A', r), r)) \wedge (\tau_1 - Cl(\tau_2 - Int(A', r), r)) \leq A'$. So $A' \in \tau_1\tau_2 - bC_r$ which implies $A \in \tau_1\tau_2 - bO_r$. Hence $\tau_1\tau_2 - RO_r \subseteq (\tau_1\tau_2 - bO_r \cap \tau_1\tau_2 - bC_r)$.

Similarly, $\tau_1\tau_2 - RC_r \subseteq (\tau_1\tau_2 - bO_r \cap \tau_1\tau_2 - bC_r)$.

The proofs of (1) and (3) are similar to (2). \square

Theorem 4.19. Let (X, τ_1, τ_2) be an LM-fbts. Then

- (1) $\tau_1\tau_2 - RO_r, \tau_1\tau_2 - SC_r$ and $\tau_1\tau_2 - bC_r$ are meet-semilattices,
- (2) $\tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r$ and $\tau_1\tau_2 - bO_r$ are join-semilattices,
- (3) $\tau_1\tau_2 - RO_r, \tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r, \tau_1\tau_2 - SC_r, \tau_1\tau_2 - bO_r$ and $\tau_1\tau_2 - bC_r$ are monoids.

Proof. (1) Let $A_1, A_2 \in \tau_1\tau_2 - RO_r$. Then $\tau_i(A_1), \tau_i(A_2) \geq r$. Thus $\tau_i(A_1 \wedge A_2) > r$. Also, $\tau_2 - Cl(A_1 \wedge A_2, r) \geq A_1 \wedge A_2$ and $\tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r)) \geq \tau_1 - Int(A_1 \wedge A_2, r) = A_1 \wedge A_2$.

Now $A_1 \geq A_1 \wedge A_2$ and $A_1 = \tau_1 - Int(\tau_2 - Cl(A_1, r), r) \geq \tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r), r)$. Also $A_2 \geq A_1 \wedge A_2$ and $A_2 = \tau_1 - Int(\tau_2 - Cl(A_2, r), r) \geq \tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r), r)$. So $A_1 \wedge A_2 \geq \tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r), r)$. Hence $A_1 \wedge A_2 = \tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r), r)$ which shows that $\tau_1\tau_2 - RO_r$ is a meet-semilattice.

Similarly, $\tau_1\tau_2 - SC_r$ and $\tau_1\tau_2 - bC_r$ are meet-semilattices.

(2) Similarly, $\tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r$ and $\tau_1\tau_2 - bO_r$ are join-semilattices.

(3) Associativity follows from (1) and (2). Again, $\underline{1} \in \tau_1\tau_2 - RO_r$ and $\underline{0} \in \tau_1\tau_2 - RC_r$ are the identity elements.

Similarly, for the rest. □

Remark 4.20. In general,

(1) $\tau_1\tau_2 - RO_r, \tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r, \tau_1\tau_2 - SC_r, \tau_1\tau_2 - bO_r$ and $\tau_1\tau_2 - bC_r$ are not lattices and hence not L-bitopologies,

(2) $\tau_1\tau_2 - RO_r, \tau_1\tau_2 - SC_r$ and $\tau_1\tau_2 - bC_r$ are not a complete meet-semilattices,

(3) $\tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r$ and $\tau_1\tau_2 - bO_r$ are not a complete join-semilattices,

(4) The partial ordering in M does not induce any ordering in the collections of $\tau_1\tau_2 - RO_r, \tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r, \tau_1\tau_2 - SC_r, \tau_1\tau_2 - bO_r$ and $\tau_1\tau_2 - bC_r$.

Definition 4.21. Let (X, τ_1, τ_2) be an LM-fbts, $r \in M$ and $A \in L^X$. Then the $\tau_1\tau_2$ -LM-fuzzy semi closure operator is defined by

$$\tau_1\tau_2 - SCl(A, r) = \bigwedge \{F \in L^X : A \leq F, F \in \tau_1\tau_2 - SC_r\}$$

and the $\tau_1\tau_2$ -LM-fuzzy semi interior operator is defined by

$$\tau_1\tau_2 - SInt(A, r) = \bigvee \{U \in L^X : U \leq A, U \in \tau_1\tau_2 - SO_r\}.$$

Definition 4.22. Let (X, τ_1, τ_2) be an LM-fbts and let $r \in M, A \in L^X$. Then the $\tau_1\tau_2$ -LM-fuzzy b-closure operator is defined by

$$\tau_1\tau_2 - bCl(A, r) = \bigwedge \{F \in L^X : A \leq F, F \in \tau_1\tau_2 - bC_r\}$$

and the $\tau_1\tau_2$ -LM-fuzzy b-interior operator is defined by

$$\tau_1\tau_2 - bInt(A, r) = \bigvee \{U \in L^X : U \leq A, U \in \tau_1\tau_2 - bO_r\}.$$

Definition 4.23. Let (X, τ_1, τ_2) be an LM-fbts, $A, B \in L^X$ and $r \in M$. Then A is called

(i) $\tau_1\tau_2$ -r-generalized fuzzy closed (or $\tau_1\tau_2$ -r-gfc) set, if $\tau_2 - Cl(A, r) \leq B$ whenever $A \leq B$ and $B \in \tau_1 - O_r$,

(ii) $\tau_1\tau_2$ -r-generalized fuzzy open (or $\tau_1\tau_2$ -r-gfo) set, if A' is a $\tau_1\tau_2$ -r-gfc set,

(iii) $\tau_1\tau_2$ -r-regular generalized fuzzy closed (or $\tau_1\tau_2$ -r-rgfc) set, if $\tau_2 - Cl(A, r) \leq B$ whenever $A \leq B$ and $B \in \tau_1 - RO_r$,

(iv) $\tau_1\tau_2$ -r-regular generalized fuzzy open (or $\tau_1\tau_2$ -r-rgfo) set, if A' is a $\tau_1\tau_2$ -r-gfc set,

(v) $\tau_1\tau_2$ -r-generalized fuzzy semi-closed (or $\tau_1\tau_2$ -r-gfsc) set, if $\tau_2 - SCl(A, r) \leq B$ whenever $A \leq B$ and $B \in \tau_1 - O_r$,

(vi) $\tau_1\tau_2$ -r-generalized fuzzy semi-open (or $\tau_1\tau_2$ -r-gfso) set, if A' is a $\tau_1\tau_2$ -r-gfsc set,

(vii) $\tau_1\tau_2$ -r-generalized fuzzy b-closed (or $\tau_1\tau_2$ -r-gfbc) set, if $\tau_2 - bCl(A, r) \leq B$ whenever $A \leq B$ and $B \in \tau_1 - O_r$,

(viii) $\tau_1\tau_2$ -r-generalized fuzzy b-open (or $\tau_1\tau_2$ -r-gfbo) set, if A' is a $\tau_1\tau_2$ -r-gfbc set.

The family of all $\tau_1\tau_2$ -r-generalized fuzzy closed (resp. $\tau_1\tau_2$ -r-regular generalized fuzzy closed, $\tau_1\tau_2$ -r-generalized fuzzy semi-closed, $\tau_1\tau_2$ -r-generalized fuzzy b-closed) are denoted by $\tau_1\tau_2 - GC_r$ (resp. $\tau_1\tau_2 - RGC_r, \tau_1\tau_2 - GSC_r, \tau_1\tau_2 - GbC_r$) and

the family of all $\tau_1\tau_2$ -r-generalized fuzzy open (resp. $\tau_1\tau_2$ -r-regular generalized fuzzy open, $\tau_1\tau_2$ -r-generalized fuzzy semi-open, $\tau_1\tau_2$ -r-generalized fuzzy b-open) are denoted by $\tau_1\tau_2 - GO_r$ (resp. $\tau_1\tau_2 - RGO_r, \tau_1\tau_2 - GSO_r, \tau_1\tau_2 - GbO_r$).

Theorem 4.24. *Let (X, τ_1, τ_2) be an LM-fbts. Then*

- (1) $\tau_1\tau_2 - RC_r \subseteq \tau_1\tau_2 - C_r \subseteq \tau_1\tau_2 - GC_r$ and $\tau_1\tau_2 - RO_r \subseteq \tau_1\tau_2 - O_r \subseteq \tau_1\tau_2 - GO_r$,
- (2) $\tau_1\tau_2 - GC_r \subseteq \tau_1\tau_2 - RGC_r \subseteq \tau_1\tau_2 - GSC_r \subseteq \tau_1\tau_2 - GbC_r$ and $\tau_1\tau_2 - GO_r \subseteq \tau_1\tau_2 - RGO_r \subseteq \tau_1\tau_2 - GSO_r \subseteq \tau_1\tau_2 - GbO_r$.

Theorem 4.25. *Let (X, τ_1, τ_2) be an LM-fbts. Then*

- (1) $\tau_1\tau_2 - GO_r, \tau_1\tau_2 - RGO_r, \tau_1\tau_2 - GSC_r$ and $\tau_1\tau_2 - GbC_r$ are meet-semilattices,
- (2) $\tau_1\tau_2 - GC_r, \tau_1\tau_2 - RGC_r, \tau_1\tau_2 - GSO_r$ and $\tau_1\tau_2 - GbO_r$ are join-semilattices,
- (2) $\tau_1\tau_2 - GO_r, \tau_1\tau_2 - GC_r, \tau_1\tau_2 - RGO_r, \tau_1\tau_2 - RGC_r, \tau_1\tau_2 - GSO_r, \tau_1\tau_2 - SC_r, \tau_1\tau_2 - GbO_r$ and $\tau_1\tau_2 - GbC_r$ are monoids.

Proof. (1) Let $A_1, A_2 \in \tau_1\tau_2 - GO_r$. Then $A'_1, A'_2 \in \tau_1\tau_2 - GC_r$. Thus $A'_1 \vee A'_2 \in \tau_1\tau_2 - GC_r$. So $(A_1 \wedge A_2)' \in \tau_1\tau_2 - GC_r$. Hence $(A_1 \wedge A_2) \in \tau_1\tau_2 - GO_r$. Therefore $\tau_1\tau_2 - GO_r$ is a meet semi-lattice.

Similarly, $\tau_1\tau_2 - RGO_r, \tau_1\tau_2 - GSC_r$ and $\tau_1\tau_2 - GbC_r$ are meet-semilattices.

(2) Let $A_1, A_2 \in \tau_1\tau_2 - GC_r$ and $B \in \tau_1\tau_2 - O_r$ such that $A_1 \vee A_2 \leq B$. Since $A_1 \leq B$ and $A_1 \in \tau_1\tau_2 - GC_r, \tau_1\tau_2 - Cl(A_1, r) \leq B$. Similarly, $\tau_1\tau_2 - Cl(A_2, r) \leq B$. Then $\tau_1\tau_2 - Cl(A_1 \vee A_2, r) = \tau_1\tau_2 - Cl(A_1, r) \vee \tau_1\tau_2 - Cl(A_2, r) \leq B$. Thus $\tau_1\tau_2 - GC_r$ is a join semi-lattice.

Similarly, $\tau_1\tau_2 - RGC_r, \tau_1\tau_2 - GSO_r$ and $\tau_1\tau_2 - GbO_r$ are join-semilattices.

(3) Associativity follows from (1), (2) and $\underline{1} \in \tau_1\tau_2 - GC_r$ and $\underline{0} \in \tau_1\tau_2 - GO_r$ are the identity elements.

Similarly, for the rest. □

Theorem 4.26. *Let (X, τ_1, τ_2) be an LM-fbts.*

- (1) If $A \in \tau_1\tau_2 - GSC_r$, then $B \in \tau_1\tau_2 - GSC_r$ for all B such that $A \leq B \leq \tau_1\tau_2 - SC_r(A, r)$.
- (2) If $A \in \tau_1\tau_2 - GbC_r$, then $B \in \tau_1\tau_2 - GbC_r$ for all B such that $A \leq B \leq \tau_1\tau_2 - bC_r(A, r)$.

Proof. (2) Let A is an $\tau_1\tau_2$ -r-gfbc and consider $B \in L^X$ such that $A \leq B \leq \tau_1\tau_2 - bC_r(A, r)$. Also, let C be a an $\tau_1\tau_2$ -r-fo in L^X such that $B \leq C$. Then clearly, $A \leq C$ and $\tau_1\tau_2 - bC_r(A, r) \leq C$. Again, note that $\tau_1\tau_2 - bC_r(B, r) = \tau_1\tau_2 - bC_r(A, r)$. Thus $\tau_1\tau_2 - bC_r(B, r) \leq C$. So B is an $\tau_1\tau_2$ -r-gfbc.

(i) The proof is similar to (1). □

Theorem 4.27. *Let (X, τ_1, τ_2) be an LM-fbts.*

(1) If $A \in \tau_1\tau_2 - GSC_r$, then for every $B \in \tau_1\tau_2 - SO_r, \tau_1\tau_2 - SC_r(A, r)\bar{q}B$ iff $A\bar{q}B$.

(2) If $A \in \tau_1\tau_2 - GbC_r$, then for every $B \in \tau_1\tau_2 - bO_r, \tau_1\tau_2 - bC_r(A, r)\bar{q}B$ iff $A\bar{q}B$.

Proof. (2) Let $B \in \tau_1\tau_2 - bO_r$ for some $r \in M$ and $A\bar{q}B$ for some $A \in LX$. Then $A \leq B'$. Since B' is an $\tau_1\tau_2$ -r-fbc set of L^X and A is an $\tau_1\tau_2$ -r-gfbc set, $\tau_1\tau_2 - bC_r(A, r)\bar{q}B$.

Conversely, let B be a $\tau_1\tau_2$ -r-fbc set of L^X such that $A \leq B, r \in M$. Then $A\bar{q}B'$. But $\tau_i - bC_r(A, r)\bar{q}B'$. Thus $\tau_1\tau_2 - bC_r(A, r) \leq B$. So A is an $\tau_1\tau_2$ -r-gfbc.

(2) The proof is similar to (1). □

Remark 4.28. Let \mathcal{O} be the collection of all $\tau_1\tau_2 - O_r$, \mathcal{C} be the collection of all $\tau_1\tau_2 - C_r$ and \mathcal{L} be the collection of all $\tau_1\tau_2 - L_r$ in an LM-fbts (X, τ_1, τ_2) . Then \mathcal{O} , \mathcal{C} and \mathcal{L} are bounded lattices where the bounds for \mathcal{O} are $\tau_1\tau_2 - O_0$ and $\tau_1\tau_2 - O_1$, the bounds of \mathcal{C} are $\tau_1\tau_2 - C_0$ and $\tau_1\tau_2 - C_1$ and that of \mathcal{L} are $\tau_1\tau_2 - L_0$ and $\tau_1\tau_2 - L_1$.

Remark 4.29. The lattices \mathcal{O} , \mathcal{C} and \mathcal{L} are neither atomic nor dual atomic. For example, let $X = \{a, b, c\}, L = M = I$ and define an LM-fuzzy bitopology on X as follows:

$$\tau_1(A) = \tau_2(A) = \begin{cases} 1 & \text{if } A \in \{\underline{0}, \underline{1}\} \\ \alpha & \text{if } A = \underline{\alpha}, \alpha \in I \setminus \{0, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly for $\alpha \in I$, $\tau_1\tau_2 - O_\alpha = \{\underline{0}\} \cup \{\beta : \beta \geq \alpha, \beta \in I\}$. Then it follows that $\mathcal{O} = \mathcal{C} = \mathcal{L} = \{\tau_1\tau_2 - O_\alpha : \alpha \in I\}$ which is neither atomic nor dual atomic.

Theorem 4.30. Let (X, τ_1, τ_2) be an LM-fbts. Then

- (1) \mathcal{O}, \mathcal{C} and \mathcal{L} are dual atomic, if M is atomic,
- (2) \mathcal{O}, \mathcal{C} and \mathcal{L} are atomic, if M is dual atomic.

Remark 4.31. Atoms and dual atoms may exist in \mathcal{O}, \mathcal{C} and \mathcal{L} without M being atomic or dual atomic. For example, let $X = \mathbb{R}$, the set of real numbers and $L = M = I$. Clearly, M is neither atomic nor dual atomic. Now, define an LM-fuzzy bitopology on X as follows:

$$\tau_1(A) = \tau_2(A) = \begin{cases} 1 & \text{if } A \in \{\underline{0}, \underline{1}\} \\ \alpha & \text{if } A = \underline{\alpha}, \alpha \in (\frac{1}{4}, \frac{2}{3}] \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{O} = \{\tau_1\tau_2 - O_0, \tau_1\tau_2 - O_1\} \cup \{\tau_1\tau_2 - O_\alpha : \alpha \in [\frac{1}{4}, \frac{2}{3}]\}$ and $\mathcal{C} = \{\tau_1\tau_2 - C_0, \tau_1\tau_2 - C_1\} \cup \{\tau_1\tau_2 - C_\alpha : \alpha \in [\frac{1}{4}, \frac{2}{3}]\}$. Also $\mathcal{L} = \{\tau_1\tau_2 - L_0, \tau_1\tau_2 - L_1\} \cup \{\tau_i - L_\alpha : \alpha \in [\frac{1}{4}, \frac{2}{3}]\}$.

5. CONCLUSIONS

As a result of the study, we have identified certain monoids that are subsets of L^X . Some of them are distributive lattices too. Now, let $\tau_1\tau_2 - \Omega_r = \{\tau_1\tau_2 - L_r, \tau_1\tau_2 - O_r, \tau_1\tau_2 - C_r, \tau_1\tau_2 - GO_r, \tau_1\tau_2 - GC_r, \tau_1\tau_2 - RGO_r, \tau_1\tau_2 - RGC_r, \tau_1\tau_2 - SO_r, \tau_1\tau_2 - SC_r, \tau_1\tau_2 - bO_r, \tau_1\tau_2 - bC_r, L^X\}$. Then $\tau_1\tau_2 - \Omega_r$ is a lattice under set inclusion whose elements are all monoids. The Hasse diagram of this lattice is shown in Figure 1.

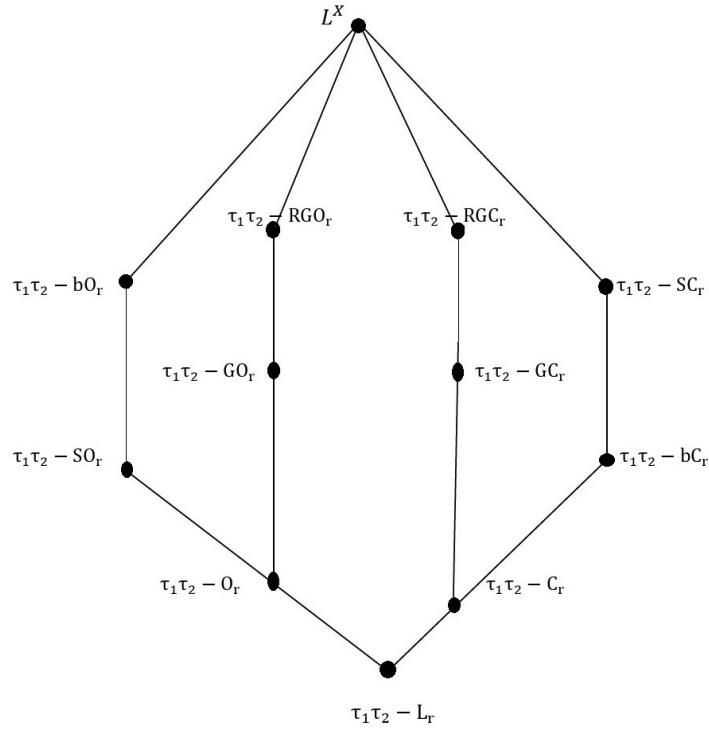


FIGURE 1. Hasse diagram of $\tau_1 \tau_2 - \Omega_r$

It is clear that $\tau_1 \tau_2 - \Omega_r$ is associative, complemented but not modular. Thus, by introducing various notions of openness and closedness in LM -fuzzy bitopological spaces, the study associates to each element of M , a lattice of monoids that are subsets of L^X .

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