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Quenching for discretization of a semilinear heat equation with singular boundary outflux

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Abstract. This paper concerns the study of the discret approximation for the following semilinear heat equation with a singular boundary outflux

$$\begin{cases} \dfrac{\partial u}{\partial t} = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -u(1,t)^{-q}, \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{cases}$$

where $p > 0$, $q > 0$.
We find some conditions under which the solution of a discrete form of above problem quenches in a finite time and estimate its discrete quenching time. We also establish the convergence of the discrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

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1. Intrroduction

We consider the semilinear heat equation with a singular boundary outflux

$$u_t = u_{xx} + (1-u)^{-p}, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -u(1,t)^{-q}, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{1.3}$$

where $p > 0$, $q > 0$. The initial value $u_0 : [0,1] \longrightarrow (0,1)$ is nonincreasing and satisfies the compatibility conditions:

$$u_0'(0) = 0,\ u_0'(1) = -u_0(1)^{-q}.$$

Selcuk and Ozalp [1] show that its solution u quenches in finite time T_q and $x = 0$ is the unique quenching point. They also show that the time derivative u_t blow-up at the quenching point and they get a quenching rate and a lower bound of the quenching time.

Definition 1.1. We say that the classical solution u of the problem (1.1)–(1.3) quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_q)$, but

$$\lim_{t \to T_q} \|u(.,t)\|_\infty = 1,$$

where $\|u(.,t)\|_\infty = \max_{0 \le x \le 1} |u(x,t)|$. The time T_q is called the *quenching time of the* u.

The theoretical study of solutions for semilinear parabolic equations with quench in a finite time has been the subject of investigations of many authors (See [2, 3, 4, 5, 6, 7, 8, 9, 1, 10] and the references cited therein). Local in time existence and uniqueness of the solution have been proved (See [10]). In [6], the authors considered semilinear parabolic problem

$$u_t = u_{xx} + f(x)(1-u)^{-p}, \quad 0 < x < 1, \quad 0 < t \le T,$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -u(1,t)^{-q}, 0 < t \le T,$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1,$$

where p and q are positive constants and $T \le \infty$.

Under some conditions, they prove three main results namely the quenching of the solution in finite time, the existence of a single quenching point $x = 0$ and the blow-up of the time derivative at the quenching point.

In recent years, more and more researchers are interested in numerical study of parabolic problems. This is the case of [11] in which the authors are interested in the numerical study of a heat equation with nonlinear boundary flux conditions using a semidiscrete form obtained by finite difference method. Under some conditions, they show that the solution of the numerical approximation for this heat equation quenches in a finite, they also establish the convergence of the semidiscrete quenching time and obtain a numerical quenching rate. Using the explicit and implicit Euler methods, they present some numerical results through tables and figures. We can also cite [12] in which The authors consider the following initial-boundary value problem:

$$u_t = u_{xx} + u^p, \quad 0 < x < 1, \quad 0 < t < \infty,$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -u(1,t)^{-q}, \quad 0 < t < \infty,$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1,$$

where $p > 0$, $q > 0$.

They work about the numerical quenching and numerical blow-up using the semidiscrete form obtained by finite difference method. Under some conditions, they prove that the solution of the semidiscrete form blows up in a finite time, and they study the convergence of semidiscrete blow-up time and estimate a semidiscrete blow up rate. They also under some conditions prove that the solution of the semidiscrete form quenches in a finite time, study the convergence of semidiscrete quenching time and estimate a semidiscrete quenching rate. The convergence of the semidiscrete scheme has also been proved. Using Hirota and Ozawa method [13], they present some numerical results which contain tables and figures for adequate values p and q which illustrate well their theoretical study. From these results, they emerged interesting results concerning the influence of the parameters p and q on the numerical quenching time. Concerning problem (1.1)–(1.3), The authors of [14] investigate about the numerical quenching phenomenon. Using the semidiscrete scheme, they show some properties of semidiscrete solution. Under some condition, they prove that the semidiscrete solution quenches in finite time and they get a upper bound of the semidiscrete quenching time. They also prove the convergence of the semidiscrete scheme and the numerical time. Using the explicit and implicit Euler methods, they illustrate their analyses by tables where they get some finite values of the numerical quenching time according to the values taken by I. They finish their study by presentation of figures with adequate values of p and q. For other previous studies on numerical approximations of parabolic system with non-linear boundary conditions, we refer to [15, 16, 17, 18].

In this paper, we will deepen the work of [14] using discrete form of problem (1.1)–(1.3). We present our work in this way: In section 2, we present some properties of the discrete solution. In sections 3 and 4, we prove some main results related to the discrete quenching time and the discrete scheme. In section 5, we give numerical results for new values of the parameters p and q.

2. Properties of the discrete scheme

Let $I \geq 3$ be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leq i \leq I$. We approximate the solution u of problem (1.1)–(1.3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \dots, U_I^{(n)})^T$ and the initial condition u_0 by the initial condition $\varphi_h = (\varphi_0, \varphi_1, \dots, \varphi_I)^T$ of the following discrete equations

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} + (1 - U_i^{(n)})^{-p}, \quad 0 \leq i \leq I-1, \tag{2.1}$$

$$\delta_t U_I^{(n)} = \delta^2 U_I^{(n)} - \frac{2}{h}(U_I^{(n)})^{-q} + (1 - U_I^{(n)})^{-p}, \tag{2.2}$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I, \tag{2.3}$$

where

$$n \geq 0, \quad p > 0, \quad q > 0,$$

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}, \quad 0 \leq i \leq I,$$

$$\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_0^{(n)} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2}, \qquad \delta^2 U_I^{(n)} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2},$$

$$\varphi_i > 0, \quad 0 \le i \le I,$$

$$\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}, \quad 0 \le i \le I-1,$$

$$\delta^+ \varphi_i < 0, \quad 0 \le i \le I-1.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches to the quenching time T_q, we need to adapt the size of the time step. We choose

$$\Delta t_n = \min\left\{\frac{h^2}{2}, \tau(1 - \|U_h^{(n)}\|_\infty)^{p+1}\right\} \text{ with } \tau \in (0,1).$$

Definition 2.1. We say that the solution $U_h^{(n)}$, $n \ge 0$ of the discrete problem (2.1)–(2.3) *quenches in finite time*, if $\|U_h^{(n)}\|_\infty < 1$ for $n \ge 0$ but $\lim_{n\to+\infty} \|U_h^{(n)}\|_\infty = 1$ and

$$T_h^{\Delta t} = \lim_{n\to+\infty} \sum_{j=0}^{n-1} \Delta t_j < +\infty.$$

We call $T_h^{\Delta t}$ the*numerical quenching time* of $U_h^{(n)}$.

Now we give some Lemmas which will be used in this work.

Lemma 2.2. *Let $b_h^{(n)}$ and $V_h^{(n)}$ be two sequences, with $n \ge 0$ and $b_h^{(n)} \le 0$, such that for $0 \le i \le I$*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + b_i^{(n)} V_i^{(n)} \ge 0,$$

$$V_i^{(0)} \ge 0.$$

Then we have

$$V_i^{(n)} \ge 0,\ 0 \le i \le I,\ n \ge 0 \text{ when } \Delta t_n \le \frac{h^2}{2}.$$

Proof. A straightforward computation shows that for

$$V_i^{(n+1)} \ge \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_i^{(n)} + \frac{\Delta t_n}{h^2}\left(V_{i+1}^{(n)} + V_{i-1}^{(n)}\right) - \Delta t_n b_i^{(n)} V_i^{(n)}, 1 \le i \le I-1$$

$$V_0^{(n+1)} \ge \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_0^{(n)} + \frac{2\Delta t_n}{h^2} V_1^{(n)} - \Delta t_n b_0^{(n)} V_0^{(n)},$$

$$V_I^{(n+1)} \ge \left(1 - 2\frac{\Delta t_n}{h^2}\right) V_I^{(n)} + \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} - \Delta t_n b_I^{(n)} V_I^{(n)}.$$

If $V_h^{(n)} \ge 0$, then using an argument of recursion, we easily see that $V_h^{(n+1)} \ge 0$, because $1 - 2\dfrac{\Delta t_n}{h^2} \ge 0$ and $-b_h^{(n)} \ge 0$. This end the proof. □

Lemma 2.3. *Let $b_h^{(n)}$, $V_h^{(n)}$ and $W_h^{(n)}$ be three sequences, with $n \geq 0$ and $b_h^{(n)} \leq 0$, such that for $0 \leq i \leq I$,*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + b_i^{(n)} V_i^{(n)} \leq \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} + b_i^{(n)} W_i^{(n)},$$

$$V_i^{(0)} \leq W_i^{(0)}.$$

Then we have

$$V_i^{(n)} \leq W_i^{(n)},\ 0 \leq i \leq I,\ n \geq 0,\ \text{when } \Delta t_n \leq \frac{h^2}{2}.$$

Proof. Define the vector $Z_h^{(n)} = W_h^{(n)} - V_h^{(n)}$. For $0 \leq i \leq I$, a straightforward calculation gives

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + b_i^{(n)} Z_i^{(n)} \geq 0.$$

Knowing that $Z_h^{(0)} \geq 0$, from Lemma 2.2, we have $Z_h^{(n)} \geq 0$, $n \geq 0$. □

Lemma 2.4. *Let $b_h^{(n)}$, $V_h^{(n)}$ and $W_h^{(n)}$ be three sequences, with $n \geq 0$ and $b_h^{(n)} \leq 0$, such that for $0 \leq i \leq I$,*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + b_i^{(n)} V_i^{(n)} < \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} + b_i^{(n)} W_i^{(n)},$$

$$V_i^{(0)} < W_i^{(0)}.$$

Then we have

$$V_i^{(n)} < W_i^{(n)},\ 0 \leq i \leq I,\ n \geq 0,\ \text{when } \Delta t_n \leq \frac{h^2}{2}.$$

Proof. Define the vector $Z_h^{(n)} = W_h^{(n)} - V_h^{(n)}$. For $0 \leq i \leq I$, a straightforward calculation gives

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + b_i^{(n)} Z_i^{(n)} > 0.$$

Knowing that $Z_h^{(0)} > 0$, from Lemma 2.2, we have $Z_h^{(n)} > 0$, $n \geq 0$. □

Lemma 2.5. *Let $U_h^{(n)}$, $n \geq 0$ be a sequence such that $\|U_h^{(n)}\|_\infty < 1$. Then we have*

$$\delta_t (1 - U_i^{(n)})^{-p} \geq p(1 - U_i^{(n)})^{-p-1} \delta_t U_i^{(n)}, \quad 0 \leq i \leq I.$$

Proof. Using Taylor's expansion, we get

$$\delta_t (1 - U_i^{(n)})^{-p} = p(1 - U_i^{(n)})^{-p-1} \delta_t U_i^{(n)} + \frac{p(p+1)}{2} \Delta t_n (1 - \theta_i^{(n)})^{-p-2} (\delta_t U_i^{(n)})^2,$$

where $\theta_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_i^{(n+1)}$, $0 \leq i \leq I$. We use the fact that $\|U_h^{(n)}\|_\infty < 1$, $n \geq 0$ to complete the proof. □

Lemma 2.6. *Let $U_h^{(n)}$, $n \geq 0$, be the solution of the discrete problem (2.1)–(2.3). Then*

$$\delta_t U_i^{(n)} \geq 0, \quad 0 \leq i \leq I.$$

Proof. Consider the vector $Z_h^{(n)}$ such that $Z_i^{(n)} = \delta_t U_i^{(n)}$, $0 \le i \le I$. Using Lemma 2.5, a straightforward calculation gives

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} - p(1 - U_i^{(n)})^{-p-1} Z_i^{(n)} \ge 0, \quad 0 \le i \le I-1,$$

$$\delta_t Z_I^{(n)} - \delta^2 Z_I^{(n)} - q\frac{2}{h}(U_I^{(n)})^{-q-1} Z_I^{(n)} - p(1 - U_I^{(n)})^{-p-1} Z_I^{(n)} \ge 0.$$

Since $Z_h^{(0)} \ge 0$, from Lemma 2.2, we have $Z_h^{(n)} \ge 0$, which implies that $\delta_t U_i^{(n)} \ge 0$, $0 \le i \le I$. □

Lemma 2.7. *Let $U_h^{(n)}$, $n \ge 0$ be the solution of the discrete problem (2.1)–(2.3). Then we have*

$$U_i^{(n)} > 0,\ 0 \le i \le I,\ n \ge 0 \text{ when } \Delta t_n \le \frac{h^2}{2}.$$

Proof. A straightforward computation shows that

$$U_i^{(n+1)} = \left(1 - 2\frac{\Delta t_n}{h^2}\right) U_i^{(n)} + \frac{\Delta t_n}{h^2}\left(U_{i+1}^{(n)} + U_{i-1}^{(n)}\right) + \Delta t_n (1 - U_i^{(n)})^{-p}, 1 \le i \le I-1$$

$$U_0^{(n+1)} = \left(1 - 2\frac{\Delta t_n}{h^2}\right) U_0^{(n)} + \frac{2\Delta t_n}{h^2} U_1^{(n)} + \Delta t_n (1 - U_0^{(n)})^{-p},$$

$$U_I^{(n+1)} = \left(1 - 2\frac{\Delta t_n}{h^2}\right) U_I^{(n)} + \frac{2\Delta t_n}{h^2} U_{I-1}^{(n)} - \frac{2\Delta t_n}{h}(U_I^{(n)})^{-q} + \Delta t_n (1 - U_I^{(n)})^{-p}.$$

If $U_h^{(n)} > 0$, then using an argument of recursion, we easily see that $U_h^{(n+1)} > 0$, because $1 - 2\frac{\Delta t_n}{h^2} \ge 0$. This end the proof. □

Lemma 2.8. *Let $U_h^{(n)}$, $n \ge 0$ be the solution of the discrete problem (2.1)–(2.3). Then we have*

$$U_{i+1}^{(n)} < U_i^{(n)}, \quad 0 \le i \le I-1. \tag{2.4}$$

Proof. Define the vector $Z_h^{(n)}$ such that $Z_i^{(n)} = U_i^{(n)} - U_{i+1}^{(n)}$, $0 \le i \le I-1$. We have

$$Z_i^{(n)} = U_i^{(n)} - U_{i+1}^{(n)}, \quad 0 \le i \le I-2,$$

$$Z_{I-1}^{(n)} = U_{I-1}^{(n)} - U_I^{(n)}.$$

By a straightforward computations, we have

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} - p(1 - \zeta_i^{(n)})^{-p-1} Z_i^{(n)} = 0, \quad 0 \le i \le I-2,$$

$$\delta_t Z_{I-1}^{(n)} - \delta^2 Z_{I-1}^{(n)} - \frac{2}{h}(U_I^{(n)})^{-q} - p(1 - \zeta_{I-1}^{(n)})^{-p-1} Z_{I-1}^{(n)} = 0.$$

Where $\zeta_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_{i+1}^{(n)}$, $0 \le i \le I-1$. Knowing that $Z_h^{(0)} > 0$, from Lemma 2.2, we have $Z_h^{(n)} > 0$, which implies that $U_{i+1}^{(n)} < U_i^{(n)}$, $0 \le i \le I-1$, and we obtain the desired result. □

3. Discrete quenching solutions

In this section, under some assumptions, we show that the solution $U_h^{(n)}$ of the discrete problem (2.1)–(2.3) quenches in a finite time and estimate its numerical quenching time. Now let us set $V_h^{(n)} = 1 - U_h^{(n)}$. The problem (2.1)–(2.3) is equivalent to

$$\delta_t V_i^{(n)} = \delta^2 V_i^{(n)} - (V_i^{(n)})^{-p}, \quad 0 \le i \le I-1, \tag{3.1}$$

$$\delta_t V_I^{(n)} = \delta^2 V_I^{(n)} + \frac{2}{h}(1 - V_I^{(n)})^{-q} - (V_I^{(n)})^{-p}, \tag{3.2}$$

$$V_i^{(0)} = \xi_i = 1 - \varphi_i, \quad 0 \le i \le I, \tag{3.3}$$

where

$$n \ge 0, \quad p > 0, \quad q > 0.$$

Lemma 3.1. *Let $V_h^{(n)}$, $n \ge 0$ be a sequence such that $\|V_h^{(n)}\|_{\inf} > 0$. Then we have*

$$\delta_t (V_i^{(n)})^{-p} \ge -p(V_i^{(n)})^{-p-1} \delta_t V_i^{(n)}, \quad 0 \le i \le I.$$

Proof. Using Taylor's expansion, we get

$$\delta_t (V_i^{(n)})^{-p} = -p(V_i^{(n)})^{-p-1} \delta_t V_i^{(n)} + \frac{p(p+1)}{2} \Delta t_n (\theta_i^{(n)})^{-p-2} (\delta_t V_i^{(n)})^2, 0 \le i \le I,$$

where $\theta_i^{(n)}$ is an intermediate value between $V_i^{(n)}$ and $V_i^{(n+1)}$, $0 \le i \le I$. We use the fact that $\|V_h^{(n)}\|_{inf} > 0$, $n \ge 0$ to complete the proof. □

Lemma 3.2. *Let $V_h^{(n)}$, $n \ge 0$ be a sequence such that $\|V_h^{(n)}\|_{\inf} > 0$. Then we have*

$$\delta^2 (V_i^{(n)})^{-p} \ge -p(V_i^{(n)})^{-p-1} \delta^2 V_i^{(n)}, \quad 0 \le i \le I.$$

Proof. Applying Taylor's expansion, we obtain

$$\begin{aligned}\delta^2 (V_i^{(n)})^{-p} = -p(V_i^{(n)})^{-p-1} \delta^2 V_i^{(n)} &+ (V_{i-1}^{(n)} - V_i^{(n)})^2 \frac{p(p+1)}{2h^2} (\theta_i^{(n)})^{-p-2} \\ &+ (V_{i+1}^{(n)} - V_i^{(n)})^2 \frac{p(p+1)}{2h^2} (\varepsilon_i^{(n)})^{-p-2}, \ 1 \le i \le I-1,\end{aligned}$$

$$\delta^2 (V_0^{(n)})^{-p} = -p(V_0^{(n)})^{-p-1} \delta^2 V_0^{(n)} + (V_1^{(n)} - V_0^{(n)})^2 \frac{p(p+1)}{h^2} (\theta_0^{(n)})^{-p-2},$$

$$\delta^2 (V_I^{(n)})^{-p} = -p(V_I^{(n)})^{-p-1} \delta^2 V_I^{(n)} + (V_{I-1}^{(n)} - V_I^{(n)})^2 \frac{p(p+1)}{h^2} (\theta_I^{(n)})^{-p-2},$$

where $\theta_0^{(n)}$ is an intermediate value between $V_0^{(n)}$ and $V_1^{(n)}$, $\theta_i^{(n)}$ is an intermediate value between $V_{i-1}^{(n)}$ and $V_i^{(n)}$, $1 \le i \le I-1$, $\theta_I^{(n)}$ is an intermediate value between $V_{I-1}^{(n)}$ and $V_I^{(n)}$, $\varepsilon_i^{(n)}$ is an intermediate value between $V_i^{(n)}$ and $V_{i+1}^{(n)}$, $1 \le i \le I-1$. The result follows taking into account the fact that $\|V_h^{(n)}\|_{\inf} > 0$. □

Theorem 3.3. *Let $U_h^{(n)}$ be the solution of (2.1)–(2.3). Suppose that there exists a constant $A \in (0,1]$ such that the initial data at (3.3) satisfies*

$$\delta^2\xi_i - \xi_i^{-p} \leq -A\xi_i^{-p}, \quad 0 \leq i \leq I-1, \tag{3.4}$$

$$\delta^2\xi_I + \frac{2}{h}(1-\xi_I)^{-q} - \xi_I^{-p} \leq -A\xi_I^{-p}. \tag{3.5}$$

Then $U_h^{(n)}$ quenches in a finite time $T_h^{\Delta t} = \sum_{n=0}^{+\infty} \Delta t_n$, which satisfies the estimate

$$T_h^{\Delta t} \leq \frac{\tau(1-\|\varphi_h\|_\infty)^{p+1}}{1-(1-\tau')^{p+1}},$$

where $\Delta t_n = \min\left\{\frac{h^2}{2}, \tau(V_{hmin}^{(n)})^{p+1}\right\}$ with $\tau \in (0,1)$, $V_{hmin}^{(n)} = (1-\|U_h^{(n)}\|_\infty)$ and

$$\tau' = A\min\{\frac{h^2(\xi_{hmin})^{-p-1}}{2}, \tau\}.$$

Proof. Introduce the vector $J_h^{(n)}$ defined as follows

$$J_i^{(n)} = \delta_t(V_i^{(n)}) + A(V_i^{(n)})^{-p}, \quad 0 \leq i \leq I, \quad n \geq 0.$$

A straightforward computation yields for $0 \leq i \leq I$ and $n \geq 0$,

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t(\delta_t V_i^{(n)} - \delta^2 V_i^{(n)}) + A\delta_t(V_i^{(n)})^{-p} - A\delta^2(V_i^{(n)})^{-p}.$$

Using (3.1)–(3.2) we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = -(1-A)\delta_t(V_i^{(n)})^{-p} - A\delta^2(V_i^{(n)})^{-p}, 0 \leq i \leq I-1,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = -(1-A)\delta_t(V_I^{(n)})^{-p} + \frac{2q}{h}(1-V_I^{(n)})^{-q-1}\delta_t V_I^{(n)} - A\delta^2(V_I^{(n)})^{-p}.$$

It follows from Lemma 3.1 and Lemma 3.2 that for $n \geq 0$,

$$\begin{aligned}&\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \\ &\leq p(1-A)(V_i^{(n)})^{-p-1}\delta_t V_i^{(n)} + Ap(V_i^{(n)})^{-p-1}\delta^2 V_i^{(n)}, 0 \leq i \leq I-1,\end{aligned}$$

$$\begin{aligned}&\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} \\ &\leq p(1-A)(V_I^{(n)})^{-p-1}\delta_t V_I^{(n)} + \frac{2q}{h}(1-V_I^{(n)})^{-q-1}\delta_t V_I^{(n)} + Ap(V_I^{(n)})^{-p-1}\delta^2 V_I^{(n)}.\end{aligned}$$

We deduce that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} - p(V_i^{(n)})^{-p-1}J_i^{(n)} \leq 0, \quad 0 \leq i \leq I-1,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(V_I^{(n)})^{-p-1}J_I^{(n)} - \frac{2q}{h}(1-V_I^{(n)})^{-q-1}\delta_t V_I^{(n)} \leq 0.$$

By inequalities (3.4) and (3.5), we have $J_h^{(0)} \leq 0$. Applying Lemma 2.2, we get $J_h^{(n)} \leq 0$ for $n \geq 0$, which implies that

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} \leq -A(V_i^{(n)})^{-p}, \quad 0 \leq i \leq I, \quad n \geq 0.$$

We get:

$$V_i^{(n+1)} \le V_i^{(n)}\left(1 - A\Delta t_n (V_i^{(n)})^{-p-1}\right), \quad 0 \le i \le I, \quad n \ge 0. \tag{3.6}$$

These estimates reveal that the sequence $V_h^{(n)}$ is nonincreasing. By induction, we obtain $V_h^{(n)} \le V_h^{(0)} = \xi_h$. Thus, the following holds

$$A\Delta t_n (V_{hmin}^{(n)})^{-p-1} \ge A \min\{\frac{h^2(\xi_{hmin})^{-p-1}}{2}, \tau\} = \tau'$$

Let i_0 be such that $V_{hmin}^{(n)} = V_{i_0}^{(n)}$. Replacing i by i_0 in (3.6), we obtain

$$V_{hmin}^{(n+1)} \le V_{hmin}^{(n)}(1-\tau'), \quad n \ge 0, \tag{3.7}$$

and by iteration, we arrive at

$$V_{hmin}^{(n)} \le V_{hmin}^{(0)}(1-\tau')^n = \xi_{hmin}(1-\tau')^n, \quad n \ge 0. \tag{3.8}$$

Since the term on the right hand side of the above equality goes to zero as n approaches infinity, we conclude that $V_{hmin}^{(n)}$ tends to zero as n approaches infinity and $\|U_h^{(n)}\|_\infty$ tends to one as n approaches infinity. Now, let us estimate the numerical quenching time. Due to (3.8) and the restriction $\Delta t_n \le \tau (V_{hmin}^{(n)})^{p+1}$, it is not hard to see that

$$\sum_{n=0}^{+\infty} \Delta t_n \le \sum_{n=0}^{+\infty} \tau \xi_{hmin}^{p+1}[(1-\tau')^{p+1}]^n.$$

Use the fact that the series on the right hand side of the above inequality converges towards

$$\frac{\tau \xi_{hmin}^{p+1}}{1-(1-\tau')^{p+1}}$$

and $\xi_{hmin} = (1 - \|\varphi_h\|_\infty)$, we obtain

$$T_h^{\Delta t} \le \frac{\tau(1-\|\varphi_h\|_\infty)^{p+1}}{1-(1-\tau')^{p+1}}.$$

□

Remark 3.4. Using Taylor's expansion, we get

$$1-(1-\tau')^{p+1} = (p+1)\tau' + o(\tau'),$$

which implies that

$$\frac{\tau}{1-(1-\tau')^{p+1}} = \frac{\tau}{\tau'}\frac{1}{(p+1)+o(1)} \le \frac{\tau}{\tau'}\frac{2}{(p+1)}.$$

If we take $\tau = h^2$, we have

$$\frac{\tau}{\tau'} = \frac{1}{A}\min\{2\xi_{hmin}^{p+1}, 1\}.$$

Then

$$\frac{\tau}{1-(1-\tau')^{p+1}} \le \frac{2\tau}{\tau'(p+1)} = \frac{2}{A(p+1)}\min\{2\xi_{hmin}^{p+1}, 1\}.$$

We conclude that $\dfrac{\tau}{1-(1-\tau')^{p+1}}$ is bounded.

Remark 3.5. From (3.8) we deduce by induction that

$$V^{(n)}_{hmin} \leq V^{(k)}_{hmin}(1-\tau')^{n-k}, \text{ for } n \geq k,$$

and we see that

$$T_h^{\Delta t} - t_k = \sum_{n=k}^{+\infty} \Delta t_n \leq \sum_{n=k}^{+\infty} \tau (V^{(k)}_{hmin})^{p+1}[(1-\tau')^{p+1}]^{n-k},$$

which implies that

$$T_h^{\Delta t} - t_k \leq \frac{\tau (V^{(k)}_{hmin})^{p+1}}{1-(1-\tau')^{p+1}}.$$

Since $V^{(k)}_{hmin} = (1 - \|U_h^k\|_\infty)$, we get

$$T_h^{\Delta t} - t_k \leq \frac{\tau (1-\|U_h^k\|_\infty)^{p+1}}{1-(1-\tau')^{p+1}}.$$

In the sequel, we take $\tau = h^2$.

4. Convergence of the discrete quenching time

In this section, under some assumptions, we show that the numerical quenching time of the discrete solution converges to the real one when the mesh size goes to zero. We denote by

$$u_h(t_n) = (u(x_0,t_n), u(x_1,t_n), ..., u(x_I,t_n))^T \text{ and } \|U_h^{(n)}\|_\infty = \max_{0\leq i\leq I} |U_i^{(n)}|.$$

In order to obtain the convergence of the numerical quenching time, we firstly prove the following theorem about the convergence of the discrete scheme.

Theorem 4.1. *Assume that the continuous problem* (1.1)–(1.3) *has a solution* $u \in C^{4,2}([0,1]\times[0,T])$ *such that* $\sup_{t\in[0,T]} \|u(.,t)\|_\infty = \zeta$, $(0<\zeta<1)$. *Suppose the initial condition at* (2.3) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \longrightarrow 0. \tag{4.1}$$

Then, for h sufficiently small, the discrete problem (2.1)–(2.3) *has a solution* $U_h^{(n)}$, $0 \leq n \leq J$, *and we have the following relation*

$$\max_{0\leq n\leq J}(\|U_h^{(n)} - u_h(t_n)\|_\infty) = O(\|\varphi_h - u_h(0)\|_\infty + h) \quad as \quad h \longrightarrow 0.$$

Where J is such that $\sum_{j=0}^{J-1} \Delta t_j \leq T$ *and* $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

Proof. For each h, the discrete problem (2.1)–(2.3) has a solution $U_h^{(n)}$. Let $N \leq J$, the greatest value of n such that there exists a positive constant β (with $\zeta < \beta < 1$) such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{\beta-\zeta}{2}, \quad n < N. \tag{4.2}$$

We know that $N \geq 1$ because of (4.1). Using the triangular inequality for $n < N$, we have

$$\|U_h^{(n)}\|_\infty \leq \|u_h(t_n)\|_\infty + \|U_h^{(n)} - u_h(t_n)\|_\infty \leq \zeta + \frac{\beta - \zeta}{2} = \frac{\beta + \zeta}{2} < 1. \tag{4.3}$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization for $n < N$. Using Taylor's expansion, we have

$$\begin{aligned}
&\delta_t e_0^{(n)} - \delta^2 e_0^{(n)} \\
&= p(1-\sigma_0^{(n)})^{-p-1} e_0^{(n)} + h\left(\frac{h}{12}u_{xxxx}(\tilde{x}_0, t_n) + \frac{2}{3}u_{xxx}(x_0, t_n)\right) - \frac{\Delta t_n}{2}u_{tt}(x_0, \tilde{t}_n), \\
&\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \\
&= p(1-\sigma_i^{(n)})^{-p-1} e_i^{(n)} + \frac{h^2}{12}u_{xxxx}(\tilde{x}_i, t_n) - \frac{\Delta t_n}{2}u_{tt}(x_i, \tilde{t}_n), 1 \leq i \leq I-1, \\
&\delta_t e_I^{(n)} - \delta^2 e_I^{(n)} \\
&= \left(p(1-\sigma_I^{(n)})^{-p-1} + \frac{2q}{h}(\mu_I^{(n)})^{-q-1}\right) e_I^{(n)} \\
&\quad + h\left(\frac{h}{12}u_{xxxx}(\tilde{x}_I, t_n) - \frac{2}{3}u_{xxx}(x_I, t_n)\right) - \frac{\Delta t_n}{2}u_{tt}(x_I, \tilde{t}_n),
\end{aligned}$$

where $\sigma_i^{(n)}$ is intermediate value between $U_i^{(n)}$ and $u(x_i, t_n)$, $0 \leq i \leq I$ and $\mu_I^{(n)}$ is intermediate value between $U_I^{(n)}$ and $u(x_I, t_n)$. Since $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$ and $u_{tt}(x,t)$ are bounded and $\Delta t_n = O(h^2)$, there exist a positive constant $K > 0$ such that

$$\delta_t e_0^{(n)} - \delta^2 e_0^{(n)} \leq C_0^{(n)} e_0^{(n)} + Kh,$$

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \leq C_i^{(n)} e_i^{(n)} + Kh^2, \quad 1 \leq i \leq I-1,$$

$$\delta_t e_I^{(n)} - \delta^2 e_I^{(n)} \leq C_I^{(n)} e_I^{(n)} + Kh,$$

where

$$C_0^{(n)} = p(1-\sigma_0^{(n)})^{-p-1},$$

$$C_i^{(n)} = p(1-\sigma_i^{(n)})^{-p-1}, \quad 1 \leq i \leq I-1,$$

$$C_I^{(n)} = p(1-\sigma_I^{(n)})^{-p-1} + \frac{2q}{h}(\mu_I^{(n)})^{-q-1}.$$

Set $M = \max_{0 \leq i \leq I}\{C_i^{(n)}\}$ and introduce the vector $Z_h^{(n)}$ defined as follows

$$Z_i^{(n)} = e^{(M+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Kh), \quad 0 \leq i \leq I, \quad n < N.$$

By a straightforward computations, we have

$$\delta_t Z_0^{(n)} - \delta^2 Z_0^{(n)} > C_0^{(n)} Z_0^{(n)} + Kh,$$

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} > C_i^{(n)} Z_i^{(n)} + Kh^2, \quad 1 \leq i \leq I-1,$$

$$\delta_t Z_I^{(n)} - \delta^2 Z_I^{(n)} > C_I^{(n)} Z_I^{(n)} + Kh,$$

$$Z_i^{(0)} > e_i^{(0)}, \quad 0 \le i \le I.$$

It follows from Lemma 2.4 that

$$Z_i^{(n)} > e_i^{(n)}, \quad 0 \le i \le I.$$

By the same way, we also prove that

$$Z_i^{(n)} > -e_i^{(n)}, \quad 0 \le i \le I,$$

which implies that

$$Z_i^{(n)} > |e_i^{(n)}|, \quad 0 \le i \le I.$$

we deduce that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \le e^{(M+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Kh), \quad n < N. \tag{4.4}$$

Now, let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (4.4), and taking into account the inequality (4.2), we obtain

$$\frac{\beta - \zeta}{2} \le \|U_h^{(N)} - u_h(t_N)\|_\infty \le e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + Kh). \tag{4.5}$$

Since $e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + Kh) \longrightarrow 0$ as $h \longrightarrow 0$, we deduce from (4.5) that $\frac{\beta - \zeta}{2} \le 0$, which is impossible. Consequently $N = J$, and we conclude the proof. □

Theorem 4.2. *Suppose that the solution u of problem (1.1)–(1.3) quenches in a finite time T_q such that $u \in C^{4,2}([0,1] \times [0, T_q))$ and the iniatial data at (2.3) satisfies*

$$||\varphi_h - u_h(0)||_\infty = o(1) \text{ as } h \longrightarrow 0.$$

Under the hypothesis of Theorem 3.3, the problem (2.1)–(2.3) has a discrete solution $U_h^{(n)}$ which quenches in a finite time $T_h^{\Delta t}$ and we have

$$\lim_{h \to 0} T_h^{\Delta t} = T_q.$$

Proof. We know from Remark 3.4 that $\frac{\tau}{1 - (1 - \tau')^{p+1}}$ is bounded.

Let $0 < \varepsilon < \frac{T_q}{2}$, there exists a constant $\eta = \beta - \zeta$ $(0 < \zeta < \beta < 1)$ such that

$$\frac{\tau(1 - \varrho)^{p+1}}{1 - (1 - \tau')^{p+1}} < \frac{\varepsilon}{2}, \quad \varrho \in [1 - \eta, 1). \tag{4.6}$$

Since u quenches in finite time T_q, there exists $T_1 \in (T_q - \frac{\varepsilon}{2}, T_q)$ and $h_0(\varepsilon) > 0$ such that $1 - \frac{\eta}{2} \le \|u(., t_n)\|_\infty < 1$ for $t_n \in [T_1, T_q)$. Let k be a positive integer such that $t_k = \sum_{n=0}^{k-1} \Delta t_n \in [T_1, T_q)$ for $h \le h_0(\varepsilon)$. It follows from Theorem 4.1 that the problem (2.1)–(2.3) has a solution $U_h^{(n)}$ which verifies $\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{\eta}{2}$ for $n \le k$, $h \le h_0(\varepsilon)$. This fact implies that

$$\|U_h^{(k)}\|_\infty \ge \|u(., t_k)\|_\infty - \|U_h^{(k)} - u_h(t_k)\|_\infty \ge 1 - \frac{\eta}{2} - \frac{\eta}{2} = 1 - \eta, \quad h \le h_0(\varepsilon).$$

From Theorem 3.3, $U_h^{(n)}$ quenches at the time $T_h^{\Delta t}$. It follows from Remark 3.5 and (4.6) that $|T_h^{\Delta t} - t_k| \leq \frac{\tau(1-\|U_h^{(k)}\|_\infty)^{p+1}}{1-(1-\tau')^{p+1}} < \frac{\varepsilon}{2}$. We deduce that for $h \leq h_0(\varepsilon)$,

$$|T_q - T_h^{\Delta t}| \leq |T_q - t_k| + |t_k - T_h^{\Delta t}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Which leads us to the result. □

5. Numerical experiments

In this section, we present some numerical approximations to the quenching time of the problem (1.1)–(1.3) in the case where $u_0(x) = 0.7 - \frac{1}{2}x^4$. We consider the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + (1 - U_i^{(n)})^{-p},\ 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n^e} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (1 - U_0^{(n)})^{-p},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n^e} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} - \frac{2}{h}(U_I^{(n)})^{-q} + (1 - U_I^{(n)})^{-p},$$

$$U_i^{(0)} = \varphi_i,\ 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n^e = \min\left\{\frac{h^2}{2}, h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}\right\}$. We also consider the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + (1 - U_i^{(n)})^{-p},\ 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (1 - U_0^{(n)})^{-p},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} - \frac{2}{h}(U_I^{(n)})^{-q} + (1 - U_I^{(n)})^{-p},$$

$$U_i^{(0)} = \varphi_i,\ 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}$. In the following tables, in rows, we present the numerical quenching times, the numbers of iterations and the orders of the approximations corresponding to meshes $16, 32, 64, 128, 256, 512$. The numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when

$$|T^{n+1} - T^n| \leq 10^{-16}.$$

The order s of the method is computed from

$$s = \frac{log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{log(2)}.$$

For the discrete initial data we take $\varphi_i = 0.7 - \dfrac{1}{2}(ih)^4$.

Table 1. Numerical quenching times obtained with the explicit Euler method $p = 4$ and $q = -log(2)/log(0.2)$

I	T^n	n	s
16	0.00048983809	1292	-
32	0.00048696537	4891	-
64	0.00048624977	18434	2.00
128	0.00048607104	69198	2.00
256	0.00048602636	258629	2.00
512	0.00048601519	961840	2.00

Table 2. Numerical quenching times obtained with the implicit Euler method $p = 4$ and $q = -log(2)/log(0.2)$

I	T^n	n	s
16	0.00049012477	1292	-
32	0.00048703076	4891	-
64	0.00048626995	18434	2.02
128	0.00048607886	69199	1.99
256	0.00048602982	258631	1.96
512	0.00048601682	961844	1.91

Next, we give some plots to illustrate our analysis. We the case where $I = 64$, $p = 4$ and $q = -log(2)/log(0.2)$.

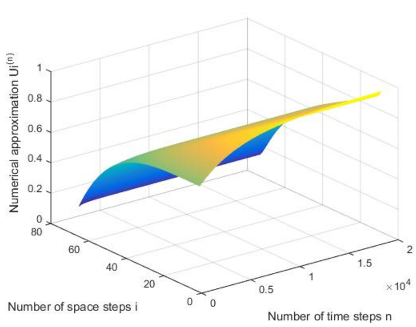

Figure 1. Evolution of the numerical solution (explicit scheme).

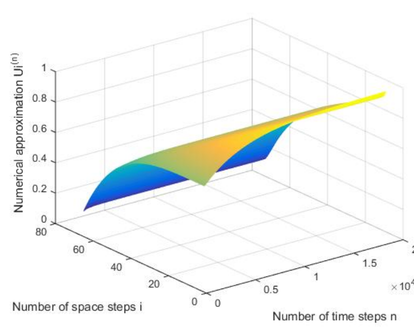

Figure 2. Evolution of the numerical solution (implicit scheme).

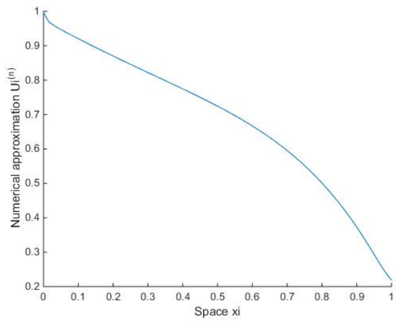

FIGURE 3. The profil of the approximation of u(x,T) where, T is the quenching time (explicit scheme).

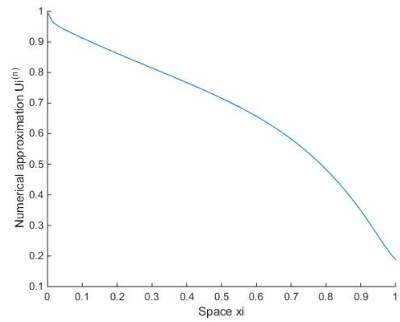

FIGURE 4. The profil of the approximation of u(x,T) where, T is the quenching time (implicit scheme).

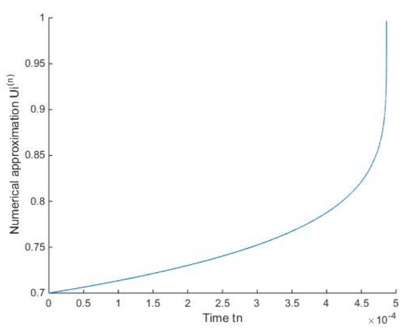

FIGURE 5. The profil of the approximation of $\|U_h^{(n)}\|_\infty$ (explicit scheme).

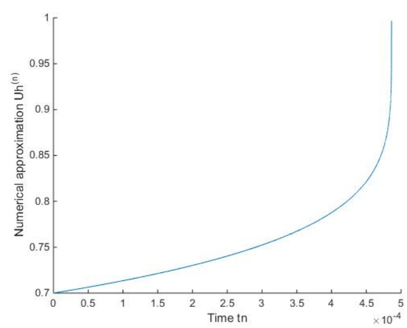

FIGURE 6. The profil of the approximation of $\|U_h^{(n)}\|_\infty$ (implicit scheme).

Remark 5.1. We can observe from figures 1–4 that the semidiscrete solution quenches in a finite time at the fisrt node, which is well known in a theoretical point of view. For figures 5–6 we see that the semidiscrete solution quenches in a finite time close to 4.9×10^{-4}.

6. CONCLUSION

In this paper, we have studied the numerical quenching of the solution of the semi-linear heat equation (1.1)–(1.3) and we have obtained good approximations of its quenching time.

We have constructed, by the finite difference method, the discrete problem (2.1)–(2.3) associated to the continuous problem (1.1)–(1.3). We have shown that under some conditions, the solution of the discrete problem (2.1)–(2.3) quenches in finite time and we have estimated its discrete quenching time. We have also established the convergence of the discrete time towards the theoretical time when the spatial and temporal discretionary steps tend towards zero. Finally, we have given some numerical experiments to illustrate our analysis.

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