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r-fuzzy-$\mathcal{I}$-supra separation axioms

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Abstract. The paper aims to introduce the notion of r-fuzzy open local function in fuzzy ideal supra topological spaces. Many of its features, characteristics and relations are studied between it and other corresponding fuzzy notions. Also study r-Fuzzy ideal supra Separation Axioms namely r-$F\mathcal{I}ST_i$-space where $i = \{0, 1, 2, 2\frac{1}{2}, 3, 4\}$ and study the relations between them. Finlay introduce the notions r-fuzzy-$\mathcal{I}$-supra separated sets and r-fuzzy-$\mathcal{I}$-connectedness and study some of the properties of r-fuzzy-$\mathcal{I}$-supra separated and r-fuzzy-$\mathcal{I}$-connectedness.

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1. Introduction

In 1965, Zadeh [1] introduced the notion of fuzzy sets. Now, they are one of the most serious and possible paths for the advancement of the set theory of Cantor. Despite the doubts and critical remarks expressed by some of the most influential mathematical logic experts in the second half of the 1960s against fuzzy sets, fuzzy sets were firmly developed as a fruitful field of study as well as a method for evaluating various objects and procedures. After that, Chang [2] in 1968 introduced the concept of fuzzy topology.

Kuratowski [3] first proposed the concept of an ideal topological space in 1966 . In an ideal topological space, he also introduced local function. In 1990, Hamlett and Jankovic [4] introduced a new topology by introduce the operator in any ideal topological space from the original ideal topological spaces.

Mashoure et al. [5], in 1983, introduced supra topological spaces. And many authors have been working in supra topological spaces lately (See [6, 7, 8, 9, 10, 11, 12].

In addition to some features of an ideal supra topological notion obtained by Shyamapada and Sukalyan [13] in 2012.

Later Abd El-Monsef and Ramadan [14] in 1987 introduced the concept of fuzzy supra topological as a natural generalization of the notion of supra topology spaces. The concept of Fuzzy Ideal Supra Topological Spaces was introduced by Fadhil Abbas [15] in 2020.

The paper aims to introduce the notion of r-fuzzy open local function in fuzzy ideal supra topological spaces. Many of its features, characteristics and relations are studied between it and other corresponding fuzzy notions. Also study r-Fuzzy ideal supra Separation Axioms namely r-$F\mathcal{I}ST_i$-space where $i = \{0, 1, 2, 2\frac{1}{2}, 3, 4\}$ and study the relations between them. Finlay introduce the notions r-fuzzy-$\mathcal{I}$-supra separated set and r-fuzzy-$\mathcal{I}$-connectedness and study some of the properties of r-fuzzy-$\mathcal{I}$-supra separated and r-fuzzy-$\mathcal{I}$-connectedness.

2. Preliminaries

Definition 2.1 ([1])**.** Let X be a non-empty set and let $I = [0, 1], I_0 = (0, 1]$. Then for $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for every $x \in X$. The family of all fuzzy sets on X denoted by I^X.

Definition 2.2 ([14])**.** A *fuzzy supra topological space* is a pair (X, S), where $S : I^X \to I$ is satisfying following axioms:
(i)$S(\underline{0}) = S(\underline{1}) = 1$,
(ii) $S(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} S(A_i)$, for every $\{A\}_{i\in\Gamma} \in I^X$.
The members of S are called *fuzzy supra open sets*. A fuzzy set A is called a *fuzzy supra closed*, ifits complement A^c is fuzzy supra open.

Definition 2.3 ([16])**.** Let $A,\ B \in I^X$. Then A is said to *be quasi-coincident with* B, if there exists at least one point $x \in X$ such that $A(x) + B(x)>1$ and denoted as AqB. Negation of such a statement is denoted as $A\bar{q}B$.

Lemma 2.4 ([2])**.** *Let* $A,\ B \in I^X$*. Then* $A \leqslant B \Leftrightarrow (A\bar{q}(1 - B))$.

Definition 2.5 ([2])**.** A fuzzy set A in a fuzzy supra topological space (X, S) is called an *s-neighbourhood* of a fuzzy point x_α, if there is $M \in S$ with $x_\alpha \in M \subseteq A$. The collection $N^S(x_\alpha)$ of all s-neighbourhoods of x_α is called the *s-neighbourhood system* of x_α.

3. r-fuzzy supra open local function

Definition 3.1. Let (X, S) be a fuzzy supra topological space. Then for every $r \in I_0, A \in I^X$, we define the operator $Cl^S : I^X \times I_0 \longrightarrow I^X$ as follows:

$$Cl^S(A, r) = \bigwedge\{F \in I^X : A \leqslant F, S(\underline{1} - F) \geq r\}.$$

For every $A,\ B \in I^X$ and $r,\ t \in I_0$ the operator Cl^S satisfies the following conditions:
(i) $Cl^S(\underline{0}, r) = \underline{0}$,
(ii) $A \leqslant Cl^S(A, r)$ for every $A \in I^X$,
(iii) $Cl^S(A, r) \vee Cl^S(B, r) \leqslant Cl^S(A \vee B, r)$ for every $A, B \in I^X$,

(iv) $Cl^S(A,r) \leqslant Cl^S(A,t)$ when $r \leqslant t$,
(v) $Cl^S(Cl^S(A,r),r) = Cl^S(A,r)$,
(vi) if $t = \bigvee\{r \in I_0 : Cl^S(A,r) = A\}$, then $Cl^S(A,r) = A$.

Definition 3.2. Let (X,S) be a fuzzy supra topological space. Then for every $r \in I_0, A \in I^X$, we define the operator $int^S : I^X \times I_0 \longrightarrow I^X$ as follows:

$$int^S(A,r) = \bigvee\{U \in I^X : A \geq U, S(U) \geq r\}.$$

For every $A,\ B \in I^X$ and $r,\ t \in I_0$ the operator int^S satisfies the following conditions:
(i) $int^S(\underline{1} - A, r) = \underline{1} - Cl^S(A,r)$,
(ii) $int^S(\underline{1}, r) = \underline{1}$,
(iii) $int^S(A,r) \leqslant A$ for every $A \in I^X$,
(iv) $int^S(A,r) \wedge int^S(B,r) = int^S(A \wedge B, r)$ for every $A, B \in I^X$,
(v) $int^S(A,r) \geq int^S(A,t)$ when $r \leqslant t$,
(vi) $int^S(int^S(A,r),r) = int^S(A,r)$,
(vii) if $t = \bigvee\{r \in I_0 : int^S(A,r) = A\}$, then $int^S(A,r) = A$.

Definition 3.3. Let φ be a subset of I^X and let $0_x \notin \varphi$. A mapping $\beta : \varphi \longrightarrow I$ is called a *fuzzy base* on X, if it satisfies the following conditions:
(i)$\beta(\underline{1}) = \underline{1}$,
(ii) $\beta(A \wedge B) \geq \beta(A) \wedge \beta(B)$, for every $A,\ B \in \varphi$.

Definition 3.4 ([13])**.** [6] A nonempty collection of fuzzy sets $\mathcal{I}$ of a set X is called a *fuzzy ideal* on X, if it satisfies the following conditions:
(i) $\mathcal{I}(\underline{0}) = 1,\ \mathcal{I}(\underline{1}) = 0$,
(ii) if $A \leqslant B$, then $\mathcal{I}(B) \leqslant \mathcal{I}(A)$ for every $A,\ B \in I^X$,
(iii) $\mathcal{I}(A \vee B) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$ for every $A,\ B \in I^X$.
If $\mathcal{I}_1$ and $\mathcal{I}_2$ are fuzzy ideals on X, we say that $\mathcal{I}_1$ is *finer than* $\mathcal{I}_2$ or $\mathcal{I}_2$ is *coarser than* $\mathcal{I}_1$, denoted by $\mathcal{I}_2 \leqslant \mathcal{I}_1$, if $\mathcal{I}_1(A) \leqslant \mathcal{I}_2(A)$ for $A \in I^X$.
The triple $(X,S,\mathcal{I})$ is called a *fuzzy ideal supra topological space.* For $\alpha \in I_0$, $(X, S_\alpha, \mathcal{I}_\alpha)$ is a fuzzy ideal supra topological space.

Definition 3.5. Let $(X,S,\mathcal{I})$ be a fuzzy ideal supra topological space and let $A \in I^X$. Then the *r-fuzzy supra open local function* $A_r^{*S}(S,\mathcal{I})$ of A is the union of all fuzzy points x_t such that if $B \in N_r^{*S}(x_t)$ and $\mathcal{I}(E) \geq r$, then at lest on point $y \in X$ for which $B(y) + A(y) - 1 > E(y)$.

In the other words, we say that a fuzzy set A *is r-fuzzy supra open locally in* $\mathcal{I}$ *at* x_t, if there exists $B \in N_r^{*S}(x_t)$ such that for every $y \in X$, $B(y) + A(y) - 1 \leqslant E(y)$, for some $\mathcal{I}(E) \geq r$. A $A_r^{*S}(S,\mathcal{I})$ is the set of fuzzy points at which A does not have the property r-fuzzy supra open locally. We will occasionally write A_r^{*S} for $A_r^{*S}(S,\mathcal{I})$, and it will cause no ambiguity.

Example 3.6. Let $(X,S,\mathcal{I})$ be a fuzzy ideal supra topological space. The simplest fuzzy ideals on X is $\mathcal{I}^0 : I^X \longrightarrow I$, where

$$\mathcal{I}^0(E) = \begin{cases} 1 & if \quad E = \underline{0} \\ 0 & otherwise. \end{cases}$$

If we take $\mathcal{I} = \mathcal{I}^0$ for every $A \in I^X$, then we have $A_r^{*S}(A) = Cl_r^S(A)$.

Theorem 3.7. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space, and let $\mathcal{I}_1$, $\mathcal{I}_2$ be two fuzzy ideals on X. Then for every $r \in I_0$ and $A,\ B \in I^X$*

(1) $\underline{0}_r^{*S} = \underline{0}$,
(2) *if* $A \leqslant B$, *then* $A_r^{*S} \leqslant B_r^{*S}$,
(3) *if* $\mathcal{I}_1 \leqslant \mathcal{I}_2$, *then* $A_r^{*S}(\mathcal{I}_2, S) \leqslant A_r^{*S}(\mathcal{I}_1, S)$,
(4) $A_r^{*S} = Cl^S(A_r^{*S}) \leqslant Cl^S(A, r)$,
(5) $(A_r^{*S})_r^{*S} \leqslant A_r^{*S}$,
(6) A_r^{*S} *is fuzzy supra closed set,*
(7) $A_r^{*S} \vee B_r^{*S} \leq (A \vee B)_r^{*S}$,
(8) *if* $\mathcal{I}(B) \geq r$, *then* $(A \vee B)_r^{*S} = A_r^{*S}$,
(9) *if* $\mathcal{I}(B) \geq r$, *then* $(B \wedge A_r^{*S}) \leqslant (B \wedge A)_r^{*S}$,
(10) $(A \wedge B)_r^{*S} \leqslant A_r^{*S} \wedge B_r^{*S}$,
(11) *if* $E \in \mathcal{I}$ *then* $E_r^{*S} = \underline{0}$,
(12) *if* $E \in \mathcal{I}$ *then* $(\underline{1} - E)_r^{*S} = \underline{1}$.

Proof. (1) The proof is obvious from the definition of r-fuzzy supra open locally function.

(2) Suppose there exist $A \in I^X$ and $r \in I_0$ such that $A_r^{*S} \nleq B_r^{*S}$ and there exist $x \in X$ and $t \in I_0$ such that

$$A_r^{*S}(x) \geq t > B_r^{*S}(x). \tag{3.1}$$

Since $B_r^{*S} < t$, there exists $U \in N_r^{*S}(x_t)$ with $\mathcal{I}(E) \geq r$ such that for every $y \in X$, we have,

$$U(y) + B(y) - 1 \leqslant E(y).$$

Since $A \leqslant B$, $U(y) + A(y) - 1 \leqslant E(y)$. Then $A_r^{*S}(x) < t$ and and this is a contradiction for equation (3.1). Thus $A_r^{*S} \leqslant B_r^{*S}$.

(3). Let $\mathcal{I}_1 \leqslant \mathcal{I}_2$ and let $x_t \in X$ any fuzzy point such that $x_t \notin A_r^{*S}(\mathcal{I}_1, S)$. Then there is at least one $U \in N_r^{*S}(x_t)$ and for every $y \in X$, $U(y) + A(y) - 1_x \leq E(y)$ for some $E \in \mathcal{I}_1$. Since $\mathcal{I}_1 \leqslant \mathcal{I}_2$, $E \in \mathcal{I}_2$. Thus $x_t \notin A_r^{*S}(\mathcal{I}_2, S)$. So $A_r^{*S}(\mathcal{I}_2, S) \leqslant A_r^{*S}(\mathcal{I}_1, S)$.

(4) For any fuzzy ideal $\mathcal{I}$ on X, it is clear that $\{\underline{0}\} \leqslant \mathcal{I}$. By (3) and Example 3.6, for any fuzzy set A in X, $A_r^{*S}(\mathcal{I}) \leqslant A_r^{*S}(\{\underline{0}\}) = Cl_r^{*S}(A)$. Now let $x_t \in Cl_r^{*S}(A_r^{*S})$. Then for every $U \in N_r^{*S}(x_t)$, there is at least one $y \in X$ such that $U(y)+A_r^{*S}(y) > 1$. Thus $A_r^{*S}(y) \neq \underline{0}$. Let $\beta = A_r^{*S}(y)$. Then clearly, $y_\beta \in A_r^{*S}$ and $\beta + U(y) > 1$. Thus $U \in N_r^{*S}(y_\beta)$. Since $y_\beta \in A_r^{*S}$, there is at least one $x_1 \in X$ such that $V(x_1) + A(x_1) - 1 > E(x_1)$ for each $V \in N_r^{*S}(y_\beta)$ and $E \in \mathcal{I}$. This is also true for U. So there is at least one $x_2 \in X$ such that $U(x_2) + A(x_2) - 1 > E(x_2)$ for each $E \in \mathcal{I}$. Since $U \in N_r^{*S}(x_t)$, $x_t \in A_r^{*S}$. Hence $A_r^{*S} = Cl_r^{*S}(A_r^{*S}) \leqslant Cl_r^{*S}(A)$.

(5) Form (4), $(A_r^{*S})_r^{*S} \leqslant Cl_r^{*S}(A_r^{*S}) = A_r^{*S}$.

(6) Let $x_t \notin A_r^{*S}(\mathcal{I}, S)$. Then there is at least one $U \in N_r^{*S}(x_t)$ such that for every $y \in X, U(y) + A(y) - 1 \leq E(y)$ for some $E \in \mathcal{I}$ and $x_t \in M$ implies $U \leq 1 - A_r^{*S}(I)$ and $1 - A_r^{*S}(\mathcal{I})$ is fuzzy supra open set. Thus A_r^{*S} is fuzzy supra closed.

(7) We have $A \leqslant A \vee B$ and $B \leqslant A \vee B$. Then form (2), $A_r^{*S} \leqslant (A \vee B)_r^{*S}$ and $B_r^{*S} \leqslant (A \cup B)_r^{*S}$. Thus $A_r^{*S} \vee B_r^{*S} \leqslant (A \vee B)_r^{*S}$.

(10) We have $(A \wedge B) \leqslant A$ and $(A \wedge B) \leqslant B$. Then from (2), $(A \wedge B)_r^{*S} \leqslant A_r^{*S}$ and $(A \wedge B)_r^{*S} \leqslant B_r^{*S}$. Thus $(A \wedge B)_r^{*S} \leqslant A_r^{*S} \wedge B_r^{*S}$.

The proofs of (8), (9), (11) and (12) are obvious. □

Example 3.8. Define $S, \mathcal{I} : I^X \longrightarrow I$ as follows:

$$S(A) = \begin{cases} 1 & if \quad A \in \{\underline{1}, \underline{0}\} \\ \frac{1}{2} & if \quad A = \underline{0.8} \\ \frac{1}{2} & if \quad A = \underline{0.7} \\ 0 & otherwise, \end{cases}$$

$$\mathcal{I}(E) = \begin{cases} 1 & if \quad E = \underline{0} \\ \frac{1}{2} & if \quad A = \underline{0.3} \\ \frac{2}{3} & if \quad \underline{0} < A < \underline{0.3} \\ 0 & otherwise. \end{cases}$$

Then $\underline{0} = (\underline{0.4}_{\frac{1}{2}}^{*S})_{\frac{1}{2}}^{*S} \neq \underline{0.4}_{\frac{1}{2}}^{*S} = \underline{0.2}$.

Theorem 3.9. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space and let $\{A_k : k \in K\} \subset I^X$. Then*

(1) $(\bigvee (A_k)_r^{*S} : k \in K) \leqslant (\bigvee A_k : k \in K)_r^{*S}$,
(2) $(\bigwedge A_k : k \in K)_r^{*S} \leqslant (\bigwedge (A_k)_r^{*S} : k \in K)$

Proof. It is obvious. □

Theorem 3.10. *Let $(X, S, \mathcal{I})$ be fuzzy ideal supra topological space and let $A \in I^X$. Then the operator $Cl_r^{*S} \colon I^X \times I_0 \to I^X$ defined by $Cl_r^{*S}(A) = A \cup A_r^{*S}$, is a fuzzy supra closure operator and hence it generates a fuzzy supra topology $S_r^{*S}(\mathcal{I}) = \{A \in I^X \colon Cl_r^{*S}(A^c) = A^c\}$ which is finer than S.*

Proof. (1) By Theorem 3.7 (1), $\underline{0}_r^{*S} = \underline{0}$. Then we have $Cl_r^{*S}(\underline{0}) = \underline{0}$.

(2) It is clear that $A \leqslant Cl_r^{*S}(A)$ for every $A \in I^X$.

(3) Let $A, \ B \in I^X$. Then we have

$$\begin{aligned} Cl_r^{*S}(A) \vee Cl_r^{*S}(B) &= (A \vee A_r^{*S}) \vee (B \vee B_r^{*S}) \\ &= (A \vee B) \vee (A_r^{*S} \vee B_r^{*S}) \leqslant (A \vee B) \vee (A \vee B)_r^{*S} \\ &= Cl_r^{*S}(A \vee B) \text{ [By Theorem 3.7 (7)).} \end{aligned}$$

Thus $Cl_r^{*S}(A) \vee Cl_r^{*S}(B) \leq Cl_r^{*S}(A \vee B)$.

(4) Let $A \in I^X$. By (2), $A \leqslant Cl_r^{*S}(A)$. Then $Cl_r^{*S}(A) \leqslant Cl_r^{*S}(Cl_r^{*S}(A))$. On the other hand, we get

$$\begin{aligned} Cl_r^{*S}(Cl_r^{*S}(A)) &= Cl_r^{*S}(A \vee A_r^{*S}) \\ &= (A \vee A_r^{*S}) \vee (A \vee A_r^{*S})_r^{*S} \leqslant A \vee A_r^{*S} \vee A_r^{*S} \\ &= Cl_r^{*S}(A). \end{aligned}$$

Thus $Cl_r^{*S}(Cl_r^{*S}(A)) \leqslant Cl_r^{*S}(A)$. So $Cl_r^{*S}(Cl_r^{*S}(A)) = Cl_r^{*S}(A)$. Hence $Cl_r^{*S}(A)$ is a fuzzy supra closure operator. Also, it is easy to show that the collection $S_r^{*S}(\mathcal{I}) = \{A \in I^X \colon Cl_r^{*S}(A^c) = A^c\}$ is a fuzzy supra topology on X which is called the *fuzzy supra topology induced by the fuzzy supra closure operator*. □

Proposition 3.11. *For any fuzzy ideal $\mathcal{I}$ on X, if $\mathcal{I} = \{\underline{0}\}$, then $Cl_r^{*S}(A) = A \vee A_r^{*S} = A \vee Cl_r^S(A) = Cl_r^S(A)$ for every $A \in I^X$. Thus $S_r^*(\{\underline{0}\}) = S$ and if*

$\mathcal{I} = I^X$, then $Cl_r^{*S}(A) = A$, because $A_r^{*S} = \underline{0}$ for every $A \in I^X$. So $S_r^{*S}(I^X)$ is a fuzzy discrete supra topology on X Since $\{\underline{0}\}$ and I^X are the tow extreme fuzzy ideal on X, for any fuzzy ideal $\mathcal{I}$ on X, we have $\{0_x\} \leqslant \mathcal{I} \leqslant I^X$. Hence we can conclude by Theorem 3.7 (2), $S_r^*(\{\underline{0}\}) \leqslant S_r^*(\mathcal{I}) \leqslant S_r^*(I^X)$, i.e., $S \leqslant S_r^*(\mathcal{I})$, for any fuzzy ideal $\mathcal{I}$ on X. In particular, we have for any tow fuzzy ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on X, $\mathcal{I}_1 \leqslant \mathcal{I}_2 \Rightarrow S_r^*(\mathcal{I}_1) \leqslant S_r^*(\mathcal{I}_2)$.

Definition 3.12. For every $(X, S, \mathcal{I})$ and $A \in I^X$, we define

$$int_r^{*S}(A) = A \wedge (\underline{1} - (\underline{1} - A)_r^{*S}).$$

Theorem 3.13. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space, let $r \in I_0$ and let $A \in I^X$. Then*

(1) $int_r^{*S}(A \vee B) \leqslant int_r^{*S}(A) \vee int_r^{*S}(B)$,
(2) $int_r^{S}(A) \leqslant int_r^{*S}(A) \leqslant Cl_r^{*S}(A) \leqslant Cl_r^{S}(A)$,
(3) $Cl_r^{*S}(\underline{1} - A) = \underline{1} - int_r^{*S}(A)$ *and* $\underline{1} - Cl_r^{*S}(A) = int_r^{*S}(\underline{1} - A)$,
(4) $int_r^{*S}(A \wedge B) = int_r^{*S}(A) \wedge int_r^{*S}(B)$.

Proof. Follows directly from definition $Cl_r^{*S}(A)$ and $int_r^{*S}(A)$. □

Theorem 3.14. *Define the mapping $\beta : I^X \to I$ on X by: for each $A \in I^X$,*

$$\beta(A) = \bigvee \{S(\beta) \wedge \mathcal{I}(E) \setminus A = B \wedge (1 - E)\}.$$

Proof. (1) Since $\mathcal{I}(\underline{0}) = \underline{1}, \beta(\underline{1}) = \underline{1}$.

(2) Suppose there exist $A_1,\ A_2 \in I^X$ such that $\beta(A_1 \wedge A_2) \not\geqslant \beta(A_1) \wedge \beta(A_2)$. Then there exist $t \in I_0$ such that

$$\beta(A_1 \wedge A_2) < t \leqslant \beta(A_1) \wedge \beta(A_2). \tag{3.2}$$

Since $\beta(A_1) \geq t$ and $\beta(A_2) \geq t$, there exist $B_1,\ B_2,\ E_1,\ E_2 \in I^X$ with $A_1 = B_1 \wedge (\underline{1} - E_1)$ and $A_2 = B_2 \wedge (\underline{1} - E_2)$ such that $\beta(A_1) \geq S(B_1) \wedge \mathcal{I}(E_1) \geq t$ and $\beta(A_2) \geq S(B_2) \wedge \mathcal{I}(E_2) \geq t$. Thus we have

$$\begin{aligned} A_1 \wedge A_2 &= (B_1 \wedge (\underline{1} - E_1)) \wedge (B_2 \wedge (\underline{1} - E_2)) \\ &= (B_1 \wedge B_2) \wedge ((\underline{1} - E_1) \wedge (\underline{1} - E_2)) \\ &= (B_1 \wedge B_2) \wedge (\underline{1} - (E_1 \vee E_2)). \end{aligned}$$

So we get

$$\begin{aligned} \beta(A_1 \wedge A_2) &\geq S(B_1 \wedge B_2) \wedge \mathcal{I}(E_1 \vee E_2) \\ &\geq S(B_1) \wedge S(B_2) \wedge \mathcal{I}(E_1) \wedge \mathcal{I}(E_2) \\ &= (S(B_1) \wedge \mathcal{I}(E_1)) \wedge (S(B_2) \wedge \mathcal{I}(E_2)) \\ &\geq t. \end{aligned}$$

It is a contradiction for equation (3.2). Hence $\beta(A_1 \wedge A_2) \geq \beta(A_1) \wedge \beta(A_2)$. □

Definition 3.15. Let $(X, S, \mathcal{I})$ be a fuzzy supra topological space. Then S is called a *fuzzy open S-compatible with $\mathcal{I}$*, denoted by $S \sim \mathcal{I}$, if for every $A \in I^X$, $x_t \in A$, and $E \in I^X$ with $\mathcal{I}(E) \geq r$, there exists $U \in N_r^S(x_t)$ such that if $U(y) + A(y) - 1 \leq E(y)$ hold for every $y \in X$, then $\mathcal{I}(A) \geq r$.

Definition 3.16. Let $\{B_k : k \in K\}$ be any indexed family of fuzzy sets in X such that $B_k q A$ for each $k \in K$, where $A \in I^X$. Then $\{B_k : k \in K\}$ is called a *r-fuzzy quasi-cover of A*, if $A(y) + \bigvee_{j \in J} B_j(y) \geq 1$ for every $y \in X$.

Definition 3.17. Let $\{B_k : k \in K\}$ be an r-fuzzy quasi-cover of A. Then $\{B_k : k \in K\}$ is called a *fuzzy quasi-supra open cover* of A, if each B_k is fuzzy supra open set. Thus in either case, $A^c \leqslant \bigvee_{j\in J} B_j$.

Theorem 3.18. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space. Then the following conditions are equivalent.*

(1) $S \sim \mathcal{I}$.
(2) *If for every $A \in I^X$ has a r-fuzzy quasi-supra open cover $\{B_k : k \in K\}$ such that for each k, $A(y) + B_j(y) - 1 \leq E(y)$ for some $\mathcal{I}(E) \geq r$ and for every $y \in X$, $\mathcal{I}(A) \geq r$.*
(3) *For every $A \in I^X$, $A \wedge A^{*S} = \underline{0}$ implies $\mathcal{I}(A) \geq r$.*
(4) *For every $A \in I^X$, $\mathcal{I}(\widetilde{A}) \geq r$, where $\widetilde{A} = \bigvee x_t$ such that $x_t \in A$ but $x_t \notin A^{*S}_r$.*
(5) *For every $S(\underline{1} - A) \geq r$, $\mathcal{I}(\widetilde{A}) \geq r$.*
(6) *For every $A \in I^X$, if A contains no $B \neq \underline{0}$ with $B \leqslant B^{*S}_r$, then $\mathcal{I}(A) \geq r$.*

Proof. (1) ⇒ (2) Let $\{B_k : k \in K\}$ be a fuzzy quasi-supra open cover of $A \in I^X$ such that for each $k \in K$, $B_k(y)+A(y)-1 \leq E(y)$ for some $\mathcal{I}(E) \geq r$ and for every $y \in X$. Since $\{B_k : k \in K\}$ is a r-fuzzy quasi-supra open cover of A, for each $x_t \in A$, there exists at least one B_{k_0} such that $x_t q B_{k_0}$ and for every $y \in X$, $B_{k_0}(y)+A(y)-1 \leq E(y)$ for some $\mathcal{I}(E) \geq r$. Obviously, $B_{k_0} \in N^S_r(x_t)$. Then by (1), we have $\mathcal{I}(A) \geq r$.

(2)⇒(1) Clear from the fact that a collection of $\{B_k : k \in K\}$ which contains at least one $B_{k_0} \in N^S_r(x_t)$, of each fuzzy point of A, constitutes a fuzzy quasi-supra open cover of A.

(2)⇒ (3) Let $A \wedge A^{*S}_r = \underline{0}$ and let $x_t \in A$ for every $x \in X$. Then $x_t \notin A^{*S}_r$. Thus there exists $U \in N^S_r(x_t)$ and $\mathcal{I}(E) \geq r$ such that for every $y \in X$, $U(y)+A(y)-1 \leqslant E(y)$. Since $U \in N^S_r(x_t)$, by (1), we have $\mathcal{I}(A) \geq r$.

(3)⇒ (1) Let for every fuzzy point $x_t \in A$, there is $U \in N^S_r(x_t)$ such that for every $y \in X$, $U(y) + A(y) - 1 \leq E(y)$ for some $\mathcal{I}(E) \geq r$. That means $x_t \notin A^{*S}_r$. Now there are two cases: either $A^{*S}_r(x) = \underline{0}$ or $A^{*S}_r(x) \neq \underline{0}$ but $t > A^{*S}_r(x) \neq \underline{0}$. Let if possible, $x_t \in A$ be such that $t > A^{*S}(x) \neq \underline{0}$. Let $t_1 = A^{*S}_r(x)$. Then the fuzzy point $x_{t_1} \in A^{*S}$ and also $x_{t_1} \in A$. This implies for each $V \in N^S_r(x_{t_1})$ and for each $\mathcal{I}(E) \geq r$, there is at least one $y \in X$ such that $V(y) + A(y) - 1_x > E(y)$. Since $x_{t_1} \in A$, this contradicts the assumption for every fuzzy point of A. Then $A^{*S}_r(x) = \underline{0}$. That means, $x_t \in A$ implies $x_t \notin A^{*S}_r$. Now this is true for every fuzzy set A in X. Thus for every fuzzy set A in X, $A \vee A^{*S}_r = \underline{0}$. So by the condition (3), we have $\mathcal{I}(A) \geq r$. Hence $S \sim \mathcal{I}$.

(3)⇒(4) Let $x_t \in \tilde{A}$. Then $x_t \in A$ but $x_t \notin A^{*S}_r$. Thus there is $U \in N^S_r(x_t)$ such that for every $y \in X$, $U(y) + A(y) - 1 \leq E(y)$ for some $\mathcal{I}(E) \geq r$. Since $\tilde{A} \leqslant A$, every $y \in X$, $U(y)+\tilde{A}(y)-1 \leq E(y)$ for some $\mathcal{I}(E) \geq r$. So $x_t \notin \tilde{A}^{*S}_r$, i.e., either $\tilde{A}^{*S}_r(x) = \underline{0}$ or $\tilde{A}^{*S}_r(x) \neq \underline{0}$ but $t > \tilde{A}^{*S}_r(x) \neq \underline{0}$. Let x_{t_1} be a fuzzy point such that $t_1 \leq \tilde{A}^{*S}_r(x) < t$, i.e., $x_{t_1} \in \tilde{A}^{*S}_r$. Then for each $V \in N^S_r(x_{t_1})$, and for each $\mathcal{I}(E) \geq r$, there is at least one $y \in X$ such that $V(y)+ \tilde{A}(y)-1 > E(y)$. Since $\tilde{A} \leqslant A$, for each $V \in N^S_r(x_{t_1})$ and for each $\mathcal{I}(E) \geq r$, there is at least one $y \in X$ such that $V(y) + A(y) - 1 > E(y)$. This implies $x_{t_1} \in A^{*S}_r$. But as $t_1 < t, x_t \in \tilde{A}$, $x_{t_1} \notin A^{*S}_r$. This is a contradiction. Thus $\tilde{A}^{*S}_r(x) = \underline{0}$, so that $x_t \in \tilde{A}$ implies $x_t \notin$

$\tilde{A}_r^{*S}$ with $\tilde{A}_r^{*S} = \underline{0}$. So we have $\tilde{A} \vee \tilde{A}_r^{*S} = \underline{0}$ for every fuzzy set A in X. Hence by the condition(3), $\mathcal{I}(\tilde{A}) \geq r$.

(4)⇒(5) Straightforward.

(4)⇒(6) Let $A \in I^X$, A contains $B \neq \underline{0}$ with $B \leqslant B_r^{*S}$. Then for every $A \in I^X$, A$=\tilde{A} \vee (A \wedge A_r^{*S})$. Thus $A_r^{*S} = (\tilde{A} \vee (A \wedge A_r^{*S}))_r^{*S} = \tilde{A}_r^{*S} \vee (A \wedge A_r^{*S})_r^{*S}$. Now by (4), we have $\mathcal{I}(\tilde{A}) \geq r$, i.e., $A_r^{*S} = \underline{0}$. So $(A \wedge A_r^{*S})_r^{*S} = A_r^{*S}$ but $A \wedge A_r^{*S} \leqslant A_r^{*S}$. Hence $A \wedge A_r^{*S} \leqslant (A \wedge A_r^{*S})_r^{*S}$. This contradicts the hypothesis about every fuzzy $A \in I^X$, if $\underline{0} = B \leqslant A$ with $B \leqslant B_r^{*S}$. Therefore $A \wedge A_r^{*S} = \underline{0}$, so that A $= \tilde{A}$ by (4), we have $\mathcal{I}(A) \geq r$.

(6)⇒(4) Since, for every $A \in I^X, A \wedge A_r^{*S} = \underline{0}$. Then by (6), as A contains no non-empty fuzzy subset B with $B \leqslant B_r^{*S}$, $\mathcal{I}(A) \geq r$.

(5)⇒(1) For every $A \in I^X, x_t \in A$, there exist an $U \in N_r^{*S}(x_t)$ such that $U(y) + A(y) - 1 \leqslant E(y)$ holds for every $y \in X$ and for some $\mathcal{I}(E) \geq r$. This implies $x_t \in A_r^{*S}$. Let $B = A \vee A_r^{*S}$ Then $B_r^{*S} = (A \vee A_r^{*S})_r^{*S} = A_r^{*S} \vee (A_r^{*S})_r^{*S}$. Thus $Cl_r^{*S}(B) = B \vee B_r^{*S} = B$. That means $S_r^*(1 - B) \geq r$. So by (5), we have $J(B) \geq r$. Again, for any $x_t \in P_t(X)$, $xt \notin B_r^{*S}$ implies $x_t \in B$ but $x_t \notin B_r^{*S} = A_r^{*S}$. Since $B = A \vee A_r^{*S}$, $x_t \in A$. Now, by hypothesis about A, we have for every $x_t \in A_r^{*S}$, $\tilde{B} = A$. Hence $\mathcal{I}(A) \geq r$, i.e., $S \sim \mathcal{I}$. □

Theorem 3.19. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space. Then the followings are equivalent and implied by $S \sim \mathcal{I}$.*

1. *For every $A \in I^X$, $A \wedge A_r^{*S} = \underline{0}$ implies $A_r^{*S} = \underline{0}$.*
2. *For every $A \in I^X$, $\tilde{A}_r^{*S} = \underline{0}$.*
3. *For every $A \in I^X$, $A \wedge A_r^{*S} = A_r^{*S}$.*

Proof. Clear from Theorem 3.14. □

Theorem 3.20. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space and let $S \sim J$. Then $A \in I^X$ is supra closed with respect to $S_r^*(\mathcal{I})$ if and only if it is the union of a fuzzy set which is supra closed with respect to S and a fuzzy set in X.*

Proof. Let $A \in I^X$ such that it is fuzzy S_r^*-supra closed set. Then $A_r^{*S} \leqslant A$ and we have A$=\tilde{A} \vee A_r^{*S}$. Since $S \sim \mathcal{I}$, $\mathcal{I}(A) \geq r$. Also A_r^{*S} is always fuzzy S-supra closed set. Conversely, let $A \in I^X$ such that $A = U \vee V$, where $Cl_r^{*S}(U) = U \leqslant A$. This means $A_r^{*S} \leqslant U \leqslant A$. Then we have $Cl_r^{*S}(A) = A \vee A_r^{*S} = A$. Thus A is fuzzy S_r^*-supra closed set. □

Corollary 3.21. *Let $S \sim \mathcal{I}$. Then $\beta(S, \mathcal{I})$ is a base for S_r^* and also $\beta(S, \mathcal{I}) = S_r^*$.*

Proof. Clear. □

4. r-fuzzy-$\mathcal{I}$-supra separation axioms

Definition 4.1. Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space, let $A \in I^X$ and let $r \in I_0$. Then A is called an *r-fuzzy-$\mathcal{I}$-supra closed set*, if $Cl_r^{*S}(A) = A$. The complement of a r-fuzzy-$\mathcal{I}$-supra closed set is called an *r-fuzzy-$\mathcal{I}$-supra open set.*

Theorem 4.2. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space, let $A \in I^X$ and let $r \in I_0$. Then*

1. *A is r-fuzzy-$\mathcal{I}$-supra closed set iff $A_r^{*S} \leqslant A$,*

(2) *A is r-fuzzy-$\mathcal{I}$-supra open set iff* $\underline{1} - A_r^{*S} \geq \underline{1} - A$,
(3) *If* $S(\underline{1} - A) \geq r$ *(resp.* $S(A) \geq r$*).*

Proof. Straightforward. □

Example 4.3. Let $S,\ \mathcal{I} : I^X \to I$ as follows: for each $B \in I^X$,

$$S(B) = \begin{cases} 1 & if \quad B \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2} & if \quad B \in \{\underline{0.3}, \underline{0.7}\} \\ 0 & otherwise, \end{cases}$$

$$\mathcal{I}(B) = \begin{cases} 1 & if \quad B = \underline{0} \\ \frac{1}{2} & if \quad \underline{0} \leqslant B < \underline{0.3} \\ 0 & otherwise. \end{cases}$$

Then we can easily check that the followings hold:
(1) $\underline{0.6}$ is a $\frac{1}{2}-$fuzzy-$\mathcal{I}$-supra closed set but $S(\underline{1} - \underline{0.6}) \neq \frac{1}{2}$,
(2) $\underline{0.2} \geq int^{*S}_{\frac{1}{2}}(\underline{0.2}) = \underline{0}$. But $\underline{0.2}$ is not $\frac{1}{2}-$fuzzy-$\mathcal{I}$-open set.

Theorem 4.4. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space and let $r \in I_0$. Then the following properties hold:*

(1) *the intersection of r-fuzzy-$\mathcal{I}$-supra closed sets is a r-fuzzy-$\mathcal{I}$-supra closed set,*
(2) *The union of r-fuzzy-$\mathcal{I}$-supra open sets is a r-fuzzy-$\mathcal{I}$-supra open set.*

Proof. (1) Let $\{A_i\}_{i \in \Gamma}$ be a class of r-fuzzy-$\mathcal{I}$-supra closed sets. Then for any $i \in \Gamma$, we have $A_i = Cl_r^{*S}(A_i)$. Then we get

$$\begin{aligned} \textstyle\bigwedge_{i \in \Gamma} A_i &= \textstyle\bigwedge_{i \in \Gamma} Cl_r^{*S}(A_i) = \bigwedge_{i \in \Gamma}(A_i \vee (A_i)_r^{*S}) \\ &\geq \textstyle\bigwedge_{i \in \Gamma} A_i \vee \bigwedge_{i \in \Gamma}(A_i)_r^{*S} \\ &\geq \textstyle\bigwedge_{i \in \Gamma} A_i \vee (\bigwedge_{i \in \Gamma} A_i)_r^{*S} \\ &= Cl_r^{*S}(\textstyle\bigwedge_{i \in \Gamma} A_i). \end{aligned}$$

So $\bigwedge_{i \in \Gamma} A_i$ is a r-fuzzy-$\mathcal{I}$-supra closed set.
(2) It is easily proved in the same manner. □

Theorem 4.5. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space and let $r \in I_0$. Then*

(1) *for every r-fuzzy-$\mathcal{I}$-supra open set A, $A\bar{q}B$ iff $A\bar{q}Cl_r^{*S}(B)$, $x_t\bar{q}Cl_r^{*S}(B)$ iff $A\bar{q}B$ for every r-fuzzy-$\mathcal{I}$-supra open set A with $x_t \in A$.*

Proof. (1) Let A be an r-fuzzy-$\mathcal{I}$-supra open set and suppose $A\bar{q}B$. Then $B \leqslant \underline{1} - A$. Since A is an r-fuzzy-$\mathcal{I}$-supra open set, $Cl_r^{*S}(B) \leqslant Cl_r^{*S}(\underline{1} - A) = \underline{1} - A$. It follows that $A\bar{q}Cl_r^{*S}(B)$.

The proof of the converse is easy.

(2) Suppose $x_t q Cl_r^{*S}(B)$. Then $A\bar{q}Cl_r^{*S}(B)$ with $x_t \in A$. Thus by (1), $A\bar{q}B$ for every r-fuzzy-$\mathcal{I}$-open set A.

Conversely, suppose $A\bar{q}B$. Then $B \leqslant \underline{1} - A$. Since A is r-fuzzy-$\mathcal{I}$-supra open set, $Cl_r^{*S}(B) \leqslant Cl_r^{*S}(\underline{1} - A) = \underline{1} - A$ and $A\bar{q}Cl_r^{*S}(B)$. Since $x_t \in A$, $x_t\bar{q}Cl_r^{*S}(B)$. □

Definition 4.6. Let $f : (X, S_1, \mathcal{I}_1) \to (Y, S_2, \mathcal{I}_2)$ be a mapping. Then f is called an:

(i) *r-fuzzy-$\mathcal{I}$-supra continuous*, if $f^{-1}(U)$ is an r-fuzzy-$\mathcal{I}$-supra open set in X for every r-fuzzy-$\mathcal{I}$-supra open set U in Y,

(ii) *r-fuzzy-$\mathcal{I}$-supra open*, if $f(U)$ is an r-fuzzy-$\mathcal{I}$-supra open set in Y for every r-every r-fuzzy-$\mathcal{I}$-supra open set U in X,

(iii) *r-fuzzy-$\mathcal{I}$-supra irresolute*, if $f^{-1}(U)$ is an r-fuzzy-$\mathcal{I}$-supra closed set in X for every r-every r-fuzzy-$\mathcal{I}$-supra closed set U in Y.

Definition 4.7. A fuzzy ideal supra topological space $(X, S, \mathcal{I})$ is called an r-$F\mathcal{I}ST_0$-space, if for x_t, y_s, there is an r-fuzzy-$\mathcal{I}$-supra open set U such that $x_t \in U$ and $y_t \notin U$.

Definition 4.8. A fuzzy ideal supra topological space $(X, S, \mathcal{I})$ is called an r-$F\mathcal{I}ST_1$-space, if for every $x_t \bar{q} y_s$, there is an r-fuzzy-$\mathcal{I}$-supra open set U such that $x_t \in U$ and $y_t \bar{q} U$.

Definition 4.9. A fuzzy ideal supra topological space $(X, S, \mathcal{I})$ is called an r-$F\mathcal{I}ST_2$-space, if for every $x_t \bar{q} y_s$, there are two r-fuzzy-$\mathcal{I}$-supra open set U and Vsuch that $x_t \in U$, $y_s \in V$ and $U \bar{q} V$.

Definition 4.10. A fuzzy ideal supra topological space $(X, S, \mathcal{I})$ is called an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space, if for every $x_t \bar{q} y_s$, there are two r-fuzzy-$\mathcal{I}$-supra open sets U and V such that $x_t \in U$, $y_s \in V$ and $Cl_r^{*S}(U) \bar{q} Cl_r^{*S}(V)$.

Definition 4.11. A fuzzy ideal supra topological space $(X, S, \mathcal{I})$ is called an r-$F\mathcal{I}ST_3$-space, if X is an r-$F\mathcal{I}ST_1$-space and $x_t \bar{q} F$, F is an r-fuzzy-$\mathcal{I}$-supra closed set, there are two r-fuzzy-$\mathcal{I}$-supra open sets U and V such that $x_t \in U$, $F \leqslant V$ and $U \bar{q} V$.

Definition 4.12. A fuzzy ideal supra topological space $(X, S, \mathcal{I})$ is called an r-$F\mathcal{I}ST_4$-space, if X is an r-$F\mathcal{I}ST_1$-space and $F \bar{q} D$, F and D are r-fuzzy-$\mathcal{I}$-supra closed sets, there are two r-fuzzy-$\mathcal{I}$-supra open sets U and V such that $F \leqslant U$, $D \leqslant V$ and $U \bar{q} V$.

Theorem 4.13. *If $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_4$-space, then $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_3$-space.*

Proof. Suppose $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_4$-space and let $x_t \bar{q} F$ such that F is an r-fuzzy-$\mathcal{I}$-supra closed set. Since $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_4$-space, $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_1$-space and $\{x_t\}$ is an r-fuzzy-$\mathcal{I}$-supra closed set. Thus there are two r-fuzzy-$\mathcal{I}$-supra open sets U and V such that $\{x_t\} \leqslant U$, $F \leqslant V$. But $\{x_t\} \leqslant U$ implies that $x_t \in U$. So $x_t \in U, F \leqslant V$ and $U \bar{q} V$. Hence $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_3$-space. □

Theorem 4.14. *If $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_3$-space, then $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space.*

Proof. Suppose $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_3$-space and let $x_t \bar{q} y_s$. Since $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_3$-space, $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_1$-space. Then there is an r-fuzzy-$\mathcal{I}$-supra open set U such that $x_t \in U$ and $y_t \bar{q} U$. Thus $x_t \in U = int_r^{*S}(U) \leqslant int_r^{*S}(\underline{1} - y_s) = \underline{1} - Cl_r^{*S}(y_s)$. So $x_t \bar{q} Cl_r^{*S}(y_s)$. By the hypothesis, there are r-fuzzy-$\mathcal{I}$-supra open sets U and V such that $x_t \in U$ and $Cl_r^{*S}(y_s) \leqslant V$ and $U \bar{q} V$. Then $U \leqslant \underline{1} - V$. Thus $Cl_r^{*S}(y_s) \leqslant Cl_r^{*S}(\underline{1} - V) = \underline{1} - V \leqslant \underline{1} - Cl_r^{*S}(y_s)$. So $Cl_r^{*S}(U) \bar{q} Cl_r^{*S}(y_s)$ with $x_t \in U$ and $y_t \in Cl_r^{*S}(y_s)$. Hence $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space. □

Theorem 4.15. *If $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space, then $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_2$-space.*

Proof. Suppose $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space. Then there are two r-fuzzy-$\mathcal{I}$-supra open sets U and V such that $x_t \in U$ and $y_s \in V$ and $Cl_r^{*S}(U)\bar{q}Cl_r^{*S}(V)$ implies that $U\bar{q}V$. Thus $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_2$-space. □

Theorem 4.16. *If $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_2$-space, then $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_1$-space.*

Proof. Suppose $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_2$-space and let $x_t\bar{q}y_s$. Since $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_2$-space, there are two r-fuzzy-$\mathcal{I}$-supra open set U and V such that $x_t \in U$ and $y_s \in V$ and $U\bar{q}V$. Then $y_s\bar{q}U$. Thus $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_1$-space. □

Theorem 4.17. *If $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_1$-space, then $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_0$-space.*

Proof. Straightforward. □

Example 4.18. Let $S,\ \mathcal{I} : I^X \to I$ as follows: for each $B \in I^X$,

$$S(B) = \begin{cases} 1 & if \quad B \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2} & if \quad B \in \{\underline{0.5}\} \\ 0 & otherwise, \end{cases}$$

$$\mathcal{I}(B) = \begin{cases} 1 & if \quad B = \underline{0} \\ \frac{2}{3} & if \quad \underline{0} \leqslant B < \underline{0.4} \\ 0 & otherwise. \end{cases}$$

Then for $\underline{0} < r \leqslant \underline{0.5}$, $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_0$-space but it is not an r-$FIST_1$-space.

Example 4.19. Let $S,\ \mathcal{I} : I^X \to I$ as follows: for each $B \in I^X$,

$$S(B) = \begin{cases} 1 & if \quad B \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2} & if \quad B \in \{\underline{0.2}, \underline{0.9}\} \\ 0 & otherwise, \end{cases}$$

$$\mathcal{I}(B) = \begin{cases} 1 & if \quad B = \underline{0} \\ \frac{1}{2} & if \quad \underline{0} \leqslant B < \underline{0.3} \\ 0 & otherwise \end{cases}$$

Then for $\underline{0} < r \leqslant \underline{0.5}$, $(X,S,\mathcal{I})$ is an r-$F\mathcal{I}ST_1$-space but it is not an r-$F\mathcal{I}ST_2$-space.

Example 4.20. Let $S,\ \mathcal{I} : I^X \to I$ as follows: for each $B \in I^X$,

$$S(B) = \begin{cases} 1 & if \quad B \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2} & if \quad \underline{0} < B < \underline{1} \\ 0 & otherwise, \end{cases}$$

$$\mathcal{I}(B) = \begin{cases} 1 & if \quad B = \underline{0} \\ \frac{2}{3} & if \quad \underline{0} < B \leqslant \underline{0.9} \\ 0 & otherwise \end{cases}$$

Then for $\underline{0} < r \leqslant \underline{0.5}$, $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_2$-space but it is not an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space. For $\underline{0} < r \leqslant \underline{0.3}$, $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space but it is not an r-$F\mathcal{I}ST_3$-space.

Example 4.21. Let S, $\mathcal{I} : I^X \to I$ as follows: for each $B \in I^X$,

$$S(B) = \begin{cases} 1 & if \quad B \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2} & if \quad \underline{0} \leqslant B \leqslant \underline{0.5} \\ 0 & otherwise, \end{cases}$$

$$\mathcal{I}(B) = \begin{cases} 1 & if \quad B = \underline{0} \\ \frac{1}{2} & if \quad \underline{0} \leqslant B < \underline{0.4} \\ 0 & otherwise. \end{cases}$$

Then for $\underline{0} < r \leqslant \underline{0.5}$, $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_3$-space but it is not an r-$F\mathcal{I}ST_4$-space.

Theorem 4.22. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space. Then $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_1$-space iff for each $x_t \bar{q} Cl_r^{*S}(y_s)$, there are two r-fuzzy-$\mathcal{I}$-supra open sets U and V such that $U \bar{q} V$, $Cl_r^{*S}(x_t) \leqslant V$ and $Cl_r^{*S}(y_s) \leqslant U$.*

Proof. Suppose $(X, S, \mathcal{I})$ is an r-$F\mathcal{I}ST_1$-space and let $x_t \bar{q} Cl_r^{*S}(y_s)$. Then by the hypothesis, there are two r-fuzzy-$\mathcal{I}$-supra open sets U and V such that $x_t \in U$, $y_s \in V$ and $U \bar{q} V$. Thus $x_t \bar{q} \underline{1} - U$ implies that $Cl_r^{*S}(x_t) \leqslant \underline{1} - U \leqslant V$. Also, $y_s \bar{q} \underline{1} - V$ implies that $Cl_r^{*S}(y_s) \leqslant \underline{1} - V \leqslant U$.

The proof of the converse is trivial. □

Theorem 4.23. *Let $f : (X, S, \mathcal{I}_X) \to (Y, \sigma, \mathcal{I}_Y)$ be an r-fuzzy-$\mathcal{I}_X$-supra continuous, bijective function and let $(X, S, \mathcal{I}_X)$ is an r-$F\mathcal{I}ST_0$-space. Then $(Y, \sigma, \mathcal{I}_Y)$ is an r-$F\mathcal{I}ST_0$-space.*

Proof. Let x_t, y_s two fuzzy points and let U be an r-fuzzy-$\mathcal{I}_Y$-supra open set in Y. Then by Definition 4.6 (i), $f^{-1}(U)$ is r-fuzzy-$\mathcal{I}_X$-supra open set in X. Put $x_t = f(z_t)$ and $y_s = f(c_s)$. Since $(X, S, \mathcal{I}_X)$ is an $r - F\mathcal{I}ST_0$-space, $z_t \in f^{-1}(U)$ and $c_s \notin f^{-1}(U)$. Since f is bijective, $x_t \in f(f^{-1}(U)) = U$ and $y_s \notin f(f^{-1}(U)) = U$. Thus $(Y, \sigma, \mathcal{I}_Y)$ is an r-$F\mathcal{I}ST_0$-space. □

Theorem 4.24. *Let $f : (X, S, \mathcal{I}_X) \to (Y, \sigma, \mathcal{I}_Y)$ be an r-fuzzy-$\mathcal{I}_X$-supra continuous, bijective function and let $(X, S, \mathcal{I}_X)$ is an r-$F\mathcal{I}ST_1$-space. Then $(Y, \sigma, \mathcal{I}_Y)$ is an r-$F\mathcal{I}ST_1$-space.*

Proof. Let $x_t \bar{q} y_s$ and let U be an r-fuzzy-$\mathcal{I}_Y$-supra open set in Y. Then by Definition 4.6 (i), $f^{-1}(U)$ is an r-fuzzy-$\mathcal{I}_X$-supra open set in X. Put $x_t = f(z_t)$ and $y_s = f(c_s)$. Since $(X, S, \mathcal{I}_X)$ is an r-$F\mathcal{I}ST_1$-space, $z_t \in f^{-1}(U)$ and $c_s \bar{q} f^{-1}(U)$. Since f is bijective, $x_t \in f(f^{-1}(U)) = U$ and $y_s \bar{q} f(f^{-1}(U)) = U$. Thus $(Y, \sigma, \mathcal{I}_Y)$ is an r-$F\mathcal{I}ST_1$-space. □

Theorem 4.25. *Let $f : (X, S, \mathcal{I}_X) \to (Y, \sigma, \mathcal{I}_Y)$ be an r-fuzzy-$\mathcal{I}_X$-supra continuous, bijective, r-fuzzy-$\mathcal{I}_X$-supra irresolute open function.*

(1) *If $(X, S, \mathcal{I}_X)$ is an r-$F\mathcal{I}ST_2$-space, then $(Y, \sigma, \mathcal{I}_Y)$ is an r-$F\mathcal{I}ST_2$-space.*
(2) *If $(X, S, \mathcal{I}_X)$ is an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space, then $(Y, \sigma, \mathcal{I}_Y)$ is an r-$F\mathcal{I}ST_{2\frac{1}{2}}$-space.*
(3) *If $(X, S, \mathcal{I}_X)$ is an r-$F\mathcal{I}ST_3$-space, then $(Y, \sigma, \mathcal{I}_Y)$ is r-$F\mathcal{I}ST_3$-space.*

(4) *If $(X, S, \mathcal{I}_X)$ is an r-$F\mathcal{I}ST_4$-space, then $(Y, \sigma, \mathcal{I}_Y)$ is r-$F\mathcal{I}ST_4$-space.*

Proof. (1) Let $y_s \bar{q} F = Cl_r^{*S}(F)$. Then by Definition 4.1, F is an r-fuzzy-$\mathcal{I}_Y$-supra closed set in Y. Thus by Definition 4.6 (ii), $f^{-1}(F)$ is an r-fuzzy-$\mathcal{I}_X$-supra closed set in X. Put $y_s = f(x_t)$. Then $x_t \bar{q} f^{-1}(F)$. Thus by the hypothesis, there are two r-fuzzy-$\mathcal{I}_X$-open sets U and V in X such that $x_t \in U$, $f^{-1}(F) \leqslant V$ and $U \bar{q} V$. Since f is bijective and r-fuzzy-$\mathcal{I}_X$-supra continuous open, $y_s \in f(U)$, $F \leqslant f(f^{-1}(F)) \leqslant f(V)$ and $f(U) \bar{q} f(V)$. So $(Y, \sigma, \mathcal{I}_Y)$ is an r-$F\mathcal{I}ST_2$-space.

(2),(3) and (4) are similarly proved. □

Definition 4.26. Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space, let $A,\ B \in I^X$ and $r \in I_0$. Then A and B are called *r-fuzzy-$\mathcal{I}$-supra separated sets,* if $Cl_r^{*S}(A) \bar{q} B$ and $A \bar{q} Cl_r^S(B)$.

Theorem 4.27. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space, let $A,\ B \in I^X$, $r \in I_0$ and let $A,\ B$ be r-fuzzy-$\mathcal{I}$-supra separated sets. If $A_1,\ B_1 \in I^X$ such that $A_1 \leqslant A$ and $B_1 \leqslant B$. Then A_1 and B_1 are r-fuzzy-$\mathcal{I}$-supra separated sets in X.*

Proof. Since $A_1 \leqslant A$ and $B_1 \leqslant B$, we have $Cl_r^{*S}(A_1) \leqslant Cl_r^{*S}(A)$ and $Cl_r^S(B_1) \leqslant Cl_r^S(B)$. Thus $Cl_r^{*S}(A) \bar{q} B$. So $Cl_r^{*S}(A_1) \bar{q} B_1$ and $A \bar{q} Cl_r^S(B)$. Hence $A_1 \bar{q} Cl_r^S(B_1)$. □

Theorem 4.28. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space and let $r \in I_0$. Then A and B be two r-fuzzy-$\mathcal{I}$-supra separated iff there exists an r-fuzzy supra open set U and an r-fuzzy-$\mathcal{I}$-open V such that $A \leqslant U$, $B \leqslant V$, $A \bar{q} V$ and $B \bar{q} U$.*

Proof. Suppose A and B are r-fuzzy-$\mathcal{I}$-supra separated fuzzy sets and let $V = \underline{1} - Cl_r^{*S}(A)$ and $U = \underline{1} - Cl_r^S(B)$. Then U is an r-fuzzy supra open and V is an r-fuzzy-$\mathcal{I}$-supra open set such that $A \leqslant U$, $B \leqslant V$, $A \bar{q} V$ and $B \bar{q} U$.

Conversely, suppose the necessary condition holds and let U be an r-fuzzy supra open and V be an r-fuzzy-$\mathcal{I}$-open set such that $A \leqslant U$, $B \leqslant V$, $(A \bar{q} V)$ and $(B \bar{q} U)$. Then $\underline{1} - U$ is an r-fuzzy supra closed and $\underline{1} - V$ is an r-fuzzy-$\mathcal{I}$-supra closed set. Thus $Cl_r^{*S}(A) \leqslant (\underline{1} - V) \leqslant (\underline{1} - B)$ and $Cl_r^S(B) \leqslant \underline{1} - U \leqslant \underline{1} - A$. So $Cl_r^{*S}(A) \bar{q} B$ and $Cl_r^S(B) \bar{q} A$. Hence A and B are r-fuzzy-$\mathcal{I}$-supra separated fuzzy sets. □

Definition 4.29. Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space and let $r \in I_0$. Then $E \in I^X$ is called an *r-fuzzy-$\mathcal{I}$-supra connected,* if it cannot be expressed as the union of two r-fuzzy-$\mathcal{I}$-supra separated sets.

Theorem 4.30. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space, let $r \in I_0$ and let A and B be r-fuzzy-$\mathcal{I}$-supra separated sets and E be a r-fuzzy-$\mathcal{I}$-supra connected set such that $E \leqslant A \vee B$. Then exactly one of the following conditions holds:*

(1) *$E \leqslant A$ and $E \wedge B = \underline{0}$,*
(2) *$E \leqslant B$ and $E \wedge A = \underline{0}$.*

Proof. We first note that when $E \wedge B = \underline{0}$, $E \leqslant A$, since $E \leqslant A \vee B$. Similarly, when $E \wedge A = \underline{0}$, we have $E \subseteq\leqslant B$. Since $E \leqslant A \vee B$, both $E \wedge A = \underline{0}$ and $E \wedge B = \underline{0}$ cannot hold simultaneously. Again if $E \wedge B \neq \underline{0}$ and $E \wedge A \neq \underline{0}$, then $E \wedge A$ and $E \wedge B$ are r-fuzzy-$\mathcal{I}$-supra separated sets such that $E = (E \wedge A) \vee (E \wedge B)$ contradicting the r-fuzzy-$\mathcal{I}$-supra connectedness of E. Thus exactly one of the conditions (1) and (2) must holds. □

Theorem 4.31. *Let $(X, S, \mathcal{I})$ be a fuzzy ideal supra topological space, let $r \in I_0$ and let $\{Y_\alpha : \alpha \in \Lambda\}$ be a collection of r-fuzzy-$\mathcal{I}$-supra connected sets. Then $Y = \bigvee\{Y_\alpha : \alpha \in \Lambda\}$ is r-fuzzy-$\mathcal{I}$-supra connected provided there exists $\beta \in \Lambda$ such that either* (1) *Y_α and Y_β are not r-fuzzy-$\mathcal{I}$-supra separated for each $\alpha \in \Lambda$ or* (2) *$Y_\alpha \wedge Y_\beta \neq \underline{0}$ for each $\alpha \in \Lambda$.*

Proof. Suppose Y is not r-fuzzy-$\mathcal{I}$-supra connected. Then $Y = A \vee B$, where A and B are r-fuzzy-$\mathcal{I}$-supra separated sets. Thus by Theorem 4.30, for an arbitrary $\alpha \in \Lambda$, either (a) $Y_\alpha \leqslant A$ with $Y_\alpha \wedge B = \underline{0}$ or (b) $Y_\alpha \leqslant B$ with $Y_\alpha \wedge A = \underline{0}$. Similarly, either (c) $Y_\beta = A$ with $Y_\beta \wedge B = \underline{0}$ or (d) $Y_\beta \leqslant B$ with $Y_\beta \wedge A = \underline{0}$. Without loss of generality, we can assume that each $\{Y_\alpha : \alpha \in \Lambda\}$ to be non-null. So exactly one of the possibilities (a) and (b) and exactly (c) and (d) will hold.

For case (2), the possibilities (a) and (b) cannot happen, and similarly (b) and (c) cannot hold simultaneously.

For case (1), if (a) and (b) hold, then $Y_\alpha = Y_\alpha \wedge A$ and $Y_\beta = Y_\beta \wedge B$ are r-fuzzy-$\mathcal{I}$-supra separated, A and B are being so. This is a contradiction.

Similarly, for case (2), the possibilities (c) and (d) together are to be ruled out. Thus in any cases, either $Y_\alpha \leqslant A$ with $Y_\alpha \wedge B = \underline{0}$ or $Y_\alpha \leqslant B$ with $Y_\alpha \wedge A = \underline{0}$ (but not both simultaneously) for each $\alpha \in \Lambda$. Now, $Y_\alpha \leqslant A$ and $Y_\alpha \leqslant B = \underline{0}$ and thus $B = \underline{0}$, a contradiction. Similarly, $Y_\alpha \leqslant B$ and $Y_\alpha \leqslant A = \underline{0}$ for all $\alpha \in \Lambda$ implies $A = \underline{0}$, again a contradiction. □

Corollary 4.32. *A fuzzy ideal supra topological space $(X, S, \mathcal{I})$ is r-fuzzy-$\mathcal{I}$-supra connected iff every pair of fuzzy points is contained in an r-fuzzy-$\mathcal{I}$-supra connected set.*

Theorem 4.33. *Let $f : (X, S, \mathcal{I}_X) \longrightarrow (Y, \sigma, \mathcal{I}_Y)$ be an r-fuzzy-$\mathcal{I}_X$-supra irresolute bijection function. If E be r-fuzzy-$\mathcal{I}_X$-supra connected set in X. Then $f(E)$ is r-fuzzy-$\mathcal{I}_Y$-supra connected set in Y.*

Proof. Suppose that $f(E)$ is not r-fuzzy-$\mathcal{I}_Y$-supra connected set in Y. Then there exist r-fuzzy-$\mathcal{I}_Y$-supra separated sets A and B in Y such that $f(E) = A \vee B$. Thus there exist r-fuzzy-$\mathcal{I}_Y$-supra open sets U and V such that $A \leqslant U$, $B \leqslant V$, $A\bar{q}U$ and $B\bar{q}V$. Now $E = f^{-1}(f(E)) = f^{-1}(A \vee B) = f^{-1}(A) \vee f^{-1}(B)$ and it can be easily verified that $f^{-1}(A) \leqslant f^{-1}(U), f^{-1}B) \leqslant f^{-1}(V)$, $f^{-1}(A)\bar{q}f^{-1}(U)$ and $f^{-1}(V)\bar{q}f^{-1}(B)$. Since f is r-fuzzy-$\mathcal{I}_X$-supra irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are r-fuzzy-$\mathcal{I}_X$-supra open set in X. Since every r-fuzzy-$\mathcal{I}_X$-supra open set is fuzzy-$\mathcal{I}_X$-supra open, it follows that $f^{-1}(V)$ is r-fuzzy-$\mathcal{I}_X$-supra open set in X. So $f^{-1}(A)$ and $f^{-1}(B)$ are r-fuzzy-$\mathcal{I}_X$-supra separated in X. Hence E is not r-fuzzy-$\mathcal{I}_X$-supra connected, which is a contradiction. □

5. Conclusions

This paper deals with the r-fuzzy open local function in fuzzy ideal supra topological spaces. Study r-Fuzzy ideal supra Separation Axioms namely r-$F\mathcal{I}ST_i$-space where $i = \{0, 1, 2, 2\frac{1}{2}, 3, 4\}$ and study the relations between them. Finlay introduce the notions r-fuzzy-$\mathcal{I}$-supra separated sets and r-fuzzy-$\mathcal{I}$-connectedness and study some of the properties of r-fuzzy-$\mathcal{I}$-supra separated and r-fuzzy-$\mathcal{I}$-connectedness.

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